Timed Continuous Petri Nets: Set Invariance in Feedback Design for Componentwise Boundedness and Weak Conservativity

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Abstract: This paper depicts a method for ensuring refined boundedness properties, namely componentwise boundedness and weak conservativity, for timed continuous Petri nets operating with infinite server semantics. The theoretical foundation, based on set-invariance, is followed by the presentation of the numerical implementation as a linear programming problem. A case study illustrates the applicability of our approach.

Keywords: timed continuous Petri nets; set invariance; feedback control; linear programming problem.

1. INTRODUCTION

Continuous Petri nets (CPNs) have been developed in order to provide a continuous approximation of the discrete behaviors of event driven systems (David and Alla, 2010). CPNs were introduced in 1987 at the *8th European Workshop on Application and Theory of Petri nets* that was held in Zaragoza, Spain. There, paper (David and Alla, 1987) presented the continuization at the net level, while paper (Silva and Colom, 1987) relaxed the state equation.

Just like in the case of discrete Petri nets, in continuous Petri nets time can be associated to places or arcs, but the timing of transitions is most often encountered in literature. For the last type of timing, in order to prescribe the token-flow through transitions, two different semantics have been proposed: *finite server semantics* (or constant speed), considered in (David and Alla, 2010), and *infinite server semantics* (or variable speed), approached in (Recalde and Silva, 2001), (Silva and Recalde. 2004). A comparison between these two types of semantics is presented in (Mahulea *et al.*, 2009) for a broad class of Petri nets.

The current paper deals with *timed continuous Petri nets* (TCPNs) operating with *infinite server semantics* (ISS), which provide a better approximation for the steady state throughput of the modeled discrete event systems than those operating with finite server semantics (Vazquez *et al.*, 2013). The dynamics of the marking of a TCPN with ISS is described by a set of switching linear systems. As such, one of the advantages offered by the usage of continuous models is represented by the application of analysis and design techniques known from control theory while retaining the graphical support provided by the Petri net topology. This approach is illustrated in recent works presenting feedback control synthesis based on linear matrix inequalities (Kara *et al.*, 2009), gradient-based controllers (Lefebvre, 2011), model predictive control or decentralized control (Julvez *et al.*, 2013), etc. Similar to the case of discrete Petri nets, in TCPNs the control actions are applied on the transitions, by modifying the token-flow through transitions; the topology of the net remains unchanged.

Our work continues the investigations developed by the same authors on some important properties of the marking vector that are detailed below. These investigations were started in (Asaftei and Matcovschi, 2011) for join-free TCPNs, and, afterwards, they were extended in (Asaftei and Matcovschi, 2012) to the case of TCPNs with general topology, as well as to CPNs with uncertain firing rates, called TCPNs with *interval firing rates*. All cited contributions focused on analysis aspects and proposed analysis tools. The current paper is interested in synthesis, by designing state-feedback architectures which are able to ensure the discussed properties for given TCPNs.

The properties we have in view are called "componentwise boundedness" and "weak-conservativity". Their rigorous definitions are given in Section 3 (Definition 1) and rely on the existence of *multi-dimensional sets* which are *invariant* with respect to the evolution of the marking vector of a TCPN. This evolution is regarded as a system trajectory whose coordinates equal the marking vector components at any time. From the geometric point of view, any trajectory initiated inside an invariant set does not leave it any longer. The geometry of an invariant set is defined in algebraic terms and, the evolution of the marking vector means that whenever the initial marking fulfils a certain algebraic condition, the condition will be satisfied by the marking at any time later. Thus, by using algebraic conditions for the marking, which are more insightful than those corresponding to the classical structural properties of discrete Petri nets, the two concepts mentioned above refine the standard "*k*-boundedness" and "conservativity", respectively. This aspect has been previously disclosed by our papers mentioned above and its mathematical characteristics will be also addressed in Section 3. However, at this point one can simply see as a basic motivation for the refinement the fact that our approach includes information about both topology and timing, whereas the definitions of the standard structural properties rely exclusively on topology. In intuitive terms, the componentwise boundedness allows the individual monitoring of each marking component by referring to concrete values of the bounds (not-generic bounds – in the sense of existence ensured by the *k*-boundedness). On the other hand, the weak conservativity considers a weighted sum of the marking components (similarly to the standard conservativity) for which an upper bound is requested (instead of the equality with a unique value).

Our paper is organized as follows. The notations and the dynamics of the marking of a TCPN with ISS are briefly presented in Section 2. The main results of our paper regarding the design of state-feedback control capable to ensure the componentwise boundedness or the weak-conservativity for a given TCPN are delivered in Section 4, starting with the theoretical foundation, and commenting on the numerical approach and its MATLAB implementation. A case study is provided in Section 5 in the form of a small assembly line.

2. MARKING DYNAMICS FOR A TCPN

The *topology* $\mathcal{N} = \langle P, T, \boldsymbol{C}^+, \boldsymbol{C}^- \rangle$ of a continuous Petri net is a directed, weighted, bipartite graph defined by the disjoint sets of places, $P = \{p_1, \dots, p_n\}$, and transitions, $T = \{t_1, \dots, t_m\}$, and by the input and output incidence matrices (with respect to places), denoted by $\boldsymbol{C}^+ = \left[C_{ij}^+\right]$ and $\boldsymbol{C}^- = \left[C_{ij}^-\right]$, $\boldsymbol{C}^+, \boldsymbol{C}^- \in \mathbb{R}_+^{n\times m}$ (where $\mathbb{R}_+ = [0, +\infty)$). Moreover, in a continuous Petri net the number of tokens from place p_i is a nonnegative number $m_i \in \mathbb{R}_+$. Thus, the fluidification of a discrete Petri net is accomplished by allowing both the weights of the arcs and the markings of the places to have continuous values, not only integer ones.

For a node $x \in P \cup T$, the preset and postset are denoted by ${}^\bullet x$ and $x^\bullet$, respectively. We assume that each transition has at least one input place ($|{}^\bullet t_j| \geq 1$ for all $j = 1, \dots, m$), meaning that there are no source transitions in the net. Another assumption we make is that the net is *pure*, i.e. it has no self-loops, so that its topology is completely characterized by the *incidence matrix* $\boldsymbol{C} = \left[C_{ij}\right] = \boldsymbol{C}^+ - \boldsymbol{C}^- \in \mathbb{R}^{n\times m}$.

A *timed continuous Petri net* (TCPN) is denoted by $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$, where $\mathcal{N}$ defines the topology of the net and $\boldsymbol{\lambda} = [\lambda_1 \dots \lambda_m]^T > 0$ is a positive vector of constant firing rates associated with the transitions in the net. The notation $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m}_0 \rangle$ is used whenever we are interested in pointing out the initial marking $\boldsymbol{m}_0 \in \mathbb{R}_+^n$ of the TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$.

For an arbitrary marking $\boldsymbol{m} = \left[m_1 \dots m_n\right]^T \in \mathbb{R}_+^n$ of the CPN $\mathcal{N} = \langle P, T, \boldsymbol{C}^+, \boldsymbol{C}^- \rangle$, the *enabling degree* of transition $t_j \in T$ is given by

$$e_j(\boldsymbol{m}) = \min_{p_i \in {}^\bullet t_j} \left\{ m_i / C_{ij}^- \right\}, \; j = 1, \dots, m. \tag{1}$$

Under *infinite server semantics* (ISS), the *token-flow* through transition t_j at an arbitrary moment $\tau \in \mathbb{R}_+$, when the marking of the net is $\boldsymbol{m}(\tau)$, equals

$$f_j(\boldsymbol{m}(\tau)) = \lambda_j \, e_j(\boldsymbol{m}(\tau)), \; j = 1, \dots, m. \tag{2}$$

The time-dependence of the marking of a TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is described by the *state-equation*

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \, \boldsymbol{f}(\boldsymbol{m}(\tau)), \tag{3}$$

or, taking (2) into account, in the equivalent form

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C}\Lambda \boldsymbol{e}(\boldsymbol{m}(\tau)), \tag{4}$$

where $\boldsymbol{f} = [f_1 \; f_2 \; \dots \; f_m]^T$ and $\boldsymbol{e} = [e_1 \; e_2 \; \dots \; e_m]^T$ are the *flow vector*, respectively the *enabling vector*, and

$$\Lambda = \operatorname{diag}\{\boldsymbol{\lambda}\} = \operatorname{diag}\{\lambda_1, \dots, \lambda_m\} \in \mathbb{R}_+^{m\times m}, \tag{5}$$

is the positive definite diagonal matrix built with the transition firing rates.

Taking equation (1) into account, we notice that the enabling degree of any transition in the net may be prescribed by each one of its input places.

In the case when every transition has exactly one input place ($|{}^\bullet t_j| = 1$ for $j = 1, \dots, m$), topology $\mathcal{N}$ is said to be *join-free* (JF). The evolution of the marking of a JF TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is described by a unique linear differential equation:

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{A}\boldsymbol{m}(\tau), \tag{6}$$

with

$$\boldsymbol{A} = \boldsymbol{C}\Lambda\Pi, \tag{7}$$

where matrix $\Pi = \left[\pi_{ji}\right] \in \mathbb{R}_+^{m\times n}$ is given by

$$\pi_{ji} = \begin{cases} 1/C_{ij}^-, & \text{if } {}^\bullet t_j = \{p_i\}, \\ 0, & \text{otherwise}, \end{cases} \quad i = 1, \dots, n, \; j = 1, \dots, m. \tag{8}$$

For a TCPN *with joins* $\mathcal{N}$, its mathematical model must take into account all possible combinations of places that dictate the enabling degree of the transitions in the net. To this respect, paper (Vazquez and Silva, 2011) introduces the concept of *configuration* of $\mathcal{N}$ defined by an m-tuple $\left(p_{k_1}, p_{k_2}, ..., p_{k_m}\right) \in {}^{\bullet}t_1 \times {}^{\bullet}t_2 \times ... \times {}^{\bullet}t_m$. For a marking $\boldsymbol{m} \in \mathbb{R}_+^n$, the configuration $\mathcal{C}_k = \left\{ \left(p_{k_1}, t_1\right), \left(p_{k_2}, t_2\right), ..., \left(p_{k_m}, t_m\right) \right\}$ is active in the case when for all $j = 1, ..., m$, the place $p_{k_j} \in {}^{\bullet}t_j$ is the one that determines the enabling degree of transition t_j, that is its marking, denoted by m_{k_j}, satisfies

$$m_{k_j} \big/ C_{k_j j}^{-} \leq m_i \big/ C_{ij}^{-} \text{, for all } p_i \in {}^{\bullet}t_j . \tag{9}$$

Place p_{k_j} is called the *critical input place* of transition t_j for the marking $\boldsymbol{m}$. The linear inequalities (9) associated with configuration $\mathcal{C}_k$ define a polyhedral region in the state space of the CPN, denoted by $\mathcal{R}_k \subseteq \mathbb{R}_+^n$. For a given topology $\mathcal{N}$, the configurations and the corresponding polyhedral regions are independent of the timing of transitions. The number of all possible configurations of $\mathcal{N}$, and consequently of all the corresponding polyhedral regions, is $\kappa = \prod_{j=1}^{m} |{}^{\bullet}t_j|$.

Taking the general model (4) into account, the dynamics of a TCPN with joins is represented by a set of linear differential equations with state dependent switching:

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{A}_k \boldsymbol{m}(\tau), \quad \boldsymbol{m}(\tau) \in \mathcal{R}_k , \tag{10}$$

where the state matrix $\boldsymbol{A}_k$ corresponding to configuration $\mathcal{C}_k$ is given by

$$\boldsymbol{A}_k = \boldsymbol{C} \Lambda \Pi_k , \tag{11}$$

and $\Pi_k = \left[\pi_{ji}^k \right] \in \mathbb{R}_+^{m \times n}$ is defined by

$$\pi_{ji}^k = \begin{cases} 1 / C_{ij}^{-} \text{ , if } (p_i, t_j) \in \mathcal{C}_k , \\ 0 \quad \text{, otherwise,} \end{cases} \quad i = 1, ..., n, \ j = 1, ..., m. \tag{12}$$

For a JF TCPN, meaning $\kappa = 1$ and $\mathcal{R}_1 = \mathbb{R}^n$, definition (12) of matrix Π_1 coincides with (8). Moreover, each row of matrix Π given by (8) contains exactly one positive element because $|{}^{\bullet}t_j| = 1$ for all $j = 1, ..., m$. Taking into account that matrix Π satisfies $\Pi \boldsymbol{C}^{-} = \boldsymbol{I}_m$, all off-diagonal elements of system matrix $\boldsymbol{A}$ (7) are non-negative, i.e. $\boldsymbol{A}$ is an *essentially non-negative* matrix, also called *Metzler* matrix (Berman and Plemmons, 1979). Consequently the dynamic linear system (6) is *positive* (Farina and Rinaldi, 2000).

Even though in the case of a TCPN with joins, the system matrices $\boldsymbol{A}_k$ given by (11) are not essentially negative any more, the dynamic system (10) is positive due to state dependent switching, as proved in (Matcovschi and Pastravanu, 2005). Thus, for any choice of the firing rates $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} > 0$, associated with the transitions of a TCPN $\mathcal{N}$, the marking of the net remains non-negative, i.e.

$$\forall \boldsymbol{m}(0) = \boldsymbol{m}_0 \geq 0 \Rightarrow \boldsymbol{m}(\tau; \boldsymbol{m}_0) \geq 0, \ \forall \tau \geq 0 . \tag{13}$$

3. COMPONENTWISE BOUNDEDNESS AND WEAK CONSERVATIVITY OF A TCPN

Definition 1.

Let $\boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} = [d_1 \dots d_n]^T > 0$, be a positive vector. We say that:

(*i*) TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is ***d**-upper bounded* if

$$\forall \boldsymbol{m}_0 \in \mathbb{R}_+^n : \ 0 \leq \boldsymbol{m}_0 \leq \boldsymbol{d} \ \Rightarrow \ 0 \leq \boldsymbol{m}(\tau; \boldsymbol{m}_0) \leq \boldsymbol{d}, \ \forall \tau \geq 0 . \tag{14}$$

(*ii*) TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is *weakly **d** -conservative* if

$$\begin{aligned} &\forall \boldsymbol{m}_0 = [m_1^0 \ \dots \ m_n^0]^T \in \mathbb{R}_+^n : \\ &d_1 m_1^0 + \dots + d_n m_n^0 \leq 1 \Rightarrow d_1 m_1(\tau) + \dots + d_n m_n(\tau) \leq 1, \forall \tau \geq 0, \end{aligned} \tag{15}$$

where $m_1(\tau)$, …, $m_n(\tau)$ represent the elements of the marking vector $\boldsymbol{m}(\tau; \boldsymbol{m}_0)$ ♦

Remark 1.

Due to the linearity of the subsystems from equation (10) and the linearity of the inequalities (9) defining the polyhedral regions $\mathcal{R}_k$, we get that if there exists a positive vector $\boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} > 0$, so that TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is ***d**-upper bounded // weakly **d** -conservative* then $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is also *c**d** -upper bounded // weakly c**d** -conservative* for all $c > 0$. ♦

The following result gives sufficient conditions for the two properties of TCPNs with general topology.

Theorem 1. (Asaftei and Matcovschi, 2012)

Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a TCPN modeled by (10). Let $\boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} > 0$, be a positive vector.

(*i*) $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is ***d** -upper bounded* if the state matrices corresponding to all the configurations of the net satisfy

$$\widehat{\boldsymbol{A}_k} \boldsymbol{d} \leq 0 , \ k = 1, \ldots, \kappa , \tag{16}$$

where the "hat" notation used for a square matrix $\boldsymbol{M} = \left[m_{ij} \right] \in \mathbb{R}^{n \times n}$ means the matrix $\widehat{\boldsymbol{M}} = \left[\hat{m}_{ij} \right] \in \mathbb{R}^{n \times n}$ built according to:

$$\begin{cases} \widehat{m}_{ii} = m_{ii}, \ i = 1, \ldots, n, \\ \widehat{m}_{ij} = | m_{ij} |, \ i \neq j, \ i, j = 1, \ldots, n. \end{cases} \tag{17}$$

(*ii*) $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is *weakly* $\boldsymbol{d}$ *-conservative* if the state matrices corresponding to all the configurations of the net satisfy

$$\widehat{\boldsymbol{A}_k}^T \boldsymbol{d} \le 0 \,,\ k = 1,\ldots,\kappa \,, \tag{18}$$

where the "hat" notation has the same meaning as in (16). ♦

In the particular case of a JF TCPN, since the state-matrix $\boldsymbol{A}$ (7) is essentially nonnegative, by applying the "hat" operator defined by (17) it results $\widehat{\boldsymbol{A}} = \boldsymbol{A}$. Consequently, from Theorem 1 we obtain that matrix $\boldsymbol{A}$ must satisfy the linear inequalities (16) or (18) for TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ to be $\boldsymbol{d}$ *-upper bounded*, respectively, *weakly* $\boldsymbol{d}$ *-conservative*. Moreover, these conditions are also necessary of in the case of a JF TCPN, as stated below.

Theorem 2. (Asaftei and Matcovschi, 2011)

Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a JF TCPN modeled by (4). Let $\boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} > 0$, be a positive vector.

(*i*) $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is $\boldsymbol{d}$ *-upper bounded* if and only if

$$\boldsymbol{A}\boldsymbol{d} \le 0 \,. \tag{19}$$

(*ii*) $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is *weakly* $\boldsymbol{d}$ *-conservative* if and only if

$$\boldsymbol{A}^T \boldsymbol{d} \le 0 \,. \quad \blacklozenge \tag{20}$$

4. FLOW CONTROL DESIGN FOR TCPNs

4.1. Theoretical Results

The TPCN framework supports the plant-operation modelling and yields a description of the plant dynamics via (switched) linear differential equations. In this section, the controller design is addressed in classical terms, relying on the mathematical model; the TCPN formalism is not expected to be extended to the designed controller.

In the following we consider that, in order to ensure the desired properties for the dynamics of a given TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$, the flow through the transitions can be controlled. As usually, this means substituting in the state-equation (3) the flow vector $\boldsymbol{f}$ by $\boldsymbol{w} = \boldsymbol{f} - \boldsymbol{u}$, where $\boldsymbol{u}$ is the control action. Therefore, the controlled flow vector is given by

$$\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau) = \Lambda \boldsymbol{e}(\boldsymbol{m}(\tau)) - \boldsymbol{u}(\tau) \,,\ \forall \tau \ge 0 \,. \tag{21}$$

Moreover, since the flow through each transition must be nonnegative, vector $\boldsymbol{w}$ must satisfy the restriction

$$\boldsymbol{w}(\tau) \ge 0 \Leftrightarrow \boldsymbol{u}(\tau) \le \Lambda \boldsymbol{e}(\boldsymbol{m}(\tau)) \,,\ \forall \tau \ge 0 \,. \tag{22}$$

The dynamics of the marking of a controlled TCPN is described by

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C}\Lambda \boldsymbol{e}(\boldsymbol{m}(\tau) - \boldsymbol{C}\boldsymbol{u}(\tau), \quad \tau \ge 0, \ \ \boldsymbol{m}(0) = \boldsymbol{m}_0 \ge 0 \,. \tag{23}$$

Given a positive vector $\boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} > 0$, our aim is to design a control law $\boldsymbol{u}(\tau)$ satisfying (22) so that the controlled TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ (23) is $\boldsymbol{d}$ *-upper bounded* or *weakly* $\boldsymbol{d}$ *-conservative*.

First, for the case of a JF TCPN whose autonomous behavior is described by (6), the control law is considered in the form of the state-feedback

$$\boldsymbol{u}(\tau) = \boldsymbol{F}\, \boldsymbol{m}(\tau) \,, \tag{24}$$

with $\boldsymbol{F} \in \mathbb{R}^{m \times n}$, so that the model of the feedback-controlled TCPN is

$$\dot{\boldsymbol{m}}(\tau) = (\boldsymbol{A} - \boldsymbol{C}\boldsymbol{F})\boldsymbol{m}(\tau), \quad \tau \ge 0, \ \ \boldsymbol{m}(0) = \boldsymbol{m}_0 \ge 0 \,. \tag{25}$$

Taking into account that we are interested only in evolutions with nonnegative state vectors, $\boldsymbol{m}(\tau) \ge 0$, the constraints (22) written for the feedback law (24) lead to

$$\boldsymbol{F}\boldsymbol{m}(\tau) \le \Lambda\Pi \boldsymbol{m}(\tau) \,,\ \forall \tau \ge 0 \ \Leftrightarrow \ \boldsymbol{F} \le \Lambda\Pi \,. \tag{26}$$

Since the system matrix in (25), $\boldsymbol{A} - \boldsymbol{C}\boldsymbol{F}$, is a general matrix, not necessarily essentially nonnegative, Theorem 2 cannot be applied. Instead, the following result can be proven, where we use the "hat" operator introduced by (17).

Theorem 3.

Let $\boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} > 0$, be a positive vector and matrix $\boldsymbol{F} \in \mathbb{R}^{m \times n}$ satisfy (26).

(*i*) The feedback-controlled JF TCPN (25) is $\boldsymbol{d}$*-upper bounded* if

$$\widehat{\boldsymbol{A} - \boldsymbol{C}\boldsymbol{F}}\, \boldsymbol{d} \le 0 \,. \tag{27}$$

(*ii*) The feedback-controlled JF TCPN (25) is *weakly* $\boldsymbol{d}$ *-conservative* if

$$\widehat{\boldsymbol{A} - \boldsymbol{C}\boldsymbol{F}}^T \boldsymbol{d} \le 0 \,. \tag{28}$$

Proof. The proof derives from the proof of Theorem 1 and Corollary 1 in (Asaftei and Matcovschi, 2012) applied in the particular cases of the Hölder *p*-norms with $p = \infty$ for part (*i*) and with $p = 1$ for part (*ii*). ♦

Next, we extend this methodology to the case of a general TCPN whose autonomous behavior is described by (10). Thus, for each configuration $\mathcal{C}_k$ (or, equivalently, for each polyhedral region $\mathcal{R}_k \subseteq \mathbb{R}^n_+$) we look for a control law

$$\boldsymbol{u}(\tau) = \boldsymbol{F}_k \boldsymbol{m}(\tau), \quad \boldsymbol{m}(\tau) \in \mathcal{R}_k \,,\ \ k = 1,\ldots,\kappa \,, \tag{29}$$

where the feedback matrices $\boldsymbol{F}_k \in \mathbb{R}^{m \times n}$, $k = 1,\ldots,\kappa$, must satisfy the constraints

$$\boldsymbol{F}_k \le \Lambda\Pi_k \,, \quad k = 1,\ldots,\kappa \,. \tag{30}$$

The feedback-controlled TCPN is modeled by the set of linear differential equations with state dependent switching

$$\dot{\boldsymbol{m}}(\tau)=(\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k)\boldsymbol{m}(\tau),\quad \boldsymbol{m}(\tau)\in\mathcal{R}_k\,. \tag{31}$$

The application of Theorem 3 to each of the subsystems of (31) leads to the following result.

Theorem 4.

Let $\boldsymbol{d}\in\mathbb{R}^n$, $\boldsymbol{d}>0$, be a positive vector and matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $k=1,\dots,\kappa$, satisfy (30).

(*i*) The feedback-controlled TCPN (31) is ***d**-upper bounded* if

$$\widehat{\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k}\,\boldsymbol{d}\le 0\,,\ k=1,\dots,\kappa\,. \tag{32}$$

(*ii*) The feedback-controlled TCPN (31) is *weakly **d**-con–servative* if

$$\widehat{\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k}^{\,T}\boldsymbol{d}\le 0\,,\ k=1,\dots,\kappa\,. \qquad \blacklozenge \tag{33}$$

Note that Theorem 4 can also be applied in the case when the state-feedback is in the form (24), with a single feedback matrix $\boldsymbol{F}_1=\dots=\boldsymbol{F}_\kappa=\boldsymbol{F}$ for all the configurations of the TCPN.

4.2. Numerical Approach

The results given in the previous section considered the feedback matrices as being known. Next we are interested in computing such matrices so that the feedback controlled TCPN to be ***d**-upper bounded* or *weakly **d**-conservative* with given parameters.

For a square matrix $\boldsymbol{M}\in\mathbb{R}^{n\times n}$, the notation $(\boldsymbol{M})^{\text{off}}$ means the matrix built from the off-diagonal entries of $\boldsymbol{M}$, with zeros on the main diagonal.

Theorem 5.

Let $\boldsymbol{d}\in\mathbb{R}^n$, $\boldsymbol{d}>0$, be a positive vector.

(*i*) The feedback-controlled TCPN (31) is ***d**-upper bounded* if matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $k=1,\dots,\kappa$, satisfy the constraints (30) and the inequalities

$$\begin{aligned}&-\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k\le-\boldsymbol{A}_k,\\&(\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k)^{\text{off}}\le(\boldsymbol{A}_k)^{\text{off}},\\&\boldsymbol{G}_k\boldsymbol{d}\le 0,\quad k=1,\dots,\kappa.\end{aligned} \tag{34}$$

(*ii*) The feedback-controlled TCPN (31) is *weakly **d**-con–servative* if matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $k=1,\dots,\kappa$, satisfy the constraints (30) and the inequalities

$$\begin{aligned}&-\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k\le-\boldsymbol{A}_k,\\&(\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k)^{\text{off}}\le(\boldsymbol{A}_k)^{\text{off}},\\&\boldsymbol{G}_k^T\boldsymbol{d}\le 0,\quad k=1,\dots,\kappa.\end{aligned} \tag{35}$$

Proof. In inequalities (34) and (35) matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $\boldsymbol{G}_k\in\mathbb{R}^{n\times n}$, $k=1,\dots,\kappa$, are considered unknown. We prove the equivalences $(32)\Leftrightarrow(34)$ and $(33)\Leftrightarrow(35)$.

The proofs for $(32)\Rightarrow(34)$ and $(33)\Rightarrow(35)$ are similar. For matrix $\boldsymbol{G}_k=\widehat{\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k}$, we have $(-\boldsymbol{G}_k)^{\text{off}}\le(\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k)^{\text{off}}$ and $\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k\le\boldsymbol{G}_k$, for all $k=1,\dots,\kappa$,. Thus, if matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $k=1,\dots,\kappa$, satisfy inequality (32) or (33), then matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $\boldsymbol{G}_k\in\mathbb{R}^{n\times n}$, $k=1,\dots,\kappa$, also satisfy inequality (34) or, respectively, (35).

Next, we prove that $(34)\Rightarrow(32)$ and $(35)\Rightarrow(33)$. If inequalities (34) or (35) are true for $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $\boldsymbol{G}_k\in\mathbb{R}^{n\times n}$, then $(-\boldsymbol{G}_k)^{\text{off}}\le(\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k)^{\text{off}}$ and $\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k\le\boldsymbol{G}_k$ also hold and imply $\widehat{\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k}\le\boldsymbol{G}_k$. Thus matrices $\boldsymbol{G}_k\in\mathbb{R}^{n\times n}$, $k=1,\dots,\kappa$, satisfying (34) or (35) are essentially nonnegative. For proving that $(34)\Rightarrow(32)$ we use $\widehat{\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k}\boldsymbol{d}\le\boldsymbol{G}_k\boldsymbol{d}\le 0$, for all $k=1,\dots,\kappa$, from which we conclude that inequalities (32) have solutions. In a similar manner, for proving that $(35)\Rightarrow(33)$ we use $\widehat{\boldsymbol{A}_k-\boldsymbol{C}\boldsymbol{F}_k}^{\,T}\boldsymbol{d}\le\boldsymbol{G}_k^T\boldsymbol{d}\le 0$, for all $k=1,\dots,\kappa$, from which we conclude that inequalities (33) have solutions. ♦

The resolution of the inequalities from Theorem 5 can be addressed as a problem of minimizing a constant objective function $J\equiv 0$ subject to inequalities (30) and (34), or (30) and (35), as constraints. Thus, the feasibility of the constrained optimization problem is equivalent to the compatibility of its constraints.

Corollary 1.

Let $\boldsymbol{d}\in\mathbb{R}^n$, $\boldsymbol{d}>0$, be a positive vector, and consider the constant objective function $J\equiv 0$.

(*i*) The feedback-controlled TCPN (31) is ***d**-upper bounded* if matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $k=1,\dots,\kappa$, solve the optimization problem

$$\begin{aligned}&\text{minimize } J\\&\quad\text{subject to}\\&\quad-\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k\le-\boldsymbol{A}_k,\\&\quad(\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k)^{\text{off}}\le(\boldsymbol{A}_k)^{\text{off}},\\&\quad\boldsymbol{F}_k\le\Lambda\Pi_k,\\&\quad\boldsymbol{G}_k\boldsymbol{d}\le 0,\quad k=1,\dots,\kappa.\end{aligned} \tag{36}$$

(*ii*) The feedback-controlled TCPN (31) is *weakly **d**-con–servative* if matrices $\boldsymbol{F}_k\in\mathbb{R}^{m\times n}$, $k=1,\dots,\kappa$, solve the optimization problem

$$\begin{aligned}&\text{minimize } J\\&\quad\text{subject to}\\&\quad-\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k\le-\boldsymbol{A}_k,\\&\quad(\boldsymbol{C}\boldsymbol{F}_k-\boldsymbol{G}_k)^{\text{off}}\le(\boldsymbol{A}_k)^{\text{off}},\\&\quad\boldsymbol{F}_k\le\Lambda\Pi_k,\\&\quad\boldsymbol{G}_k^T\boldsymbol{d}\le 0,\quad k=1,\dots,\kappa.\end{aligned} \qquad \blacklozenge \tag{37}$$

4.3. Notes on the MATLAB implementation

In the following we detail the MATLAB implementation used for solving the constrained optimization problem (36) with respect to $\boldsymbol{F}_k \in \mathbb{R}^{m\times n}$, $\boldsymbol{G}_k \in \mathbb{R}^{n\times n}$, $k=1,\dots,\kappa$, for a given positive vector $\boldsymbol{d}=\left[d_1 \ \dots \ d_n\right]^T \in \mathbb{R}^n$, $\boldsymbol{d}>0$.

Let $\boldsymbol{M}(:,i)$ be the i-th column of a matrix $\boldsymbol{M}\in\mathbb{R}^{m\times n}$, for $i=1,\dots,n$, and let $\text{vect}(\boldsymbol{M})\in\mathbb{R}^{mn}$ be the vector obtained by arranging the columns of $\boldsymbol{M}$ one after another, i.e.

$$\text{vect}(\boldsymbol{M}) = \begin{bmatrix} \boldsymbol{M}(:,1) \\ \vdots \\ \boldsymbol{M}(:,n) \end{bmatrix} \in \mathbb{R}^{mn}. \tag{38}$$

Consider an arbitrary index $k=1,\dots,\kappa$, and construct the column vectors (considered unknown in problem (36))

$$\begin{aligned} &\boldsymbol{f}_k = \text{vect}(\boldsymbol{F}_k)\in\mathbb{R}^{mn},\ \boldsymbol{g}_k = \text{vect}(\boldsymbol{G}_k)\in\mathbb{R}^{n^2}, \\ &\boldsymbol{v}_k = \begin{bmatrix} \boldsymbol{f}_k \\ \boldsymbol{g}_k \end{bmatrix} \in \mathbb{R}^{(m+n)n}. \end{aligned} \tag{39}$$

The first two constraints from (36) may be written in the equivalent forms

$$-\boldsymbol{C}\boldsymbol{F}_k - \boldsymbol{G}_k \le -\boldsymbol{A}_k \Leftrightarrow \begin{cases} -\boldsymbol{C}\boldsymbol{F}_k(:,1) - \boldsymbol{G}_k(:,1) \le -\boldsymbol{A}_k(:,1) \\ \dots \\ -\boldsymbol{C}\boldsymbol{F}_k(:,n) - \boldsymbol{G}_k(:,n) \le -\boldsymbol{A}_k(:,n) \end{cases} \tag{40}$$

$$(\boldsymbol{C}\boldsymbol{F}_k - \boldsymbol{G}_k)^{\text{off}} \le \boldsymbol{A}_k^{\text{off}} \Leftrightarrow \begin{cases} (\boldsymbol{C}\boldsymbol{F}_k - \boldsymbol{G}_k)^{\text{off}}(:,1) \le \boldsymbol{A}_k^{\text{off}}(:,1) \\ \dots \\ (\boldsymbol{C}\boldsymbol{F}_k - \boldsymbol{G}_k)^{\text{off}}(:,n) \le \boldsymbol{A}_k^{\text{off}}(:,n) \end{cases} \tag{41}$$

Let

$$\boldsymbol{a}_k = \text{vect}(\boldsymbol{A}_k),\ \tilde{\boldsymbol{a}} = \text{vect}(\boldsymbol{A}^{\text{off}}) \in \mathbb{R}^{n^2} \tag{42}$$

$$\boldsymbol{N} = \underbrace{\begin{bmatrix} \boldsymbol{C} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{C} \end{bmatrix}}_{n \text{ times}}. \tag{43}$$

$$\boldsymbol{P} = \left[\boldsymbol{N} \mid \boldsymbol{I}_{n^2}\right],\ \boldsymbol{Q} = \left[\boldsymbol{N} \mid -\boldsymbol{I}_{n^2}\right] \in \mathbb{R}^{n^2\times(m+n)n}, \tag{44}$$

where $\boldsymbol{I}_{n^2}$ represents the identity matrix of order n^2, and denote by $\tilde{\boldsymbol{Q}}$ the matrix obtained from $\boldsymbol{Q}$ by placing zeros on the lines 1, $n+2$, $2n+3$, …, n^2. Inequalities (40) and (41) can be expressed as

$$-\boldsymbol{C}\boldsymbol{F}_k - \boldsymbol{G}_k \le -\boldsymbol{A}_k \Leftrightarrow -\boldsymbol{P}\boldsymbol{v}_k \le -\boldsymbol{a}_k, \tag{45}$$

$$(\boldsymbol{C}\boldsymbol{F}_k - \boldsymbol{G}_k)^{\text{off}} \le \boldsymbol{A}_k^{\text{off}} \Leftrightarrow \tilde{\boldsymbol{Q}}\boldsymbol{v}_k \le \tilde{\boldsymbol{a}}_k. \tag{46}$$

The third constraint from (36) is equivalent to

$$\boldsymbol{F}_k \le \Lambda\Pi_k \Leftrightarrow \begin{cases} \boldsymbol{F}_k(:,1) \le \Lambda\Pi_k(:,1) \\ \dots \\ \boldsymbol{F}_k(:,n) \le \Lambda\Pi_k(:,n) \end{cases} \Leftrightarrow \boldsymbol{f}_k \le \boldsymbol{b}_k, \tag{47}$$

where

$$\boldsymbol{b}_k = \boldsymbol{L}\cdot\text{vect}(\Pi_k)\in\mathbb{R}^{mn},\ \boldsymbol{L} = \underbrace{\begin{bmatrix} \Lambda & & 0 \\ & \ddots & \\ 0 & & \Lambda \end{bmatrix}}_{n \text{ times}} \in \mathbb{R}^{mn\times mn}. \tag{48}$$

The last constraint from (36) takes the form

$$\boldsymbol{G}\boldsymbol{d} \le 0 \Leftrightarrow d_1\boldsymbol{G}(:,1) + \dots + d_n\boldsymbol{G}(:,n) \le 0 \Leftrightarrow \boldsymbol{R}\boldsymbol{g} \le 0, \tag{49}$$

where

$$\boldsymbol{R} = \left[d_1\boldsymbol{I}_n \ \ \dots \ \ d_n\boldsymbol{I}_n\right] \in \mathbb{R}^{n\times n^2}. \tag{50}$$

Inequalities (47) and (49) lead to

$$\begin{cases} \boldsymbol{F}_k \le \Lambda\Pi_k \\ \boldsymbol{G}_k\boldsymbol{d} \le 0 \end{cases} \Leftrightarrow \begin{bmatrix} \boldsymbol{I}_{mn} & 0 \\ 0 & \boldsymbol{R} \end{bmatrix}\boldsymbol{v}_k \le \begin{bmatrix} \boldsymbol{b}_k \\ 0 \end{bmatrix} \Leftrightarrow \boldsymbol{S}\boldsymbol{v}_k \le \boldsymbol{c}_k, \tag{51}$$

with

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I}_{mn} & 0 \\ 0 & \boldsymbol{R} \end{bmatrix} \in \mathbb{R}^{(m+1)n\times(m+n)n},\ \boldsymbol{c}_k = \begin{bmatrix} \boldsymbol{b}_k \\ 0 \end{bmatrix} \in \mathbb{R}^{(m+1)n}. \tag{52}$$

To state inequalities (45), (46) and (51) for all $k=1,\dots,\kappa$ in a compact form, let us consider the column vectors

$$\begin{aligned} &\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_\kappa \end{bmatrix} \in \mathbb{R}^{\kappa(m+n)n},\ \boldsymbol{a}_e = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_\kappa \end{bmatrix},\ \tilde{\boldsymbol{a}}_e = \begin{bmatrix} \tilde{\boldsymbol{a}}_1 \\ \vdots \\ \tilde{\boldsymbol{a}}_\kappa \end{bmatrix} \in \mathbb{R}^{\kappa n^2}, \\ &\boldsymbol{c}_e = \begin{bmatrix} \boldsymbol{c}_1 \\ \vdots \\ \boldsymbol{c}_\kappa \end{bmatrix} \in \mathbb{R}^{\kappa(m+1)n}, \end{aligned} \tag{53}$$

and the matrices

$$\boldsymbol{P}_e = \underbrace{\begin{bmatrix} \boldsymbol{P} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{P} \end{bmatrix}}_{\kappa \text{ times}},\ \tilde{\boldsymbol{Q}}_e = \underbrace{\begin{bmatrix} \tilde{\boldsymbol{Q}} & & 0 \\ & \ddots & \\ 0 & & \tilde{\boldsymbol{Q}} \end{bmatrix}}_{\kappa \text{ times}} \in \mathbb{R}^{\kappa n^2\times\kappa(m+n)n},$$

$$\boldsymbol{S}_e = \underbrace{\begin{bmatrix} \boldsymbol{S} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{S} \end{bmatrix}}_{\kappa \text{ times}} \in \mathbb{R}^{\kappa(m+1)n\times\kappa(m+n)n}. \tag{54}$$

Taking these notations into account, the constrained optimization problem (36) is equivalent to the following *linear programming problem* (LPP) which can be solved by a call to the MATLAB function **linprog**

$$\begin{aligned}&\text{minimize } J(\boldsymbol{v})\\ &\quad \text{subject to}\\ &\quad -\boldsymbol{P}_e \boldsymbol{v} \le -\boldsymbol{a}_e,\\ &\quad \tilde{\boldsymbol{Q}}_e \boldsymbol{v} \le \tilde{\boldsymbol{a}}_e,\\ &\quad \boldsymbol{S}_e \boldsymbol{v} \le \boldsymbol{c}_e,\end{aligned} \tag{55}$$

for the objective function

$$J : \mathbb{R}^{\kappa(m+n)n} \to \mathbb{R},\ J(\boldsymbol{v}) = 0\,. \tag{56}$$

For solving the constrained optimization problem (37), we notice that the last constraint can be written as

$$\boldsymbol{G}_k^T \boldsymbol{d} \le 0 \Leftrightarrow \boldsymbol{d}^T \boldsymbol{G}_k \le 0 \Leftrightarrow \begin{cases} d_1 \boldsymbol{G}_k(:,1) \le 0 \\ \dots \\ d_n \boldsymbol{G}_k(:,n) \le 0 \end{cases} \Leftrightarrow \boldsymbol{R}'\boldsymbol{g}_k \le 0\,, \tag{57}$$

where

$$\boldsymbol{R}' = \begin{bmatrix} d_1 \boldsymbol{I}_n & & 0 \\ & \ddots & \\ 0 & & d_n \boldsymbol{I}_n \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}\,. \tag{58}$$

Inequalities (48) and (59) lead to

$$\begin{cases} \boldsymbol{F}_k \le \Lambda \Pi_k \\ \boldsymbol{G}_k^T \boldsymbol{d} \le 0 \end{cases} \Leftrightarrow \begin{bmatrix} \boldsymbol{I}_{mn} & 0 \\ 0 & \boldsymbol{R}' \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_k \\ \boldsymbol{g}_k \end{bmatrix} \le \begin{bmatrix} \boldsymbol{b}_k \\ 0 \end{bmatrix} \Leftrightarrow \boldsymbol{S}'\boldsymbol{v}_k \le \boldsymbol{c}'_k\,, \tag{59}$$

where

$$\boldsymbol{S}' = \begin{bmatrix} \boldsymbol{I}_{mn} & 0 \\ 0 & \boldsymbol{R}' \end{bmatrix} \in \mathbb{R}^{(m+n)n \times (m+n)n},\quad \boldsymbol{c}'_k = \begin{bmatrix} \boldsymbol{b}_k \\ 0 \end{bmatrix} \in \mathbb{R}^{(m+n)n}\,. \tag{60}$$

With

$$\boldsymbol{S}'_e = \underbrace{\begin{bmatrix} \boldsymbol{S}' & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{S}' \end{bmatrix}}_{\kappa \text{ times}} \in \mathbb{R}^{\kappa(m+n)n \times \kappa(m+n)n},\ \boldsymbol{c}'_e = \begin{bmatrix} \boldsymbol{c}'_1 \\ \vdots \\ \boldsymbol{c}'_\kappa \end{bmatrix} \in \mathbb{R}^{\kappa(m+n)n}\,, \tag{61}$$

the constrained optimization problem (37) is equivalent to the following LPP

$$\begin{aligned}&\text{minimize } J(\boldsymbol{v})\\ &\quad \text{subject to}\\ &\quad -\boldsymbol{P}_e \boldsymbol{v} \le -\boldsymbol{a}_e,\\ &\quad \tilde{\boldsymbol{Q}}_e \boldsymbol{v} \le \tilde{\boldsymbol{a}}_e,\\ &\quad \boldsymbol{S}'_e \boldsymbol{v} \le \boldsymbol{c}'_e.\end{aligned} \tag{62}$$

It is obvious that if the LPP (55) // (62) is feasible and $\boldsymbol{v}^* \in \mathbb{R}^{\kappa(m+n)n}$ is its solution, then the feedback matrices $\boldsymbol{F}_k \in \mathbb{R}^{m \times n}$, $k = 1,\dots,\kappa$, that ensure the property of $\boldsymbol{d}$ *-upper boundedness // weakly* $\boldsymbol{d}$ *-conservativity* for the feedback-controlled TCPN (31) can be easily constructed from the corresponding elements of $\boldsymbol{v}^*$.

5. CASE STUDY

Let us consider the TCPN whose topology $\mathcal{N}$ is sketched in Figure 1 representing a small manufacturing cell. The rates of the two processes represented, as usually, by transitions t_1 and t_2 are 1 and, respectively 0.5. There are two configurations for this TCPN, $\mathcal{C}_1 = \{(p_2,t_1),(p_1,t_2)\}$ and $\mathcal{C}_2 = \{(p_3,t_1),(p_1,t_2)\}$, active in the polyhedral regions $\mathcal{R}_1 = \left\{\boldsymbol{m} \in \mathbb{R}^3 \mid \frac{m_2}{2} < m_3\right\}$ and $\mathcal{R}_2 = \left\{\boldsymbol{m} \in \mathbb{R}^3 \mid \frac{m_2}{2} \ge m_3\right\}$, respectively, whose characteristic matrices are

$$\Pi_1 = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{bmatrix},\ \Pi_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \end{bmatrix}. \tag{63}$$

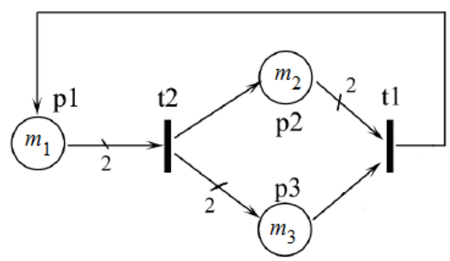

Fig.1. Topology of the TCPN considered as case-study.

The state-equation describing the time-dependence of the marking of TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ with $\boldsymbol{\lambda} = [1\ \ 0.5]^T$ is

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{A}_k \boldsymbol{m}(\tau),\quad \boldsymbol{m}(\tau) \in \mathcal{R}_k,\quad k = 1,2\,, \tag{64}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.25 & -1 & 0 \\ 0.5 & -0.5 & 0 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} -0.5 & 0 & 1 \\ 0.25 & 0 & -2 \\ 0.5 & 0 & -1 \end{bmatrix}. \tag{65}$$

For $\boldsymbol{d} = [1\ \ 0.5\ \ 1]^T$ the study of $\boldsymbol{d}$ *-upper boundedness* of the considered TCPN is based on Theorem 1. Since $\widehat{\boldsymbol{A}_1 \boldsymbol{d}} = [-0.25\ \ -0.25\ \ 0.75]^T$ and $\widehat{\boldsymbol{A}_2 \boldsymbol{d}} = [0.5\ \ 2.25\ \ -0.5]^T$, inequalities (16) are not satisfied, therefore no conclusion can be drawn. For ensuring the $\boldsymbol{d}$ *-upper boundedness* we consider the marking based feedback control law (29) and apply Corollary 1 (i). The solution to the LPP (36) is

$$\boldsymbol{F}_1 = \begin{bmatrix} -1.3546 & -9.6248 & -2.2258 \\ -1.2952 & -5.0624 & -0.2450 \end{bmatrix},$$

$$\boldsymbol{F}_2 = \begin{bmatrix} -1.2771 & -9.6479 & -1.1262 \\ -1.2183 & -4.8239 & -0.2333 \end{bmatrix}. \quad (66)$$

The problem of ensuring the *weakly* $\boldsymbol{d}$ *-conservativity* can be approached in a similar manner.

6. CONCLUSIONS

The current paper formulates sufficient conditions for ensuring refined structural properties of a TCPN through state-feedback. The implementation of the proposed method as a linear programming problem in MATLAB is also presented. Its applicability is illustrated through a case study.

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