# Adaptive finite time tracking control for the perturbed nonlinear systems by using backstepping technique and a fast tracking differentiator

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**Abstract**: In this work, the fast tracking control issue is researched for the pure-feedback nonlinear systems in the case of compound disturbances. Based on the stability theory, a new nonlinear tracking-differentiator containing two nonlinear terminal terms and one linear term is developed, which provides a faster convergence rate. The chatter phenomenon can be well attenuated owing to the application of continuous functions contained in tracking-differentiator. Next, the robust tracking control scheme is devised by adopting the combination of tracking-differentiator and adaptive backstepping technique, which not only ensures that the system states are all bounded, but also makes the system output track the reference signal in a limited time. In addition, owing to the utilization of newly designed tracking-differentiator, the problem of complexity explosion occurring in the backstepping control can be nicely addressed, and the recursive design procedure can be well simplified. In the end, a numerical example is given to show the effectiveness of obtained results.

*Keywords*: Pure-feedback nonlinear systems, tracking-differentiator(TD), backstepping control, finite time stability.

## 1. INTRODUCTION

The stabilization and tracking control of nonlinear systems is a hot topic and has aroused a great deal of research interests of scholars from different discipline fields. A mass of different control schemes have been applied to the many practical systems including multi-agent systems (Zheng et al., 2023; Rehak et al., 2020), robot manipulators (Shi et al., 2018; Abdul-Adheem et al., 2020), biological population system (Lamrani et al., 2023), permanent-magnet synchronous motor (PMSM) (Zou et al., 2020) and so on. Therefore, the research on the control of nonlinear systems has a theoretical and practical significance.

The backstepping control is a systematical recursive design approach and a powerful control technique, which has received much attention from researchers (Peng et al., 2019; Liu et al., 2019; Tian et al., 2023). Combing barrier Lyapunov function and backstepping method, some adaptive controllers were constructed by (Hua et al., 2018) to ensure the boundedness of tracking error and state trajectories of closed-loop system. The asymptotical tacking control issue of MIMO nonlinear systems with unknown dead-zones was considered in (Wang et al., 2019) by using backstepping fuzzy control. The adaptive neural control and backstepping approach were exploited in (Gao C et al., 2020) to deal with the finite-time tracking control problem of pure-feedback systems. The practical stability of nonlinear systems in a strict-feedback form was investigated in (Kim et al., 2019) by adopting backstepping criteria integrated with Lyapunov-based model predictive control. By the backstepping technique and super-twisting algorithm, article (Wu et al., 2019) put forward a sensorless strategy to realize speed regulation of interior permanent magnet synchronous motors. The asymptotical tracking control problem of strict-feedback nonlinear systems was resolved in (Parsa et al., 2021) by using backstepping method and command filters.

It can be observed from above research results that backstepping method plays an important role in controller design. Nevertheless, in the design procedure of backstepping approach, the “explosion of complexity” and singularity of actual controller will be exposed due to the repeated derivation of virtual controllers. To reduce the complexity of actual controller, the fuzzy observer (Peng et al., 2019; Wang et al., 2019), super-twisting algorithm (Wu et al., 2019) and command filter (Parsa et al., 2021) are presented to approximate the virtual controller and its derivative. However, the design procedure in above is rather complicated and there are so many parameters to be determined, which is a great disadvantage to practical engineering. Besides, the control schemes developed in article (Peng et al., 2019; Liu et al., 2019; Tian et al., 2023; Hua et al., 2018; Wang et al., 2019; Gao C. et al., 2020; Kim et al., 2019; Wu et al., 2019; Parsa et al., 2021) can only ensure that the signals of closed-loop system are all asymptotically bounded, and the output tracking error can only asymptotically converge to a small neighborhood of the origin. Then, the control designs mentioned above may not be suitable for systems with high tracking performance. So, how to devise a valid control scheme with faster convergence rate to avoid the ``explosion of complexity" for the uncertain nonlinear systems is an important and significant work.

Recently, the tracking differentiator (TD) has became a powerful tool in function estimation because of its simplicity and high-precision approximation. Various composite control criteria consisting of TDs have been proposed and applied to the tracking control and stabilization of diffferent systems

(Bu et al., 2015; Wang et al., 2021; Wang et al., 2022; Levant et al., 2020; Gao Y et al., 2020). It is noted that TD has a good robustness to the uncertainties and disturbances owing to its simple structure and noise suppression ability. In addition, duo to the influence of physical structure and changeable environment, the operation of systems will be inevitably interfered, thus it is necessary to take the inner impactors and external disturbances into consideration when designing control scheme of nonlinear systems.

Out of above consideration, this paper intends to explore the tracking control problem of nonlinear systems with disturbances by designing a new fast backsteppting control criterion. An adaptive anti-interference tracking strategy is proposed by using a new TD and backstepping congtrol, the strict theoretical analysis of control design is provided for the stability of closed-loop systems. The stability analysis shows that all the signals of perturbed system are finite time bounded and the tracking error can be arbitrarily small. Moreover, in each step of control design, the newly constructed TD is used to approximate the virtual controller and its derivative, which eliminates the problem of ``explosion complexity" occurred in the control scheme. Simulation results show the feasibility of given control scheme.

The rest of this paper is organized as follows. In Section 2, a new TD is presented. The design of new tracking control based on TD is presented in Section 3. In Section 4, simulation studies are made to demonstrate the main results. The conclusions are proposed in Section 5.

## 2. PRELIMINARIES

A nonlinear systems with disturbances is described as:

$$\begin{cases} \dot{x}_1 = f_1(\overline{x}_1) + g_1(\overline{x}_1)x_2 + d_1(t), \\ \dot{x}_2 = f_2(\overline{x}_2) + g_2(\overline{x}_2)x_3 + d_2(t), \\ \quad \vdots \qquad \vdots \qquad \vdots \\ \dot{x}_i = f_i(\overline{x}_i) + g_i(\overline{x}_i) + d_i(t), \\ \quad \vdots \qquad \vdots \qquad \vdots \\ \dot{x}_{n-1} = f_{n-1}(\overline{x}_{n-1}) + g_{n-1}(\overline{x}_{n-1})x_n + d_{n-1}(t), \\ \dot{x}_n = f_n(\overline{x}_n) + g_n(\overline{x}_n)u + d_n(t), \\ y = x_1, \end{cases} \tag{1}$$

where $\overline{x}_i = (x_1, x_2, \cdots, x_i)^T \in R^i$, $(i = 1, 2, \cdots, n)$ denotes the system's state vector. $f_i(\overline{x}_i)$ and $f_n(\overline{x}_n)$ are the continuous nonlinear functions, $g_i(\overline{x}_i)$ and $g_n(\overline{x}_n)$ stand for the control gain functions, $d_i(t)$ denotes the compound disturbances consisting of inner and exogenous uncertainties, $y$ is the output.

**Remark 1.** There are many practical systems including hypersonic vehicle system (Bu et al., 2015; Gao Y et al., 2020), electromechanical motor system (Parsa et al., 2021), hydraulic actuator system (Wang et al., 2022) and so on, can be modeled by system (1). If some feasible control strategies are devised for system (1), then the stability control problems of above practical systems can be well addressed. Thus, the study of system (1) is valuable and the control designs can be applied to the areas of tracking control and system stabilization of aforementioned practical systems.

For system (1), the main objective of this paper is to synthesize a control scheme so that output $y$ can track the preset reference trajectory $r(t)$ as accurate as possible, meanwhile the tracking error finite-timely converges to a small neighborhood centered on the origin in facing of compound disturbances.

**Assumption 1.** Suppose that all the states of system (1) are measurable, the output signal $y(t)$ and reference signal $r(t)$ are bounded, continuous and differential with respected to $t$.

**Assumption 2.** Suppose that functions $g_i(\overline{x}_i) \neq 0$ $(i = 1, 2, \cdots, n)$ and satisfy $|g_i(\overline{x}_i)| \leq m_i$, here $m_i > 0$ $(i = 1, 2, \cdots, n)$.

**Assumption 3.** Suppose that disturbances $d_i(t)$ $(i = 1, 2, \cdots, n)$ satisfy $|d_i(t)| \leq D_i$, here $D_i \geq 0$ $(i = 1, 2, \cdots, n)$.

**Lemma 1** (Ou et al, 2021). For any $x_i \in R$ $(i = 1, 2, \cdots, n)$, $0 < \alpha \leq 1$, it holds:

$$\left(\sum_{i=1}^{n} |x_i|\right)^{\alpha} \leq \sum_{i=1}^{n} |x_i|^{\alpha} \leq n^{1-\alpha}\left(\sum_{i=1}^{n} |x_i|\right)^{\alpha}.$$

**Lemma 2** (Bu et al (2015)). An original system (2) is described by:

$$\begin{cases} \dot{y}_1 = y_2, (9) \\ \dot{y}_2 = F(y_1, y_2), \end{cases} \tag{2}$$

and a tracking differentiator system (3) is given by:

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = M^2 F(z_1 - v(t), \dfrac{z_2}{M}). \end{cases} \tag{3}$$

If any solution $(y_1(t), y_2(t))$ of system (2) satisfies $\lim_{t\to\infty} y_1(t) = 0$, $\lim_{t\to\infty} y_2(t) = 0$, then the solution $z_1(t)$ of system (3) holds that

$$\lim_{M\to\infty} \int_0^T |z_1(t) - v(t)| dt = 0, \tag{4}$$

where $y = (y_1, y_2) \in R^2$ is the state vector of (2), $z = (z_1, z_2) \in R^2$ is the state vector of (3), $v(t)$ is a bounded and integrable function, $T$ is a positive constant, $M$ is a proper positive constant.

**Lemma 3.** A new nonlinear system is presented as below:

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -a_1 y_1 - a_2 y_1^p - a_3 y_1^q \\ \quad -b_1 y_2 - b_2 y_2^p - b_3 y_2^q, \end{cases} \tag{5}$$

the system (5) is asymptotically stable at the origin, that is,

$$\lim_{t\to\infty} y_1(t) = 0, \quad \lim_{t\to\infty} y_2(t) = 0,$$

where $y = (y_1, y_2) \in R^2$ is the state vector, $a_1, a_2, a_3$, $b_1, b_2, b_3$ are the positive real numbers, constants $p, q$ such that $0 < p = \frac{p_1}{p_2} < 1 < q = \frac{q_1}{q_2} < \infty$, $p_1, p_2$ and $q_1, q_2$ are the positive odd constants.

**Proof:** For the system (5), a new Lyapunove function $V$ is constructed by:

$$V = \frac{a_1}{2} y_1^2 + \frac{a_2}{p+1} y_1^{p+1} + \frac{a_3}{q+1} y_1^{q+1} + \frac{1}{2} y_2^2.$$

The derivative of $V$ with regard to time is calculated as:

$$\begin{aligned} \dot{V} &= a_1 y_1 \dot{y}_1 + a_2 y_1^p \dot{y}_1 + a_3 y_1^q \dot{y}_1 + y_2 \dot{y}_2 \\ &= a_1 y_1 y_2 + a_2 y_1^p y_2 + a_3 y_1^q y_2 \\ &\quad + y_2(-a_1 y_1 - a_2 y_1^p - a_3 y_1^q) \\ &\quad + y_2(-b_1 y_2 - b_2 y_2^p - b_3 y_2^q) \\ &= -(b_1 y_2^2 + b_2 y_2^{p+1} + b_3 y_2^{q+1}). \end{aligned} \tag{6}$$

Noting that $p, q$ are the ratios of positive odd integers, then $p+1$ and $q+1$ are the positive even rational numbers. Accordingly, one gets

$$\dot{V} = -(b_1 y_2^2 + b_2 y_2^{p+1} + b_3 y_2^{q+1}) \le 0\,. \tag{7}$$

If $\dot{V} = 0$, then $y_2 = 0$. Substituting $y_2 = 0$ into the equations $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$, one obtains $y_1 = 0, y_2 = 0$. Thus when $\dot{V} = 0$, the $(y_1, y_2) = (0, 0)$ is the unique equilibrium. Based on the LaSalle invariance principle one canconclude that $\lim_{t\to\infty} y_1(t) = 0, \lim_{t\to\infty} y_2(t) = 0$. Thus the proof of Lemma 3 is over.

**Remark 2.** For the system (5), if system states are far away from the origin, i.e $||z|| >> 1$, the term $||z||+||z||^q$ dominates the convergence rate and provide a faster response performance. If system states close to the origin, i.e $||z|| << 1$, theterm $||z|| + ||z||^p$ dominate the convergence rate and provide a faster response performance. In a word, no matter where the system states are, the system (5) has a fast convergence rate.

**Lemma 4.** A new nonlinear tracking differentiator is given by:

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = M^2(-a_1(z_1 - v(t)) \\ \quad - a_2(z_1 - v(t))^p - a_3(z_1 - v(t))^q \\ \quad - b_1 \dfrac{z_2}{M} - b_2(\dfrac{z_2}{M})^p - b_3(\dfrac{z_2}{M})^q), \end{cases} \tag{8}$$

then the solution $z_1(t)$ of system (8) satisfies

$$\lim_{M\to\infty} \int_0^T | z_1(t) - v(t) | \, dt = 0, \tag{9}$$

where $z = (z_1, z_2) \in R^2$ is the state vector, $a_1, a_2, a_3$, $b_1, b_2, b_3$ are the positive real numbers, constants $p, q$ such that $0 < p = \frac{p_1}{p_2} < 1 < q = \frac{q_1}{q_2} < \infty$, $p_1, p_2$ and $q_1, q_2$ are the positive odd constants. $v(t)$ is a bounded and integrable function, $T$ is a positive constant, $M$ is a proper positive constant.

**Proof:** In view of the Lemma 2 and Lemma 3, it is easy to obtain the conclusion of Lemma 4.

**Remark 3.** According to the results of articles (Bu et al., 2015; Wang et al., 2021; Wang et al., 2022; Levant et al., 2020), one can choose a proper constant $M > 0$, then the $z_1(t)$ will weakly converge to $v(t)$, and the derivative of $v(t)$, i.e $\dot{v}(t)$, can be obtained by $z_2(t)$. Thus, based on the Lemma 4, the $v(t)$ and $\dot{v}(t)$ can be respectively approximated by $z_1(t)$ and $z_2(t)$ quickly.

**Remark 4.** To show the better convergence performance of the proposed TD, a comparison among TD (8), LinearTD (Wang et al., 2022) and Slide mode-TD (Levant, 2003) is given.

The linear-TD (LTD) from (Wang et al., 2022) is formulated as

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_1 = -\bar{h}_1(z_2 - v(t)) - \bar{h}_2 z_2, \end{cases}$$

Where $\bar{h}_1 > 0$ and $\bar{h}_2 > 0$ are adjustable parameters.

The slide mode-TD (SMTD) from (Levant, 2003) is given by

$$\begin{cases} \dot{z}_1 = -\bar{h}_1 | z_1 - v(t) |^{\frac{1}{2}} sign(z_1 - v(t)) + z_2, \\ \dot{z}_1 = -\bar{h}_1 sign(z_2 - \dot{z}_1), \end{cases}$$

Where $\bar{h}_1 > 0$ and $\bar{h}_2 > 0$ are adjustable parameters.

The input signal v(t) is taken as sin(t). The parameters of the aforementioned differentiators are listed in Table 1. The simulation results are reflflected in Fig.1. It is obvious from Fig.1 that the convergence rate and stability performance of proposed TD (8) are better than that of Linear-TD (Wang et al., 2022) and SMTD (Levant, 2003).

**Table 1. Different tracking differentiators.**

| Differentiator | The value of parameters |
|---|---|
| TD from Eq.(8) | $a_i = b_i = 1 (\mathrm{i} = 1,2,3)$, $M = 80$, $p = \frac{3}{5}$, $q = \frac{5}{3}$ |
| LTD in (Wang et al (2022)) | $\bar{h}_1 = 1, \bar{h}_2 = 1$ |
| SMTD in (Levant (2003)) | $\bar{h}_1 = 1, \bar{h}_2 = 1$ |

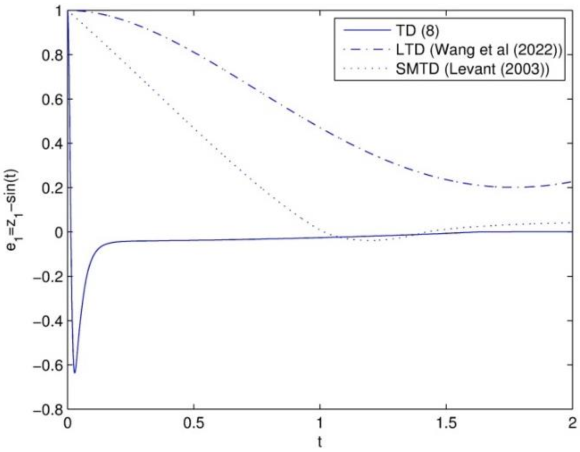


Fig. 1. The tracking performance of different tracking differentiators in Table 1.

## 3. TRACKING CONTROL STRATEGY

In this section, to quantify the control objective, the adaptive technique and backstepping approach are adopted to the design of track control for uncertain system (1). The recursive control design will be addressed in the step by step manner, which is derived as follows.

**Step 1.** To reduce the complexity of controllers, a tracking differentiator is devised as below:

$$\begin{cases} \dot{z}_{11} = z_{12}, \\ \dot{z}_{12} = M^2(-a_1(z_{11} - r(t)) - a_2(z_{11} - r(t))^p \\ \qquad - a_3(z_{11} - r(t))^q - b_1 \dfrac{z_{12}}{M} \\ \qquad - b_2(\dfrac{z_{12}}{M})^p - b_3(\dfrac{z_{12}}{M})^q), \end{cases} \tag{10}$$

where constants $a_1, a_2, a_3$, $b_1, b_2, b_3$, $p, q$ are the same as that in Lemma 4. Thus the reference signal $r(t)$ and its derivative $\dot{r}(t)$ can be obtained by differentiators $z_{11}$ and $z_{12}$, respectively. In what follows, some new variables are defined as:

$$e_1 = x_1 - z_{11}, \quad \xi_1 = z_{11} - r(t), \tag{11}$$

then the derivative of $e_1$ is given by

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 - \dot{z}_{11} \\ &= f_1 + g_1 x_2 + d_1 - z_{12} \\ &= f_1 + g_1(x_2 - x_{2d}) + g_1 x_{2d} + d_1 - z_{12}. \end{aligned} \tag{12}$$

The virtual controller $x_{2d}$ and gain adaptive law $\eta_1$ are designed as below:

$$x_{2d} = g_1^{-1}(-k_{11}e_1 - \eta_1 sign(e_1) - f_1 + z_{12}), \tag{13}$$

$$\dot{\eta}_1 = -k_{11}(\eta_1 - k_{12}) - l_1 sign(\eta_1 - k_{12}) + |e_1|, \tag{14}$$

where $x_{2d}$ denotes the virtual controller, $\eta_1$ represents the gain adaptive law.

The Lyapunov function is selected as:

$$V_1 = \frac{1}{2}e_1^2 + \frac{1}{2}(\eta_1 - k_{12})^2. \tag{15}$$

Calculating the derivative of $V_1$ gives

$$\begin{aligned} \dot{V}_1 &= e_1\dot{e}_1 + (\eta_1 - k_{12})\dot{\eta}_1 \\ &= e_1(f_1 + g_1(x_2 - x_{2d}) + g_1 x_{2d} \\ &\quad + d_1 - z_{12}) + (\eta_1 - k_{12})\dot{\eta}_1 \\ &= e_1(f_1 + g_1(x_2 - x_{2d}) + d_1 - z_{12} \\ &\quad - k_{11}e_1 - \eta_1 sign(e_1) - f_1 + z_{12}) \\ &\quad + (\eta_1 - k_{12})(-k_{11}(\eta_1 - k_{12}) \\ &\quad - l_1 sign(\eta_1 - k_{12}) + |e_1|) \\ &\le -k_{11}e_1^2 + D_1|e_1| - \eta_1|e_1| + g_1 e_1(x_2 - x_{2d}) \\ &\quad - k_{11}(\eta_1 - k_{12})^2 - l_1|\eta_1 - k_{12}| + (\eta_1 - k_{12})|e_1| \\ &= -k_{11}e_1^2 - (k_{12} - D_1)|e_1| + g_1 e_1(x_2 - x_{2d}) \\ &\quad - k_{11}(\eta_1 - k_{12})^2 - l_1|\eta_1 - k_{12}| \\ &\le -k_{11}(e_1^2 + (\eta_1 - k_{12})^2) \\ &\quad - l_1(|e_1| + |\eta_1 - k_{12}|) + g_1 e_1(x_2 - x_{2d}), \end{aligned} \tag{16}$$

where $k_{12} - D_1 \ge l_1 > 0$.

**Step 2.** Design a tracking differentiator as below:

$$\begin{cases} \dot{z}_{21} = z_{22}, \\ \dot{z}_{22} = M^2(-a_1(z_{21} - x_{2d}) \\ \qquad - a_2(z_{21} - x_{2d})^p - a_3(z_{21} - x_{2d})^q \\ \qquad - b_1 \dfrac{z_{22}}{M} - b_2(\dfrac{z_{22}}{M})^p - b_3(\dfrac{z_{22}}{M})^q), \end{cases} \tag{17}$$

where constants $a_1, a_2, a_3$, $b_1, b_2, b_3$, $p, q$ are the same as that in Lemma 4. Thus $x_{2d}$, and $\dot{x}_{2d}$ can be approached by $z_{21}$ and $z_{22}$, respectively. In what follows, define some new variables as:

$$e_2 = x_2 - z_{21}, \quad \xi_2 = z_{21} - x_{2d}. \tag{18}$$

Accordingly, the Eq.(16) can be further expressed as

$$\begin{aligned}\dot{V}_1 \le &-k_{11}(e_1^2 + (\eta_1 - k_{12})^2) \\ &-l_1(|e_1| + |\eta_1 - k_{12}|) + g_1 e_1 (x_2 - x_{2d}) \\ &= -k_{11}(e_1^2 + (\eta_1 - k_{12})^2) - l_1(|e_1| + |\eta_1 - k_{12}|) \\ &\quad + g_1 e_1 (x_2 - z_{21} + z_{21} - x_{2d}) \\ &= -k_{11}(e_1^2 + (\eta_1 - k_{12})^2) - l_1(|e_1| + |\eta_1 - k_{12}|) \\ &\quad + g_1 e_1 e_2 + g_1 e_1 \xi_2.\end{aligned} \tag{19}$$

Noting that $e_2 = x_2 - z_{21}$, The derivative of error variable $e_2$ is obtained:

$$\begin{aligned}\dot{e}_2 &= \dot{x}_2 - \dot{z}_{21} \\ &= f_2 + g_2 x_3 + d_2 - z_{22} \\ &= f_2 + g_2(x_3 - x_{3d}) + g_2 x_{3d} + d_2 - z_{22}.\end{aligned} \tag{20}$$

To stabilize $e_2$, the virtual controller $x_{3d}$ and gain adaptive law $\eta_2$ are respectively designed as below:

$$x_{3d} = g_3^{-1}(-k_{21}e_2 - \eta_2 sign(e_2) - f_2 + z_{22} - g_1 e_1), \tag{21}$$

$$\dot{\eta}_2 = -k_{21}(\eta_2 - k_{22}) - l_2 sign(\eta_2 - k_{22}) + |e_2|. \tag{22}$$

where $x_{3d}$ and $\eta_2$ are the virtual controller and gain adaptive law, respectively.

A Lyapunov function is defined by

$$V_2 = V_1 + \frac{1}{2}e_2^2 + \frac{1}{2}(\eta_2 - k_{22})^2. \tag{23}$$

The derivative of $V_2$ can be computed as:

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + e_2\dot{e}_2 + (\eta_2 - k_{22})\dot{\eta}_2 \\ &= \dot{V}_1 + e_2(f_2 + g_2(x_3 - x_{3d}) + g_2 x_{3d} + d_2 - z_{22}) \\ &\quad + (\eta_2 - k_{22})\dot{\eta}_2 \\ &= \dot{V}_1 + e_2(f_2 + g_2(x_3 - x_{3d}) + d_2 - z_{22} \\ &\quad - k_{21}e_2 - \eta_2 sign(e_2) - f_2 + z_{22} - g_1 e_1) \\ &\quad + (\eta_2 - k_{22})(-k_{21}(\eta_2 - k_{22}) \\ &\quad - l_2 sign(\eta_2 - k_{22}) + |e_2|) \\ &= \dot{V}_1 - k_{21}e_2^2 + d_2 e_2 - \eta_2 |e_2| \\ &\quad + g_2 e_2 (x_3 - x_3 d) - g_1 e_1 e_2 - k_{21}(\eta_2 - k_{22})^2 \\ &\quad - l_2 |\eta_2 - k_{22}| + (\eta_2 - k_{22})|e_2| \\ &\le \dot{V}_1 - k_{21}e_2^2 + D_2|e_2| - \eta_2|e_2| \\ &\quad + g_2 e_2 (x_3 - x_3 d) - g_1 e_1 e_2 - k_{21}(\eta_2 - k_{22})^2 \\ &\quad - l_2 |\eta_2 - k_{22}| + (\eta_2 - k_{22})|e_2| \\ &= \dot{V}_1 - k_{21}e_2^2 - (k_{22} - D_2)|e_2| \\ &\quad + g_2 e_2 (x_3 - x_3 d) - g_1 e_1 e_2 \\ &\quad - k_{21}(\eta_2 - k_{22})^2 - l_2 |\eta_2 - k_{22}| \\ &\le \dot{V}_1 - k_{21}(e_2^2 + (\eta_2 - k_{22})^2) \\ &\quad - l_2(|e_2| + |\eta_2 - k_{22}|) \\ &\quad + g_2 e_2 (x_3 - x_3 d) - g_1 e_1 e_2,\end{aligned} \tag{24}$$

where $k_{22} - D_2 \ge l_2 > 0$.

Taking Eq.(19) into above Eq.(24) gives:

$$\begin{aligned}\dot{V}_2 \le &-k_{11}(e_1^2 + (\eta_1 - k_{12})^2) - l_1(|e_1| + |\eta_1 - k_{12}|) \\ &+ g_1 e_1 e_2 + g_1 e_1 \xi_2 \\ &- k_{21}(e_2^2 + (\eta_2 - k_{22})^2) - l_2(|e_2| + |\eta_2 - k_{22}|) \\ &+ g_2 e_2 (x_3 - x_3 d) - g_1 e_1 e_2 \\ = &-\sum_{j=1}^{2} k_{j1}(e_j^2 + (\eta_j - k_{j2})^2) \\ &-\sum_{j=1}^{2} l_j(|e_j| + |\eta_j - k_{j2}|) \\ &+ g_1 e_1 \xi_2 + g_2 e_2 (x_3 - x_{3d}).\end{aligned} \tag{25}$$

**Step i** ($2 \le i \le n-1$). The tracking differentiator for the $i$-th subsystem of (1) is designed as follows:

$$\begin{cases}\dot{z}_{i1} = z_{i2}, \\ \dot{z}_{i2} = M^2(-a_1(z_{i1} - x_{id}) \\ \quad - a_2(z_{i1} - x_{id})^p - a_3(z_{i1} - x_{id})^q \\ \quad - b_1 \dfrac{z_{i2}}{M} - b_2(\dfrac{z_{i2}}{M})^p - b_3(\dfrac{z_{i2}}{M})^q),\end{cases} \tag{26}$$

where constants $a_1, a_2, a_3$, $b_1, b_2, b_3$, $p, q$ are the same as that in Lemma 4. Thus $x_{id}$ and its derivative $\dot{x}_{id}$ can be obtained by $z_{i1}$ and $z_{i2}$, respectively. In what follows, define some new variables as:

$$e_i = x_i - z_{i1}, \quad \xi_i = z_{i1} - x_{id}, \tag{27}$$

then the derivative of $e_i$ is

$$\begin{aligned}\dot{e}_i &= \dot{x}_i - \dot{z}_{i1} \\ &= f_i + g_i x_{i+1} + d_i - z_{i2} \\ &= f_i + g_i(x_{i+1} - x_{i+1,d}) + g_i x_{i+1,d} + d_i - z_{i2}.\end{aligned} \tag{28}$$

The virtual controller $x_{i+1,d}$ and gain adaptive law $\eta_i$ are respectively designed as below:

$$x_{i+1,d} = g_i^{-1}(-k_{i1}e_i - \eta_i sign(e_i) - f_i + z_{i2} - g_{i-1}e_{i-1}), \tag{29}$$

$$\dot{\eta}_i = -k_{i1}(\eta_2 - k_{i2}) - l_i sign(\eta_i - k_{i2}) + |e_i|, \tag{30}$$

The $i$-th Lyapunov function is chosen to be:

$$V_i = V_{i-1} + \frac{1}{2}e_i^2 + \frac{1}{2}(\eta_i - k_{i2})^2. \tag{31}$$

Similar to the procedure of step 1 and 2, we have:

$$\begin{aligned}\dot{V}_i &= \dot{V}_{i-1} + e_i\dot{e}_i + (\eta_i - k_{i2})\dot{\eta}_i \\ &= \dot{V}_{i-1} + e_i(f_i + g_i(x_{i+1} - x_{i+1,d}) + g_i x_{i+1,d} \\ &\quad + d_i - z_{i2}) + (\eta_i - k_{i2})\dot{\eta}_i \\ &= \dot{V}_{i-1} + e_i(f_i + g_i(x_{i+1} - x_{i+1,d}) + d_i - z_{i2} \\ &\quad - k_{i1}e_i - \eta_i sign(e_i) - f_i + z_{i2} - g_{i-1}e_{i-1}) \\ &\quad + (\eta_i - k_{i2})(-k_{i1}(\eta_2 - k_{i2}) \\ &\quad - l_i sign(\eta_i - k_{i2}) + |e_i|) \\ &= \dot{V}_{i-1} - k_{i1}e_i^2 + d_i e_i - \eta_i |e_i| \\ &\quad - g_{i-1}e_{i-1}e_i + g_i e_i(x_{i+1} - x_{i+1,d}) \\ &\quad - k_{i1}(\eta_2 - k_{i2})^2 - l_i|\eta_i - k_{i2}| + (\eta_i - k_{i2})|e_i| \\ &\le \dot{V}_{i-1} - k_{i1}e_i^2 - (k_{i2} - D_i)|e_i| \\ &\quad - g_{i-1}e_{i-1}e_i + g_i e_i(x_{i+1} - x_{i+1,d}) \\ &\quad - k_{i1}(\eta_2 - k_{i2})^2 - l_i|\eta_i - k_{i2}| \\ &\le \dot{V}_{i-1} - k_{i1}(e_i^2 + (\eta_2 - k_{i2})^2) \\ &\quad - l_i(|e_i| + |\eta_i - k_{i2}|) \\ &\quad - g_{i-1}e_{i-1}e_i + g_i e_i(x_{i+1} - x_{i+1,d}).\end{aligned} \tag{32}$$

where $k_{i2} - D_i \ge l_i > 0$.

Taking the expression of $\dot{V}_{i-1}$ into the above equation, one gets:

$$\begin{aligned}\dot{V}_i &\le -\sum_{j=1}^{i-1} k_{j1}(e_j^2 + (\eta_j - k_{j2})^2) \\ &\quad - \sum_{j=1}^{i-1} l_j(|e_j| + |\eta_j - k_{j2}|) \\ &\quad + \sum_{j=2}^{i-1} g_{j-1}e_{j-1}\xi_j + g_{i-1}e_{i-1}(x_i - x_{id}) \\ &\quad - k_{i1}(e_i^2 + (\eta_2 - k_{i2})^2) - l_i(|e_i| + |\eta_i - k_{i2}|) \\ &\quad - g_{i-1}e_{i-1}e_i + g_i e_i(x_{i+1} - x_{i+1,d}) \\ &= -\sum_{j=1}^{i} k_{j1}(e_j^2 + (\eta_j - k_{j2})^2) \\ &\quad - \sum_{j=1}^{i} l_j(|e_j| + |\eta_j - k_{j2}|) \\ &\quad + \sum_{j=2}^{i-1} g_{j-1}e_{j-1}\xi_j + g_{i-1}e_{i-1}e_i + g_{i-1}e_{i-1}\xi_i \\ &\quad - g_{i-1}e_{i-1}e_i + g_i e_i(x_{i+1} - x_{i+1,d}) \\ &= -\sum_{j=1}^{i} k_{j1}(e_j^2 + (\eta_j - k_{j2})^2) \\ &\quad - \sum_{j=1}^{i} l_j(|e_j| + |\eta_j - k_{j2}|) \\ &\quad + \sum_{j=2}^{i} g_{j-1}e_{j-1}\xi_j + g_i e_i(x_{i+1} - x_{i+1,d}),\end{aligned} \tag{33}$$

where $x_i - x_{id} = x_i - z_{i1} + z_{i1} - x_{id} = e_i + \xi_i$.

**Step n.** For the $n$-th subsystem of (1), the tracking differentiator is designed as follows:

$$\begin{cases} \dot{z}_{n1} = z_{n2}, \\ \dot{z}_{n2} = M^2(-a_1(z_{n1} - x_{nd}) \\ \qquad - a_2(z_{n1} - x_{nd})^p - a_3(z_{n1} - x_{nd})^q \\ \qquad - b_1\dfrac{z_{n2}}{M} - b_2(\dfrac{z_{n2}}{M})^p - b_3(\dfrac{z_{i2}}{M})^q), \end{cases} \tag{34}$$

where constants $a_1, a_2, a_3$, $b_1, b_2, b_3$, $p, q$ are the same as that in Lemma 4. Thus $x_{nd}$ and its derivative $\dot{x}_{nd}$ can be obtained by $z_{n1}$ and $z_{n2}$, respectively. Next, introducing some new variables as:

$$e_n = x_n - z_{n1}, \quad \xi_n = z_{n1} - x_{nd}. \tag{35}$$

Based on the Eq.(33) and Eq.(35), one can deduce that:

$$\begin{aligned}\dot V_{n-1} &\le -\sum_{j=1}^{n-1} k_{j1}(e_j^2+(\eta_j-k_{j2})^2) \\ &\quad -\sum_{j=1}^{n-1} l_j(|e_j|+|\eta_j-k_{j2}|) \\ &\quad +\sum_{j=2}^{n-1} g_{j-1}e_{j-1}\xi_j + g_{n-1}e_{n-1}(x_n-x_{nd}) \\ &= -\sum_{j=1}^{n-1} k_{j1}(e_j^2+(\eta_j-k_{j2})^2) \\ &\quad -\sum_{j=1}^{n-1} l_j(|e_j|+|\eta_j-k_{j2}|) \\ &\quad +\sum_{j=2}^{n} g_{j-1}e_{j-1}\xi_j + g_{n-1}e_{n-1}e_n, \end{aligned} \tag{36}$$

where $x_n - x_{nd} = x_n - z_{n1} + z_{n1} - x_{nd} = e_n + \xi_n$.

Besides, the derivative of $e_n$ from (1) and (35) is

$$\dot e_n = \dot x_n - \dot z_{n1} = f_n + g_n u + d_n - z_{n2}. \tag{37}$$

Through the n-1 steps, it can be found that actual controller u appears in Eq.(37), thus there is only need to design actual controller u instead of another virtual controller in the next derivation. To realize tracking control of (1), the actual controller $u$ and gain adaptive law $\eta_n$ are respectively designed as below:

$$u = g_n^{-1}(-k_{n1}e_n - \eta_n sign(e_n) - f_n + z_{n2} - g_{n-1}e_{n-1}), \tag{38}$$

$$\dot\eta_n = -k_{n1}(\eta_2 - k_{n2}) - l_n sign(\eta_n - k_{n2}) + |e_n|. \tag{39}$$

And the corresponding Lyapunov function is chosen as

$$V_n = V_{n-1} + \frac{1}{2}e_n^2 + \frac{1}{2}(\eta_n - k_{n2})^2. \tag{40}$$

Similar to the step $i$, the derivative of $V_n$ can be computed as:

$$\begin{aligned}\dot V_n &= \dot V_{n-1} + e_n\dot e_n + (\eta_n - k_{n2})\dot\eta_n \\ &= \dot V_{n-1} + e_n(f_n + g_n u + d_n - z_{n2}) + (\eta_n - k_{n2})\dot\eta_n \\ &= \dot V_{n-1} + e_n(f_n + d_n - z_{n2} - k_{n1}e_n \\ &\quad -\eta_n sign(e_n) - f_n + z_{n2} - g_{n-1}e_{n-1}) \\ &\quad +(\eta_n - k_{n2})(-k_{n1}(\eta_2 - k_{n2}) \\ &\quad -l_n sign(\eta_n - k_{n2}) + |e_n|) \\ &= \dot V_{n-1} - k_{n1}e_n^2 + d_n e_n - \eta_n|e_n| - g_{n-1}e_{n-1}e_n \\ &\quad -k_{n1}(\eta_2 - k_{n2})^2 - l_n|\eta_2 - k_{n2}| + (\eta_n - k_{n2})|e_n| \\ &\le \dot V_{n-1} - k_{n1}e_n^2 - (k_{n2} - D_n)|e_n| - g_{n-1}e_{n-1}e_n \\ &\quad -k_{n1}(\eta_2 - k_{n2})^2 - l_n|\eta_2 - k_{n2}| \\ &\le \dot V_{n-1} - k_{n1}(e_n^2 + (\eta_2 - k_{n2})^2) \\ &\quad -l_n(|e_n| + |\eta_2 - k_{n2}|) - g_{n-1}e_{n-1}e_n. \end{aligned} \tag{41}$$

where $k_{n2} - D_n \ge l_n > 0$.

Putting Eq.(35) into the above inequality, it follows:

$$\begin{aligned}\dot V_n &\le -\sum_{j=1}^{n-1} k_{j1}(e_j^2+(\eta_j-k_{j2})^2) \\ &\quad -\sum_{j=1}^{n-1} l_j(|e_j|+|\eta_j-k_{j2}|) \\ &\quad +\sum_{j=2}^{n} g_{j-1}e_{j-1}\xi_j + g_{n-1}e_{n-1}e_n \\ &\quad -k_{n1}(e_n^2+(\eta_2-k_{n2})^2) \\ &\quad -l_n(|e_n|+|\eta_2-k_{n2}|) - g_{n-1}e_{n-1}e_n \\ &= -\sum_{j=1}^{n} k_{j1}(e_j^2+(\eta_j-k_{j2})^2) \\ &\quad -\sum_{j=1}^{n} l_j(|e_j|+|\eta_j-k_{j2}|) + \sum_{j=2}^{n} g_{j-1}e_{j-1}\xi_j. \end{aligned} \tag{42}$$

Keeping in mind that the Assumption 2 is satisfied, thus by the Young's inequality one gets:

$$\begin{aligned}\sum_{j=2}^{n} g_{j-1}e_{j-1}\xi_j &\le \sum_{j=2}^{n}\frac{1}{2}(g_{j-1}^2e_{j-1}^2+\xi_j^2) \\ &\le \sum_{j=1}^{n}\frac{1}{2}m_j^2e_j^2 + \sum_{j=1}^{n}\frac{1}{2}\xi_j^2. \end{aligned} \tag{43}$$

Combing Eq.(42) and Eq.(43), one can derive that

$$\begin{aligned}\dot V_n &\le -\sum_{j=1}^{n}(k_{j1}-\frac{m_j^2}{2})(e_j^2+(\eta_j-k_{j2})^2) \\ &\quad -\sum_{j=1}^{n} l_j(|e_j|+|\eta_j-k_{j2}|) + \sum_{j=1}^{n}\frac{1}{2}\xi_j^2 \\ &\le -\sum_{j=1}^{n} h_j(e_j^2+(\eta_j-k_{j2})^2) \\ &\quad -\sum_{j=1}^{n} l_j(|e_j|+|\eta_j-k_{j2}|) + \delta \end{aligned}$$

$$\le -h\sum_{j=1}^{n}(e_j^2+(\eta_j-k_{j2})^2) \\ -l\sum_{j=1}^{n}(|e_j|+|\eta_j-k_{j2}|)+\delta, \tag{44}$$

where $k_{j1}-\frac{m_j^2}{2}\ge h_j>0$, $\delta=\sum_{j=1}^{n}\frac{1}{2}\xi_j^2$, $h=min\{h_j,\ j=1,2,\cdots,n\}$, $l=min\{l_j,\ j=1,2,\cdots,n\}$.

By means of Lemma 1, we can deduce that

$$\sum_{j=1}^{n}(e_j^2+(\eta_j-k_{j2})^2)=2V_n, \tag{45}$$

$$\begin{aligned}\sum_{j=1}^{n}(|e_j|+|\eta_j-k_{j2}|) &=\sum_{j=1}^{n}((|e_j|^2)^{\frac{1}{2}}+(|\eta_j-k_{j2}|^2)^{\frac{1}{2}}) \\ &\ge\sum_{j=1}^{n}(|e_j|^2+|\eta_j-k_{j2}|^2)^{\frac{1}{2}} \\ &\ge(\sum_{j=1}^{n}(|e_j|^2+|\eta_j-k_{j2}|^2))^{\frac{1}{2}}=V_n^{\frac{1}{2}}.\end{aligned} \tag{46}$$

Consequently, the inequality (44) can be rearranged as:

$$\dot{V}_n\le -2hV_n-lV_n^{\frac{1}{2}}+\delta. \tag{47}$$

At this stage, the design procedure of adaptive back stepping control has been completed. We will go on the stability analysis of system (1), and the main results are summarized in the following Theorem 1.

**Theorem 1.** For the nonlinear system (1), the virtual controllers are defined by (13) (21) (29), actual controller is defined by (38), adaptive laws are given by (14) (22) (30) (39), the tracking differentiators are designed as (10) (17) (26) (34). If the following conditions hold,

$$\begin{cases} k_{j1}-\dfrac{m_j^2}{2}\ge h_j>0\ (j=1,2,\cdots,n), \\ k_{j2}-D_j\ge l_j>0\ (j=1,2,\cdots,n), \\ 2h>\rho_1>0, l>\rho_2>0, \end{cases} \tag{48}$$

then the states of system (1) are all finite time bounded, and the tracking error between $y$ and $r(t)$ can be rendered to a small neighbourhood $\Omega_3$ of origin in finite time $t_3$. Here $\Omega_1=\{e=(e_1,e_2,\cdots,e_n)^T\ |V_n\le\frac{\delta}{\rho_1}\}$, $\Omega_2=\{e=(e_1,e_2,\cdots,e_n)^T\ |V_n^{\frac{1}{2}}\le\frac{\delta}{\rho_2}\}$, $\Omega_3=\Omega_1\cap\Omega_2$. $t_3=\max\{t_1,t_2\}$, $t_1$ and $t_2$ satisfy

$$\begin{cases} t_1\le\dfrac{2}{2h-\rho_1}\ln(1+\dfrac{2h-\rho_1}{l}V_n(0)^{\frac{1}{2}}), \\ t_2\le\dfrac{1}{h}\ln(1+\dfrac{2h}{l-\rho_2}V_n(0)^{\frac{1}{2}}), \end{cases} \tag{49}$$

$V_n(0)$ denotes the initial value at time $t=0$. $\delta=\sum_{j=1}^{n}\frac{1}{2}\xi_j^2$, $h=min\{h_j,\ j=1,2,\cdots,n\}$, $l=min\{l_j,\ j=1,2,\cdots,n\}$. $h_j$, $l_j$, $\rho_1$, $\rho_2$ are appropriate positive constants.

**Proof:** See the Appendix A

## 4. NUMERICAL SIMULATION

In this section, an illustrate example by using a chemical reactor system (Jing et al (2019)) is presented to verify the effectiveness of adaptive backstepping control strategy. The chemical reactor system is structured as below:

$$\begin{cases} \dot{x}_1=-(\dfrac{1}{\alpha_1}+\alpha_2)x_1+\dfrac{1-\alpha_3}{\alpha_4}x_2+d_1(t), \\ \dot{x}_2=-(\dfrac{1}{\beta_1}+\beta_2+\dfrac{\alpha_3}{\beta_3})x_2+\dfrac{\beta_4}{\beta_3}x_1+\dfrac{\beta_5}{\beta_3}u+d_2(t), \\ y=x_1 \end{cases} \tag{50}$$

where $\alpha_i(i=1,2,3,4)$ $\beta_j\ (j=1,2,3,4,5)$ are the system parameters, whose physical meanings can be referred to article (Jing et al., 2019). The system parameters and disturbances are as follows: $\alpha_1=1$, $\alpha_2=1$, $\alpha_3=0.5$, $\alpha_4=0.5$, $\underline{e}ta_1=1$, $\beta_2=1$, $\beta_3=0.5$, $\beta_4=0.5$, $\beta_5=0.5$. $d_1=0.1cos(x_1)$, $d_2=0.2cos(x_2)$. The initial condition is given as $(x_1(0),x_2(0))^T=(1,1)^T$, and the reference signal is assumed to be $r(t)=sin(t)$. According to the control scheme in Section 3, the control parameters are taken as $a_1=1,a_2=1,a_3=1,$ $b_1=1,b_2=1,b_3=1,$ $p=3/5,q=5/3,M=80,$ $k_{11}=5,k_{12}=5,l_1=4,$ $k_{21}=5,k_{22}=5,l_2=4.$ The tracking error curves of system (50) and the response curves of adaptive laws are reflected in Fig.2 and Fig.3, respectively. From the simulation results in Figs.2-3, one can obviously observe that all the signals are fifinite-time bounded and output signal can converge to the reference signal quickly, which illustrates the validity of theoretical analysis and control method.

In what follows, to further show the superiority of proposed scheme, a simulation comparison between the proposed method and control method in (Wang et al., 2022) is carried out. The reference signal r(t) is set as sin(t).

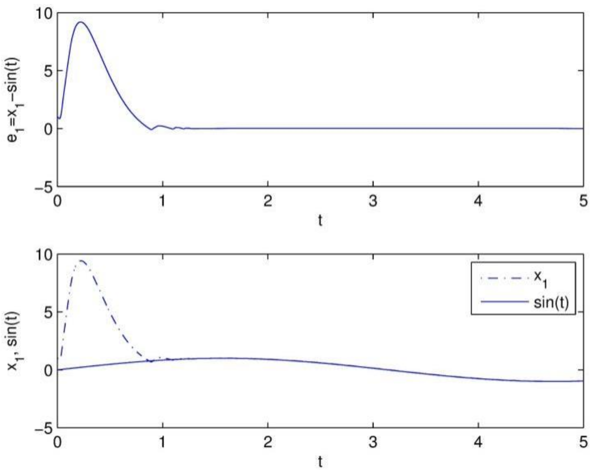


Fig. 2. The tracking trajectories of system (50).

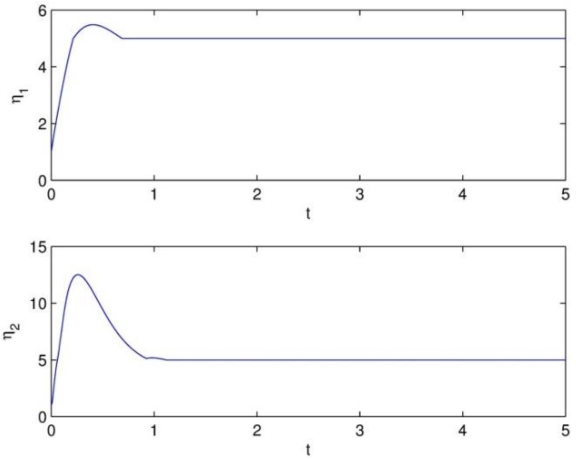


Fig. 3. The time curves of adaptive laws of system (50).

The simulation results are exhibited in Fig.4 under the same initial condition x(0) = (1, 1). It can be seen from Fig.4 that the convergence rate that state x1 converges to signal r(t) in this paper is faster than that in (Wang et al., 2022)), and the stability of error $e_1 = x_1 - \sin(t)$ in this paper is better than that in (Wang et al (2022)) as time goes by, which illustrates the advantages of proposed tracking control scheme.

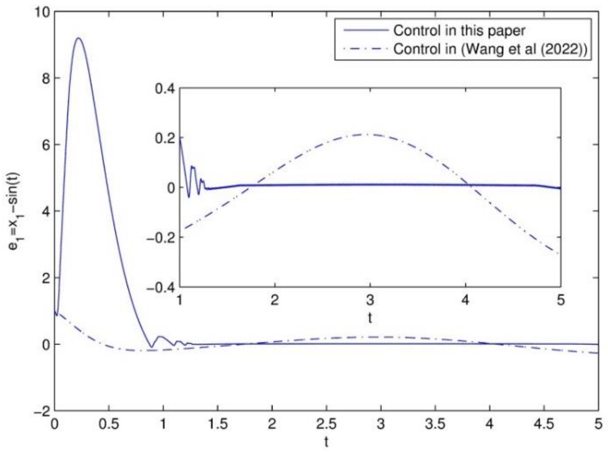


Fig. 4. The comparison between control method in this paper and control method in (Wang et al., 2022).

## 5. CONCLUSIONS

A new tacking differentiator with fast convergence rate is firstly presented in this article by using nonlinear system theory. It can not only offer a excellent convergence performance, but also mitigate the adverse effects of system chatter caused by switch functions. In the follows, a robust adaptive control scheme for pure feedback nonlinear systems with compound disturbances is developed with the help of backstepping control and newly presented tracking differentiator. In the tracking control scheme, the tracking differentiator is embedded into each recursive procedure to approximate the virtual controller and its derivative. Accordingly, the complexity of back-stepping controller can be reduced to a certain degree. The signals of closed loop system are all finite-time bounded, and the output signal can recover the target signal quickly. Finally, numerical studies on a practical pure feedback system clarifies and verifies the obtained approach.

## Appendix A

**Proof of Theorem 1:** Based on Eq.(47), two inequalities are obtained as:

$$\begin{aligned}\dot{V}_n &\le -2hV_n - lV_n^{\frac{1}{2}} + \delta \\ &= -(2h-\rho_1)V_n - lV_n^{\frac{1}{2}} - (\rho_1 V_n - \delta),\end{aligned}$$

and

$$\begin{aligned}\dot{V}_n &\le -2hV_n - lV_n^{\frac{1}{2}} + \delta \\ &= -2hV_n - (l-\rho_2)V_n^{\frac{1}{2}} - (\rho_2 V_n^{\frac{1}{2}} - \delta).\end{aligned}$$

Accordingly, the discussions of stability of system (1) under two cases are given as below.

**Case 1.** Let $\Omega_1 = \{e = (e_1, e_2, \cdots, e_n)^T \mid V_n \le \frac{\delta}{\rho_1}\}$ , $2h > \rho_1 > 0$. If $e \notin \Omega_1$, then $V_n > \frac{\delta}{\rho_1} \ge 0$ and therefore

$$\dot{V}_n \le -(2h-\rho_1)V_n - lV_n^{\frac{1}{2}} < 0, \qquad (51)$$

As a result function $V_n$ is monotonically decreasing with respect to $t$ until $V_n \le \frac{\delta}{\rho_1}$. Next, we show that after time $t_1$, the error states $e_j$ will fall into the set $\Omega_1$. It is easy to derive from Eq.(50) that

$$\frac{dV_n}{(2h-\rho_1)V_n + lV_n^{\frac{1}{2}}} \le -dt \quad \Rightarrow t \le \int_{V_n(t)}^{V_n(0)} \frac{dV_n}{(2h-\rho_1)V_n + lV_n^{\frac{1}{2}}}.$$

Consequently, there is a time $t_1$ such that

$$\begin{aligned}t_1 &\le \frac{2}{2h-\rho_1}\ln\left(\frac{(2h-\rho_1)V_n(0)^{\frac{1}{2}} + l}{(2h-\rho_1)V_n(t)^{\frac{1}{2}} + l}\right) \\ &\le \frac{2}{2h-\rho_1}\ln\left(1 + \frac{2h-\rho_1}{l}V_n(0)^{\frac{1}{2}}\right),\end{aligned} \qquad (52)$$

where $V_n(0)$ denotes the initial value at time $t=0$.

**Case 2.** Let $\Omega_2=\{e=(e_1,e_2,\cdots,e_n)^T \mid V_n^{\frac{1}{2}} \le \frac{\delta}{\rho_2}\}$ , $l>\rho_2>0$. If $e \notin \Omega_2$, then $V_n^{\frac{1}{2}} > \frac{\delta}{\rho_2} \ge 0$ and therefore

$$\dot{V}_n \le -2hV_n-(l-\rho_2)V_n^{\frac{1}{2}}<0, \tag{53}$$

It follows that function $V_n$ is monotonically decreasing with respect to $t$ until $V_n^{\frac{1}{2}} \le \frac{\delta}{\rho_2}$. Next, we show that after time $t_2$ , the error states $e_j$ will fall into the set $\Omega_2$. It is easy to derive from Eq.(52) that

$$\frac{dV_n}{2hV_n+(l-\rho_2)V_n^{\frac{1}{2}}} \le -dt \quad \Rightarrow t \le \int_{V_n(t)}^{V_n(0)} \frac{dV_n}{2hV_n+(l-\rho_2)V_n^{\frac{1}{2}}}.$$

Consequently, there is a time $t_2$ such that

$$\begin{aligned} t_2 &\le \frac{1}{h}\ln(\frac{2hV_n(0)^{\frac{1}{2}}+l-\rho_2}{2hV_n(t)^{\frac{1}{2}}+l-\rho_2}) \\ &\le \frac{1}{h}\ln(1+\frac{2h}{l-\rho_2}V_n(0)^{\frac{1}{2}}), \end{aligned} \tag{54}$$

where $V_n(0)$ denotes the initial value at time $t=0$.

Taking Case 1 and Case 2 into consideration, let $t_3=\max\{t_1,t_2\}$ , $\Omega_3=\Omega_1\cap\Omega_2$ , then $e_j$ , $\eta_j-k_{j2}$ $(j=1,2,\cdots,n)$ will be attracted to the set $\Omega_3$ after $t \ge t_3$, thus signals of closed-loop system are allbounded. Furthermore if the control parameters are appropriately designed, then the sets $\Omega_1$ and $\Omega_2$ will be small enough. And then we have $x_j \to z_{j1}$, $\eta_j \to k_{j2}$ $(j=1,2,\cdots,n)$ after $t \ge t_3$. Since $z_{11} wr(t)$, hence $y=x_1 \to r(t)$ after $t \ge t_3$. This completes the proof of Theorem 1.

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