On Constructing Fuzzy Classifiers from Interval-Valued Data in Case of Unstable Clustering Structure

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Abstract: The theoretical paper deals in the preliminary way with the problem of extracting fuzzy classification rules from the interval-valued data in the case of unstable clustering structure of the training data set. The corresponding technique is based on a heuristic method of possibilistic clustering. Types of clustering structures are considered. The description of basic concepts of the heuristic method of possibilistic clustering based on the allotment concept is provided. A technique of the interval-valued data preprocessing is also given. An extended technique of constructing of fuzzy rules based on clustering results is described. An illustrative example of the method's application to the Ichino and Yaguchi's oil data set is carried out. An analysis of the experimental results is given and some conclusions are forwarded.

Keywords: fuzzy control; fuzzy inference system; uncertainty; possibilistic clustering; fuzzy rule.

1. INTRODUCTION

Some preliminary remarks are considered in the first subsection. Types of clustering structures are described in the second subsection. Related works are considered briefly in the third subsection.

1.1 Fuzzy clustering and fuzzy rules

Fuzzy set theory was proposed by (Zadeh 1965). Since the fundamental paper was published, fuzzy set theory has been applied to many areas such as learning, decision-making, classification and control.

Fuzzy rule-based classifiers have been widely used on control and classification problems. An overview of the state of the art in fuzzy control is given, mainly, in (Kacprzyk 1997 and Babuška 1998). A beautiful survey of design and analysis of fuzzy control systems in the framework of industrial applications is presented by (Precup and Hellendoorn 2011). Fuzzy classifiers are based on fuzzy inference systems. The fact was demonstrated, for example, by (Michels et al. 2006). One key feature of fuzzy inference systems is their comprehensibility because each fuzzy classification rule is clearly interpretable. The interest in using fuzzy inference systems arises from the fact that those systems consider both accuracy and relevance of the classification result at the same time.

There are a number of approaches to learning fuzzy rules from data based on techniques of evolutionary or neural computation, mostly aiming at optimizing parameters of fuzzy rules. From other hand, fuzzy clustering seems to be a very appealing method for learning fuzzy rules since there is a close and canonical connection between fuzzy clusters and fuzzy rules. The fact was shown by (Höppner et al. 1999). The idea of deriving fuzzy classification rules from the data can be formulated as follows: the training data set is divided into homogeneous group and a fuzzy rule is associated to each group.

Let us consider some methods of fuzzy rules extracting from the data using fuzzy clustering algorithms. Some basic definitions must be given in the first place.

The training set contains n data pairs. Each pair is made of a m_1-dimensional input-vector and a c-dimensional output-vector. We assume that the number of rules in the fuzzy inference system rule base is c. So, (Mamdani and Assilian's 1975) fuzzy rule l within the fuzzy inference system is written as follows:

$$\begin{array}{l} If\ \hat{x}^1\ is\ B_l^1\ and\ \ldots and\ \hat{x}^{m_1}\ is\ B_l^{m_1} \\ \qquad\qquad\qquad\qquad , \\ then\ y_1\ is\ C_1^l\ and \ldots and\ y_c\ is\ C_c^l \end{array} \tag{1}$$

where $B_l^{t_1}$, $t_1 \in \{1,\ldots,m_1\}$ and C_l^l, $l \in \{1,\ldots,c\}$ are fuzzy sets that define an input and output space partitioning. A fuzzy inference system which is described by a set of fuzzy classification rules with the form (1) is the multiple inputs, multiple outputs system.

The principal idea of extracting fuzzy classification rules based on fuzzy clustering was outlined by (Höppner et al. 1999) and the idea is the following. Each fuzzy cluster is assumed to be assigned to one class for classification and the membership grades of the data to the clusters determine the degree to which they can be classified as a member of the corresponding class. So, with a fuzzy cluster that is assigned to the some class we can associate a linguistic rule. The fuzzy cluster is projected into each single dimension leading to a fuzzy set on the real numbers. An approximation of the fuzzy set by projecting only the data set and computing the convex hull of this projected fuzzy set or approximating it by a trapezoidal or triangular membership function is used for the fuzzy rules obtaining.

Objective function-based fuzzy clustering algorithms are the most widespread methods in fuzzy clustering. However, objective function-based fuzzy clustering algorithms are sensitive to initial partition selection and fuzzy rules depend on the selection of the fuzzy clustering method. Moreover, a possibilistic approach to clustering was proposed by (Krishnapuram and Keller 1993) and all algorithms of possibilistic clustering are also objective functions-based algorithms.

On the other hand, a heuristic approach to possibilistic clustering was outlined by (Viattchenin 2004) and the approach was developed in other publications. Moreover, a method of the rapid prototyping fuzzy inference systems based on deriving fuzzy classification rules from the data was proposed by (Viattchenin 2010a).

1.2 Types of clustering structures

Most fuzzy clustering techniques are designed for handling crisp data with their class membership functions. However, the data can be uncertain. Different types of uncertainty can be characterizing the initial data which must be processed by clustering algorithms. For example, a brief review of types of uncertainty of the initial data is given by (Kreinovich and Kosheleva 2009). An interval uncertainty of the initial data is a basic type of uncertainty in clustering problems.

Let $X = \{x_1, \ldots, x_n\}$ be a set of n objects in an m_1-dimensional feature space with coordinate axis labels $(x^1, \ldots, x^{t_1}, \ldots, x^{m_1}\}$. Each object x_i is represented as vector of intervals $x_i = (\hat{x}_i^1, \ldots, \hat{x}_i^{t_1}, \ldots, \hat{x}_i^{m_1})$ where $\hat{x}_i^{t_1} = [\hat{x}_i^{t_1(\min)}, \hat{x}_i^{t_1(\max)}]$. So, the interval-valued data table $\hat{X}_{n \times m_1} = [\hat{x}_i^{t_1}]$ is made up of n rows representing the n objects to be clustered, and m_1 columns representing the m_1 interval variables. In other words, each cell of this table contains an interval $\hat{x}_i^{t_1} = [\hat{x}_i^{t_1(\min)}, \hat{x}_i^{t_1(\max)}]$, $i \in \{1, \ldots, n\}$, $t_1 \in \{1, \ldots, m_1\}$.

The initial data matrix can be represented as a set of two matrices $\hat{X}_{n \times m_1}^{t_2} = [\hat{x}_i^{t_1(t_2)}]$, $i = 1, \ldots, n$, $t_2 \in \{\min, \max\}$ and the "plausible" number c of fuzzy clusters can be different for each matrix $\hat{X}_{n \times m_1}^{t_2} = [\hat{x}_i^{t_1(t_2)}]$, $t_2 \in \{\min, \max\}$. A clustering structure of the data set depends on the type of the initial data.

Three types of the clustering structure were defined by (Viattchenin 2011). Firstly, if the number of clusters c is some constant for each matrix $\hat{X}_{n \times m_1}^{t_2} = [\hat{x}_i^{t_1(t_2)}]$, $t_2 \in \{\min, \max\}$ and the coordinates of prototypes $\{\tau^1, \ldots, \tau^c\}$ of the clusters $\{A^1, \ldots, A^c\}$ are constant, then the clustering structure called stable. Secondly, if the actual number of clusters c is some constant for each matrix $\hat{X}_{n \times m_1}^{t_2} = [\hat{x}_i^{t_1(t_2)}]$, $t_2 \in \{\min, \max\}$ and the coordinates of prototypes of the clusters are not constant, then the clustering structure called quasi-stable. Thirdly, if the number of clusters c is different for matrices $\hat{X}_{n \times m_1}^{t_2} = [\hat{x}_i^{t_1(t_2)}]$, $t_2 \in \{\min, \max\}$ then the corresponding clustering structure called unstable.

Detection of most "plausible" fuzzy clusters in the sought clustering structure for the uncertain data set X can be considered as a final aim of classification and the construction of the set of values of most possible number of fuzzy clusters with corresponding possibility degrees is the important step in this way. The corresponding technique of constructing the stable clustering structure, which corresponds to most natural allocation of objects among fuzzy clusters for the interval-valued data set is proposed by Viattchenin (2011).

1.3 Related works

The idea of application of interval-valued data for control and pattern recognition is not new. In the first place, (Gorzałczany 1988) outlines an idea of control algorithms construction on the basis of verbal models of the controlled objects and the interval-valued fuzzy method of approximate inference was employed for the aim.

In the second place, a fuzzy logic controller with interval-valued inference mechanism for the control of distributed parameter system was proposed by (Li and Zhang 1996). Fuzzifier, fuzzy composition, rules, inference engine, type-reducer, and defuzzifier are six modules of the fuzzy logic controller. The interval-valued rule inference operates on the interval-valued fuzzy set that contains spatial information and produces control action.

In the third place, necessity of using interval-valued data for control and possible ways for the problem solving are discussed by (Kreinovich and Nguyen 1997).

In the fourth place, (Korvin, Hu, and Chen 2004) propose a method of generating rules for interval-valued fuzzy observations, which based on interval arithmetic and obtaining interval rule matrix from interval-valued training data set.

In the fifth place, (Burduk 2009) considers the problem of pattern recognition based on Bayes rule. In his model of classification interval-valued fuzzy observations are used, where a probability of the interval-valued fuzzy event is represented by the real number as upper and lower probability.

In the sixth place, a problem of the interval-valued fuzzy control was considered in (Zeng and Wang 2010), where the inference algorithm of interval-valued fuzzy inference and mathematical model of interval-valued fuzzy control are proposed on the basis of the idea of the interpolation mechanism of fuzzy control.

The method of the rapid prototyping fuzzy inference systems based on heuristic possibilistic clustering was extended for the case of the interval-valued data by (Viattchenin 2010b). However, the method was outlined for cases of stable and quasi-stable clustering structures of the training data set. The aim of the paper is a consideration of the methodology of extracting fuzzy rules from the interval-valued data for the case of unstable clustering structure.

So, the contents of this paper is as follows: in the second section basic concepts of the heuristic method of possibilistic clustering and techniques of the interval-valued data preprocessing are considered, in the third section the extended methodology of constructing the fuzzy inference system is proposed, in the fourth section an example of application of the proposed methodology to the well-known (Ichino and Yaguchi's 1994) oil data set is given, in the fifth section some final remarks are stated.

2. HEURISTIC POSSIBILISTIC CLUSTERING AND INTERVAL-VALUED DATA PREPROCESSING

Basic concepts of the heuristic method of possibilistic clustering are defined in the first subsection. Some techniques of the interval-valued data preprocessing are considered in the second subsection.

2.1 Basic concepts of the heuristic method of possibilistic clustering

Heuristic algorithms of fuzzy clustering display low level of complexity and high level of essential clarity. Some heuristic clustering algorithms are based on a definition of the cluster concept and the aim of these algorithms is cluster detection conform to a given definition. Such algorithms are called algorithms of direct classification or direct clustering algorithms.

An outline for a new heuristic method of fuzzy clustering was presented by (Viattchenin 2004), where a basic version of direct clustering algorithm was described and the algorithm was called the D-AFC(c)-algorithm in (Viattchenin 2007). The D-AFC(c)-algorithm can be considered as a direct algorithm of possibilistic clustering and the algorithm is the basis of the family of other heuristic algorithms of possibilistic clustering. Direct heuristic algorithms of possibilistic clustering can be divided into two types: relational versus prototype-based. These algorithms were developed in other publications.

Let us remind the basic concepts of the heuristic method of possibilistic clustering. Let $X = \{x_1,...,x_n\}$ be the initial set of elements and $T : X \times X \to [0,1]$ some binary fuzzy relation on X with $\mu_T(x_i, x_j) \in [0,1]$, $\forall x_i, x_j \in X$ being its membership function. Fuzzy tolerance is the fuzzy binary intransitive relation which possesses the symmetry property and the reflexivity property. The notions of powerful fuzzy tolerance, feeble fuzzy tolerance and strict feeble fuzzy tolerance were considered in (Viattchenin 2004), as well. In this context the classical fuzzy tolerance in the sense of given definition was called usual fuzzy tolerance. However, the essence of the method here considered does not depend on the kind of fuzzy tolerance. That is why the method herein is described for any fuzzy tolerance T.

Let α be the α-level value of the fuzzy tolerance T, $\alpha \in (0,1]$. Columns or rows of the fuzzy tolerance matrix are fuzzy sets $\{A^1,...,A^n\}$ on X. Let A^l, $l \in \{1,\ldots,n\}$ be a fuzzy set on X with $\mu_{A^l}(x_i) \in [0,1]$, $\forall x_i \in X$ being its membership function. The α-level fuzzy set $A^l_{(\alpha)} = \{(x_i, \mu_{A^l}(x_i)) \mid \mu_{A^l}(x_i) \geq \alpha, x_i \in X\}$ is fuzzy α-cluster. So, $A^l_{(\alpha)} \subseteq A^l$, $\alpha \in (0,1]$, $A^l \in \{A^1,\ldots,A^n\}$ and $\mu_{A^l}(x_i)$ is the membership degree of the element $x_i \in X$ for some fuzzy α-cluster $A^l_{(\alpha)}$, $\alpha \in (0,1]$, $l \in \{1,\ldots,n\}$. The membership degree will be denoted μ_{li} in further considerations. Value of α is the tolerance threshold of fuzzy α-cluster elements. The membership degree of the element $x_i \in X$ for some fuzzy α-cluster $A^l_{(\alpha)}$, $\alpha \in (0,1]$, $l \in \{1,\ldots,n\}$ can be defined as a

$$\mu_{li} = \begin{cases} \mu_{A^l}(x_i), & x_i \in A^l_\alpha \\ 0, & otherwise \end{cases}, \tag{2}$$

where the α-level $A^l_\alpha = \{x_i \in X \mid \mu_{A^l}(x_i) \geq \alpha\}$, $\alpha \in (0,1]$ of a fuzzy set A^l is the support of the fuzzy α-cluster $A^l_{(\alpha)}$.

In other words, if columns or lines of fuzzy tolerance T matrix are fuzzy sets $\{A^1,...,A^n\}$ on X, then fuzzy α-clusters $\{A^1_{(\alpha)},...,A^n_{(\alpha)}\}$ are fuzzy subsets of fuzzy sets $\{A^1,...,A^n\}$ for some value the tolerance threshold α, $\alpha \in (0,1]$.

The value zero for a fuzzy set membership function is equivalent to non-belongingness of an element to a fuzzy set. That is why values of tolerance threshold α are considered in the interval $(0,1]$.

The value of membership function of each element of the fuzzy α-cluster is the degree of similarity of the object to some typical object of fuzzy α-cluster. Moreover, membership degree defines a possibility distribution function for some fuzzy α-cluster $A^l_{(\alpha)}$, $\alpha \in (0,1]$, and this possibility distribution function is denoted $\pi_l(x_i)$.

Let $\{A^1_{(\alpha)},...,A^n_{(\alpha)}\}$ be the family of fuzzy α-clusters for some α. The point $\tau^l_e \in A^l_\alpha$, for which

$$\tau^l_e = \arg\max_{x_i} \mu_{li}, \ \forall x_i \in A^l_\alpha \tag{3}$$

is called a typical point of the fuzzy α-cluster $A^l_{(\alpha)}$, $\alpha \in (0,1]$, $l \in [1,n]$.

A fuzzy α-cluster can have several typical points. That is why symbol e is the index of the typical point. A set $K(A^l_{(\alpha)}) = \{\tau^l_1, \dots, \tau^l_{|l|}\}$ of typical points of the fuzzy cluster $A^l_{(\alpha)}$ is a kernel of the fuzzy cluster and $card\left(K(A^l_{(\alpha)})\right) = |l|$ is a cardinality of the kernel. Obviously, if the fuzzy cluster have an unique typical point, then $|l| = 1$.

Let $R^{\alpha}_z(X) = \{A^l_{(\alpha)} \mid l = \overline{1,c},\, 2 \le c \le n\}$ be a family of fuzzy α-clusters for some value of tolerance threshold α, which are generated by a fuzzy tolerance T on the initial set of elements $X = \{x_1, ..., x_n\}$. If condition

$$\sum_{l=1}^{c} \mu_{li} > 0,\ \forall x_i \in X \tag{4}$$

is met for all $A^l_{(\alpha)}$, $l = \overline{1,c}$, $c \le n$, then the family is the allotment of elements of the set $X = \{x_1, ..., x_n\}$ among fuzzy α-clusters $\{A^l_{(\alpha)}, l = \overline{1,c}, 2 \le c \le n\}$ for some value of the tolerance threshold α. It should be noted that several allotments $R^{\alpha}_z(X)$ can exist for some tolerance threshold α. That is why symbol z is the index of an allotment.

The allotment among fuzzy α-clusters in the sense of the expression (4) can be considered as a special case of the possibilistic partition which was introduced by (Krishnapuram and Keller 1993). So, fuzzy α-clusters in the sense of the expression (2) are elements of the possibilistic partition. However, the concept of allotment among fuzzy α-clusters will be used in further considerations. The next concept introduced should be paid attention to, as well.

Allotment $R^{\alpha}_I(X) = \{A^l_{(\alpha)} \mid l = \overline{1,n},\, \alpha \in (0,1]\}$ of the set of objects among n fuzzy α-clusters for some threshold α is the initial allotment of the set $X = \{x_1, ..., x_n\}$. In other words, if initial data are represented by a matrix of some fuzzy T then lines or columns of the matrix are fuzzy sets $A^l \subseteq X$, $l = \overline{1,n}$ and α-level fuzzy sets $A^l_{(\alpha)}$, $l = \overline{1,n}$, $\alpha \in (0,1]$ are fuzzy α-clusters. These fuzzy α-clusters constitute an initial allotment for some tolerance threshold and they can be considered as clustering components.

If some allotment $R^{\alpha}_z(X) = \{A^l_{(\alpha)} \mid l = \overline{1,c},\, c \le n\}$ is considered appropriate for the formulation of a concrete problem, then this allotment is an adequate allotment. In particular, if condition

$$\sum_{l=1}^{c} card(A^l_{\alpha}) \ge card(X),\ \forall A^l_{(\alpha)} \in R^{\alpha}_z(X),$$
$$\alpha \in (0,1],\ card(R^{\alpha}_z(X)) = c \tag{5}$$

and condition

$$card(A^l_{\alpha} \cap A^m_{\alpha}) \le w, \forall A^l_{(\alpha)}, A^m_{(\alpha)}, l \ne m, \alpha \in (0,1] \tag{6}$$

are met for all fuzzy α-clusters $A^l_{(\alpha)}$, $l = \overline{1,c}$ of some allotment $R^{\alpha}_z(X) = \{A^l_{(\alpha)} \mid l = \overline{1,c},\, c \le n\}$ then the allotment is the allotment among particularly separate fuzzy α-clusters and $w \in \{0, \dots, n\}$ is the maximum number of elements in the intersection area of different fuzzy α-clusters. If $w = 0$ in conditions (5) and (6) then the allotment is the allotment among fully separate fuzzy α-clusters.

The adequate allotment $R^{\alpha}_z(X)$ for some value of tolerance threshold $\alpha \in (0,1]$ is a family of fuzzy α-clusters which are elements of the initial allotment $R^{\alpha}_I(X)$ for the value of α and the family of fuzzy α-clusters satisfy the conditions (5) and (6). Several adequate allotments can exist. Thus, the problem consists in the selection of the unique adequate allotment $R^*_c(X)$ from the set B of adequate allotments, $B = \{R^{\alpha}_z(X)\}$, which is the class of possible solutions of the concrete classification problem and $B = \{R^{\alpha}_z(X)\}$ depends on the parameters of the classification problem. In particular, the number c of fuzzy α-clusters is the parameter of the D-AFC(c)-algorithm.

The selection of the unique adequate allotment from the set $B = \{R^{\alpha}_z(X)\}$ of adequate allotments must be made on the basis of evaluation of allotments. The criterion

$$F(R^{\alpha}_z(X), \alpha) = \sum_{l=1}^{c} \frac{1}{n_l} \sum_{i=1}^{n_l} \mu_{li} - \alpha \cdot c, \tag{7}$$

where c is the number of fuzzy α-clusters in the allotment $R^{\alpha}_z(X)$ and $n_l = card(A^l_{\alpha})$, $A^l_{(\alpha)} \in R^{\alpha}_z(X)$ is the number of elements in the support of the fuzzy α-cluster $A^l_{(\alpha)}$, can be used for evaluation of allotments.

Maximum of the criterion (7) corresponds to the best allotment of objects among c fuzzy α-clusters. So, the classification problem can be characterized formally as determination of the solution $R^*_c(X)$ satisfying

$$R^*_c(X) = \arg \max_{R^{\alpha}_z(X) \in B} F(R^{\alpha}_z(X), \alpha), \tag{8}$$

where $B = \{R^{\alpha}_z(X)\}$ is the set of adequate allotments corresponding to the formulation of a concrete classification problem.

Thus, the problem of cluster analysis can be defined as the problem of discovering the allotment $R^*_c(X)$, resulting from the classification process and detection of fixed or unknown number c of fuzzy α-clusters can be considered as the aim of classification.

The basic relational D-AFC(c)-algorithm is described, for example, in (Viattchenin 2010a) and the algorithm is a basis of the method of derivation of fuzzy rules from the data. The allotment $R_c^*(X) = \{A_{(\alpha)}^l \mid l = \overline{1,c}\}$ among the a priori given number c of particularly separate fuzzy α-clusters and the corresponding value of tolerance threshold α are the results of classification obtained from the D-AFC(c)-algorithm. Moreover, different validity measures for the D-AFC(c)-algorithm were proposed in (Viattchenin 2010c).

2.2 Notes on the interval-valued data preprocessing

Relational heuristic algorithms of possibilistic clustering can be applied directly to the data given as a matrix of some fuzzy tolerance $T = [\mu_T(x_i, x_j)]$, $i, j = 1,\ldots,n$. This means that it can be used with the objects by attributes data by choosing a suitable metric to measure similarity. However, the initial data should be normalized. A method of the interval-valued data normalization was defined in Viattchenin (2011) as follows:

$$x_i^{t_1(t_2)} = \frac{\hat{x}_i^{t_1(t_2)} - \min\limits_{i,t_2} \hat{x}_i^{t_1(t_2)}}{\max\limits_{i,t_2} \hat{x}_i^{t_1(t_2)} - \min\limits_{i,t_2} \hat{x}_i^{t_1(t_2)}}, \tag{9}$$

where $t_1 = 1,\ldots,m_1$, $t_2 \in \{\min, \max\}$. So, each object x_i, $i = 1,\ldots,n$ can be considered as an interval-valued fuzzy set and $\mu_{x_i}(x^{t_1}) = [\mu_{x_i}(x^{t_1(\min)}), \mu_{x_i}(x^{t_1(\max)})]$, $i = 1,\ldots,n$, $t = 1,\ldots,m$ is its membership function, where $\mu_{x_i}(x^{t_1(\min)}) \in [0,1]$, $\mu_{x_i}(x^{t_1(\max)}) \in [0,1]$.

Different distances and similarity measures for interval-valued fuzzy sets were proposed in some publications. Firstly, some methods for measuring distances between interval-valued fuzzy sets were proposed by (Burillo and Bustince 1996). For example, the normalized Euclidean distance was defined in as follows:

$$d_I(x_i, x_j) = \sqrt{\frac{1}{2m_1} \sum_{t_1=1}^{m_1} \left\{ \begin{array}{l} \left(\mu_{x_i}(x^{t_1(\min)}) - \mu_{x_j}(x^{t_1(\min)})\right)^2 + \\ + \left(\mu_{x_i}(x^{t_1(\max)}) - \mu_{x_j}(x^{t_1(\max)})\right)^2 \end{array} \right\}}, \tag{10}$$

for all $i, j = 1,\ldots,n$.

Secondly, the normalized Euclidean distance between interval-valued fuzzy sets based on Hausdorff metric was defined by (Grzegorzewski 2004) as follows:

$$e_I(x_i, x_j) = \sqrt{\frac{1}{m_1} \sum_{t_1=1}^{m_1} \max \left\{ \begin{array}{l} \left(\mu_{x_i}(x^{t_1(\min)}) - \mu_{x_j}(x^{t_1(\min)})\right)^2, \\ \left(\mu_{x_i}(x^{t_1(\max)}) - \mu_{x_j}(x^{t_1(\max)})\right)^2 \end{array} \right\}}, \tag{11}$$

for all $i, j = 1,\ldots,n$.

Thirdly, a similarity measure between interval-valued fuzzy sets was defined by(Ju and Yuan 2007) as follows:

$$s_I(x_i, x_j) = 1 - \frac{1}{\sqrt[\lambda]{m_1}} \sqrt[\lambda]{\sum_{t_1=1}^{m_1} \left| \begin{array}{l} \dfrac{\mu_{x_i}(x^{t_1(\min)}) + \mu_{x_i}(x^{t_1(\max)})}{2} - \\ - \dfrac{\mu_{x_j}(x^{t_1(\min)}) + \mu_{x_j}(x^{t_1(\max)})}{2} \end{array} \right|^{\lambda}}, \tag{12}$$

for all $i, j = 1,\ldots,n$ and $1 \le \lambda < \infty$.

Moreover, dissimilarity coefficients between the objects can be constructed on a basis of generalizations of distances between fuzzy sets which were proposed by (Viattchenin 2009). In particular, a generalization of the squared normalized Euclidean distance for interval-valued fuzzy sets can be described by the expression:

$$\varepsilon_I(x_i, x_j) = \frac{1}{m_1} \sum_{t_1=1}^{m_1} \left(\frac{1}{2^2} \sum_{t_2 \in \{\min, \max\}} \left(\mu_{x_i}(x^{t_1(t_2)}) - \mu_{x_j}(x^{t_1(t_2)})\right)^2 \right), \tag{13}$$

for all $i, j = 1,\ldots,n$.

The matrix of fuzzy intolerance relation $I = [\mu_I(x_i, x_j)]$, $i, j = 1,\ldots,n$ is a result of application of formulae (10), (11), (13) to the family of interval-valued fuzzy sets $\{x_1,\ldots,x_n\}$. The matrix of fuzzy tolerance $T = [\mu_T(x_i, x_j)]$, $i, j = 1,\ldots,n$ can be obtained after application of complement operation

$$\mu_T(x_i, x_j) = 1 - \mu_I(x_i, x_j), \ \forall i, j = 1,\ldots,n, \tag{14}$$

to the matrix of fuzzy intolerance $I = [\mu_I(x_i, x_j)]$.

3. AN OUTLINE OF THE APPROACH

The technique of derivation of fuzzy rules from the interval-valued data is described in the first subsection. A generalized methodology for constructing fuzzy inference systems in the case of unstable clustering structure is proposed in the second subsection of the section.

3.1 Extracting fuzzy rules from the interval-valued data

Let us consider the extended method of extracting fuzzy rules from the interval-valued data which was proposed by (Viattchenin 2010b). In the following, we will consider that the Mamdani-type fuzzy inference system is a multiple inputs, multiple outputs system.

The antecedent of a fuzzy rule in a fuzzy inference system defines a decision region in the m_1-dimensional feature space. Let us consider a fuzzy rule (1) where $B_l^{t_1}$, $t_1 = 1,\ldots,m_1$, $l \in \{1,\ldots,c\}$ is a fuzzy set associate with the attribute variable $\hat{x}^{t_1}$. Let $B_l^{t_1}$ be characterized by the trapezoidal membership function $\gamma_{B_l^{t_1}}(\hat{x}^{t_1})$ which is presented in Fig. 1. So, the fuzzy set $B_l^{t_1}$ can be defined by four parameters, $B_l^{t_1} = (\underline{a}_{(l)}^{t_1}, \underline{m}_{(l)}^{t_1}, \overline{m}_{(l)}^{t_1}, \overline{a}_{(l)}^{t_1})$.

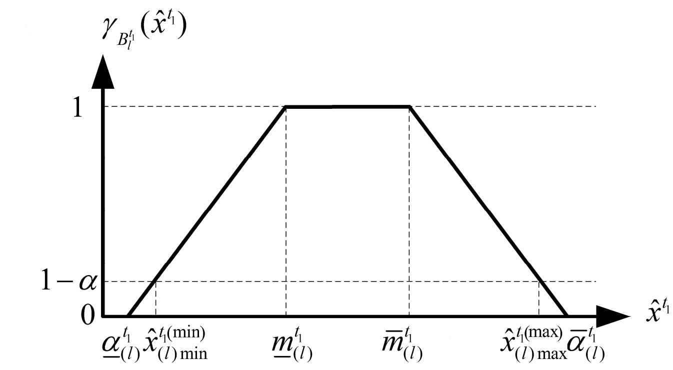

Fig. 1. A trapezoidal membership function for an antecedent fuzzy set.

Let us consider a technique of antecedents learning. We apply the D-AFC(c)-algorithm to the given interval-valued data and then obtain for each fuzzy α-cluster $A_{(\alpha)}^l$, $l \in \{1,\ldots,c\}$ a kernel $K(A_{(\alpha)}^l)$ and a support A_α^l. The value of tolerance threshold $\alpha \in (0,1]$, which corresponds to the allotment $R_c^*(X) = \{A_{(\alpha)}^1,\ldots,A_{(\alpha)}^c\}$, is the additional result of classification. The situation of the interval-valued data can be described by the expression $\hat{x}_i^{t_1} = (\hat{x}_i^{t_1(\min)}, \hat{x}_i^{t_1(\max)})$, $t_1 = 1,\ldots,m_1$, $i = 1,\ldots,n$. In particular, the interval $[\hat{x}_{(l)\min}^{t_1(\min)}, \hat{x}_{(l)\max}^{t_1(\min)}]$ of values of every attribute $\hat{x}^{t_1} = (\hat{x}^{t_1(\min)}, \hat{x}^{t_1(\max)})$, $t_1 \in \{1,\ldots,m_1\}$ for the support A_α^l should be calculated. We calculate the interval $[\hat{x}_{(l)\min}^{t_1(\min)}, \hat{x}_{(l)\max}^{t_1(\min)}]$ of values of every attribute $\hat{x}^{t_1}$, $t_1 \in \{1,\ldots,m_1\}$ for the support A_α^l. The value $\hat{x}_{(l)\min}^{t_1(\min)}$ can be obtained as follows

$$\hat{x}_{(l)\min}^{t_1(\min)} = \min_{x_i \in A_\alpha^l} \hat{x}^{t_1(\min)}, \ \forall t_1 \in \{1,\ldots,m_1\}, \ \forall l \in \{1,\ldots,c\}, \quad (15)$$

and the value $\hat{x}_{(l)\max}^{t_1(\max)}$, $t_1 \in \{1,\ldots,m_1\}$ can be calculated using a formula

$$\hat{x}_{(l)\max}^{t_1(\max)} = \max_{x_i \in A_\alpha^l} \hat{x}^{t_1(\max)}, \ \forall t_1 \in \{1,\ldots,m_1\}, \ \forall l \in \{1,\ldots,c\}. \quad (16)$$

The parameter $\underline{a}_{(l)}^{t_1}$ can be calculated using the conditions

$$\gamma_{B_l^{t_1}}(\hat{x}_{(l)\min}^{t_1(\min)}) = (1-\alpha), \ \gamma_{B_l^{t_1}}(\underline{a}_{(l)}^{t_1}) = 0, \quad (17)$$

and the parameter $\overline{a}_{(l)}^{t_1}$ can be obtained from the conditions

$$\gamma_{B_l^{t_1}}(\hat{x}_{(l)\max}^{t_1(\max)}) = (1-\alpha), \ \gamma_{B_l^{t_1}}(\overline{a}_{(l)}^{t_1}) = 0. \quad (18)$$

We calculate the value $\hat{\underline{x}}_{(l)}^{t_1(\min)}$ for all typical points $\tau_e^l \in K(A_{(\alpha)}^l)$ of the fuzzy α-cluster $A_{(\alpha)}^l$, $l \in \{1,\ldots,c\}$ as follows:

$$\hat{\underline{x}}_{(l)}^{t_1(\min)} = \min_{\tau_e^l \in K(A_{(\alpha)}^l)} \hat{x}^{t_1(\min)}, \ \forall e \in \{1,\ldots,|l|\}, \quad (19)$$

and the value $\hat{\overline{x}}_{(l)}^{t_1(\max)}$ can be obtained from the equation

$$\hat{\overline{x}}_{(l)}^{t_1(\max)} = \max_{\tau_e^l \in K(A_{(\alpha)}^l)} \hat{x}^{t_1(\max)}, \ \forall e \in \{1,\ldots,|l|\}. \quad (20)$$

Thus, the parameter $\underline{m}_{(l)}^{t_1}$ can be calculated from the conditions

$$\gamma_{B_l^{t_1}}(\hat{\underline{x}}_{(l)}^{t_1(\min)}) = \gamma_{B_l^{t_1}}(\underline{m}_{(l)}^{t_1}) = 1, \quad (21)$$

and the parameter $\overline{m}_{(l)}^{t_1}$ can be obtained as following:

$$\gamma_{B_l^{t_1}}(\hat{\overline{x}}_{(l)}^{t_1(\max)}) = \gamma_{B_l^{t_1}}(\overline{m}_{(l)}^{t_1}) = 1. \quad (22)$$

Let us consider a technique of consequents learning. The variables y_l, $l = 1,\ldots,c$ are the consequents of fuzzy rules (1), represented by the fuzzy sets C_l^l, $l = 1,\ldots,c$ with the membership functions $\gamma_{C_l^l}(y_l)$. Fuzzy sets C_l^l, $l = 1,\ldots,c$ can be defined on the interval of memberships $[0,1]$ and these fuzzy sets can be presented as follows: $C_l^l = (\alpha, \underline{\mu}_l, \overline{\mu}_l, 1)$, where α is the tolerance threshold, $\underline{\mu}_l = \min_{x_i \in A_\alpha^l} \mu_{li}$ and $\overline{\mu}_l = \max_{x_i \in A_\alpha^l} \mu_{li}$. The membership function can be interpreted as a high membership. The case is presented in Fig. 2.

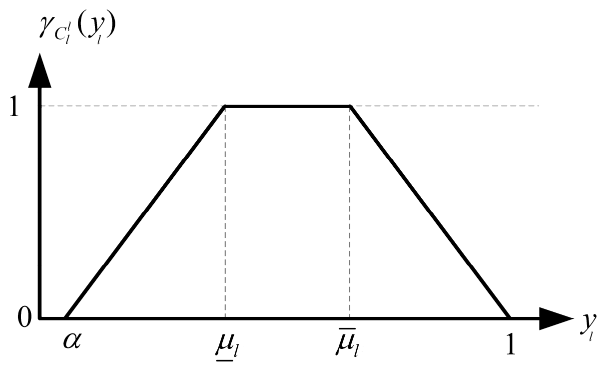

Fig. 2. A membership function for a consequent fuzzy set in a case of a high membership.

From other hand, if $A_{(\alpha)}^l$ and $A_{(\alpha)}^m$, $l \neq m$ are two particularly separated fuzzy α-clusters, and then a condition $w \neq 0$ is met in the equation (6). So, a fuzzy set $C_m^l = (0, 1-\overline{\mu}_m, 1-\underline{\mu}_m, 1-\alpha)$ is the consequent for the variable y_m of the l-th fuzzy rule for a case of low membership. The corresponding case is illustrated by Fig. 3.

Thus, trapezoidal membership functions $\gamma_{C_l^l}(y_l)$ for the fuzzy sets C_l^l, $l = 1,\ldots,c$ can be constructed on a basis of the clustering results. The empty set $A_\alpha^l = \varnothing$, $l \in \{1,\ldots,c\}$ can be correspond to some output variable y_l, $l \in \{1,\ldots,c\}$. So, the empty fuzzy set C_l^l will be correspond to the output variable

y_l, $l \in \{1,\ldots,c\}$ and $\gamma_{C_l^l}(y_l)=0$ is the membership function of the corresponding fuzzy set C_l^l.

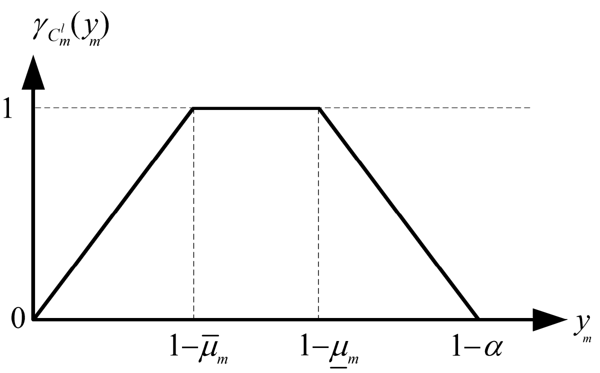

Fig. 3. A membership function for a consequent fuzzy set in a case of a low membership.

That is why the technique of extracting fuzzy rules from the interval-valued data can be summarized as a three-step procedure:

1. The matrix of tolerance coefficients $T=[\mu_T(x_i,x_j)]$, $i,j=1,\ldots,n$ can be constructed from the normalized initial data by choosing a suitable distance for interval-valued fuzzy sets;
2. The D-AFC(c)-algorithm should be applied to the matrix of tolerance coefficients $T=[\mu_T(x_i,x_j)]$, $i,j=1,\ldots,n$ for the fixed number c of fuzzy α-clusters and the corresponding allotment $R_c^*(X)$ will be constructed;
3. Fuzzy rules should be constructed for each fuzzy α-cluster $A_{(\alpha)}^l \in R_c^*(X)$ as follows:
   - 3.1. Antecedents of fuzzy rules should be constructed for each fuzzy α-cluster $A_{(\alpha)}^l \in R_c^*(X)$ according to formulae (15) – (22);
   - 3.2. Consequents y_l, $l=1,\ldots,c$ of fuzzy rules should be constructed for corresponding antecedents as described above.

So, the number of obtained fuzzy rules is equal to the number c of fuzzy α-cluster in the sought allotment $R_c^*(X)$.

3.2 A technique of extracting fuzzy rules from the data in the case of unstable clustering structure

A generalized scheme of prototyping of the fuzzy inference system from the interval-valued data in case of unstable clustering structure can be described shortly as follows: a stationary clustering structure should be constructed on the first step and fuzzy rules must be derived on the second step according to the described technique. There is a two-step procedure for constructing fuzzy inference systems in the case of the interval-valued data set:

1. The allotment $R_c^*(X)$ among unknown number c of fuzzy α-clusters should be constructed according to the technique of constructing the stable clustering structure using a suitable distance for interval-valued fuzzy sets ;
2. Fuzzy classification rules should be extracted from the training interval-valued data set according to the corresponding method.

The effective technique of constructing the stable clustering structure for the uncertain data set was described in (Viattchenin 2011).

The proposed technique can be generalized for a case of the three-way data very simply. An application of the proposed technique to the problem of constructing fuzzy inference system will be illustrated on the well-known interval-valued data example in the next section.

4. AN ILLUSTRATIVE EXAMPLE

The (Ichino and Yaguchi's 1994) interval-valued oil data set is described in the first subsection. The second subsection of the section includes the results of the data clustering obtained from the technique of constructing the stable clustering structure. The performance of the designed fuzzy inference system for some testing data set is also considered in the second subsection of the section.

4.1 The oil data set

Let us consider the set of interval-valued data which were considered by (Ichino and Yaguchi 1994). The data are presented in Table 1. The data set consists of the specific gravity, iodine value, and saponification value measured for 8 types of oils.

Table 1. The oil data set

Oils	Specific gravity	Iodine value	Saponification value
Linseed oil	0.930-0.935	170-204	118-196
Perilla oil	0.930-0.935	192-208	188-197
Cottonseed oil	0.916-0.918	99-113	189-198
Sesame oil	0.920-0.926	104-116	187-193
Camellia oil	0.916-0.917	80-82	189-193
Olive oil	0.914-0.919	79-90	187-196
Beef tallow	0.860-0.870	40-48	190-199
Hog fat	0.858-0.864	53-77	190-202

The analysis of types of oils in Table 1 highlights that the first six oils are vegetable and the remaining two are animal.

4.2 The results

Let us consider results of the numerical experiment. So, results of the data clustering must be considered in the first place.

The data matrix $\hat{X}_{8\times3} = [\hat{x}_i^{t_1}]$, $\hat{x}_i^{t_1} = [\hat{x}_i^{t_1(\min)}, \hat{x}_i^{t_1(\max)}]$ was normalized according to formula (9) and the procedure of constructing the set of values of most possible number c of fuzzy α-clusters was applied to the interval-valued data set. So, the set $\{2,\ldots,4\}$ of values of most possible number c of fuzzy α-clusters in sought allotment $R_c^*(X)$ was obtained. That is why the clustering structure of the oil data set is unstable, and the fact was demonstrated in (Viattchenin 2011). The technique of constructing the stable clustering structure was applied to the normalized oil data using generalization of the squared normalized Euclidean distance for interval-valued fuzzy sets (13). We obtain that the actual number of fuzzy α-clusters is equal 2. The corresponding allotment $R_c^*(X)$ among two fully separated fuzzy α-clusters was obtained for the tolerance threshold $\alpha = 0.8184$. Membership functions of two classes of the allotment $R_c^*(X)$ are presented in Fig. 4.

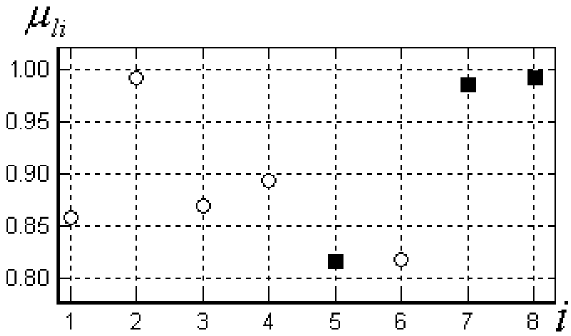

Fig. 4. Membership functions of two classes obtained from the D-AFC(c)-algorithm using the formula (13).

Membership values of the first class are represented in Fig. 4 by ○, membership values of the second class are represented by ■ and values which equal zero are not shown in the figure. The first class corresponds to the vegetable oils and the second class corresponds to the animal oils. The first class is formed by 5 elements and the second class includes 3 elements. So, the fifth element is the misclassified object. Results obtained from the technique of constructing the stable clustering structure for measures (11), (12) and (13) are considered in detail by (Viattchenin 2011).

Let us consider the performance of the generated fuzzy inference system. The testing data are presented in Table 2.

Table 2. The testing data set

Oils	Specific gravity	Iodine value	Saponification value
Linseed oil	0.930	190	170
Perilla oil	0.930	200	190
Cottonseed oil	0.916	100	192
Sesame oil	0.920	110	192
Camellia oil	0.916	81	192
Olive oil	0.916	81	187
Beef tallow	0.865	45	195
Hog fat	0.860	70	195

The technique for deriving fuzzy rules from fuzzy α-clusters was applied to the initial interval-valued data. So, membership functions $\gamma_{B_l^{t_1}}(\hat{x}^{t_1})$ and $\gamma_{C_l^l}(y_l)$ for corresponding fuzzy sets $B_l^{t_1}$ and C_l^l, $t_1 = 1,\ldots,3$, $l = 1,2$, were constructed immediately. A meaningful linguistic label can be assigned to each fuzzy set $B_l^{t_1}$, $t_1 = 1,\ldots,3$, $l = 1,2$ and linguistic labels Vegetable and Animal are associated with corresponding output variables y_l, $l = 1,2$. Fuzzy sets C_2^1 and C_1^2 are empty fuzzy sets because corresponding elements of the obtained allotment $R_c^*(X)$ are fully separated fuzzy α-clusters. The graph of the performance of the designed fuzzy inference system is shown in Fig. 5.

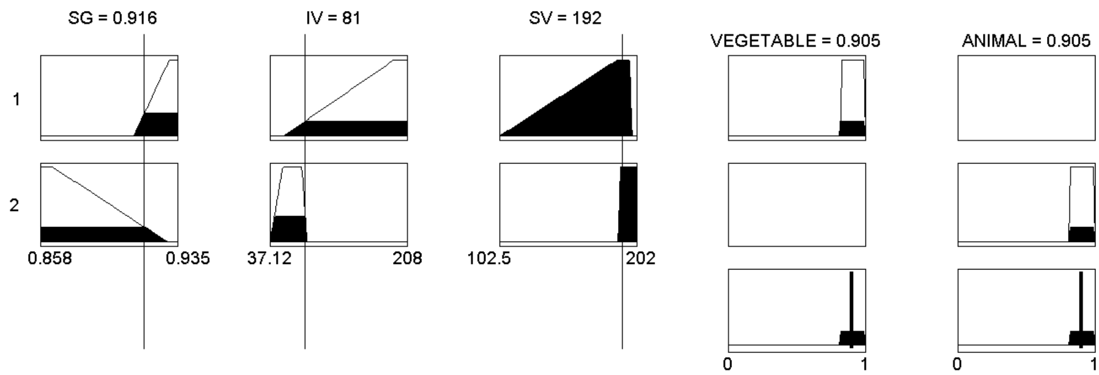

Fig. 5. The performance of the designed fuzzy inference system.

The corresponding fuzzy rule base induced by the D-AFC(c)-algorithm clustering result can be seen in the figure, where labels SG l, IV l, and SV l denote, respectively, specific gravity, iodine value, and saponification value, and $l = 1,2$ is the number of rule.

Let us consider values of the average of the range of output variables, which are the result of the defuzzification process. These values for all objects of the testing data set obtained from the generated fuzzy inference system are presented in Table 3.

Table 3. Results of classification

Oils	Values of the average of the range of output variables	
	VEGETABLE	ANIMAL
Linseed oil	0.908	0.500
Perilla oil	0.909	0.500
Cottonseed oil	0.906	0.500
Sesame oil	0.907	0.500
Camellia oil	0.905	0.905
Olive oil	0.905	0.500
Beef tallow	0.500	0.905
Hog fat	0.500	0.905

The example of classification of the camellia oil is presented in Fig. 5. It should be noted that camellia oil is misclassified object in the result of the initial oils data clustering. From other hand, Fig. 5 shows that the camellia oil is the element of both classes and corresponding values of an average of the range of output variables are equal 0.905. These values can be interpreted as high belonging of the object to both classes. So, the fuzzy inference system is accurate in comparison with clustering results.

Output values are equal 0.5 sometimes, because total area is zero in the defuzzification procedure for corresponding output variable. The value can be interpreted as uncertain belonging of the object to the corresponding class.

5. CONCLUSIONS

Some final remarks are formulated in the first subsection. The second subsection deals with the perspectives on future investigations.

5.1 Discussions

Many techniques to design fuzzy inference systems from data are available; they all take advantage of the property of fuzzy inference systems to be universal approximators.

This paper presents the technique of extracting fuzzy rules from the from the interval-valued training data set via heuristic possibilistic clustering in the case of unstable clustering structure of the training data set.

The results obtained with the proposed modeling approach for the (Ichino and Yaguchi's 1994) data set case illustrate the effectiveness of the proposed technique for derivation of fuzzy classification rules from the interval-valued data. Evidently, that the results are correlated with the results, obtained from the D-AFC(c)-algorithm. So, the fuzzy inference system is accurate. From other hand, the result which is obtained from fuzzy inference system is easily interpreted. Thus, the obtained model is suitable for interpretation since the rules consequents are the same or close to the actual class labels, such that each rule can be taken to describe all classes.

Of course, the presented example is the brief illustration for the application of the proposed approach. However, the results obtained with the proposed approach for (Guru, Kiranagi, and Nagabhushan's 2004) city temperature symbolic data set are satisfactory also.

It should be noted, that the proposed approach to deriving fuzzy classification rules from the interval-valued data is simpler from mathematical and intuitive positions than the method which was proposed, for example, by (Korvin, Hu, and Chen 2004).

5.2 Perspectives

Constructing a rule base from fuzzy clusters gives a first approximation for the data which can be used as a basis for further improvements. So, a technique of fuzzy rules tuning can be developed.

From other hand, the D-AFC(c)-algorithm can be applied for classification the three-way data. So, the proposed technique of designing fuzzy inference systems can be generalized for the case of the three-way training data set. Moreover, the proposed technique can be extended for constructing adaptive fuzzy classifiers.

These perspectives for investigations are of great interest both from the theoretical point of view and from the practical one as well.

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