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Two Design Procedures for a Time-Delay Control System

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Abstract: This paper deals with two design procedures of a control system for a proportional or integrator time-delay plant using a suitable placement of the closed-loop poles to have a small response time, a zero or small overshoot and a given magnitude ratio of the controller output to a step reference or disturbance. To obtain these performances, the both controller design procedures provide a unique solution for the closed-loop pole placement, such that the open-loop system is of simple integrator type (with one pole in origin) to have zero steady-state error to a step input. Three numerical examples for time-delay plants of proportional and integrator type are given to illustrate the effectiveness of the proposed control procedures.

*Keywords***:** pole polynomial, stable semi-proper controller, time-delay plant, simple integrator, magnitude ratio.

1. INTRODUCTION

The design of linear control systems is frequently examined in the state space domain and in the input-output domain [2,5,7,8], by either classical approach of a state feedback pole placement and of a suitable state-estimator, or H_2 optimal synthesis of a state feedback and of a Kalman state-estimator. In frequency domain, the control system design by H_2 or H_∞ optimal method is, as a rule, parametric, aiming to find the best tuning parameters of a given controller structure [1,6,9,10].

This paper extends the design method of a control linear system without time-delay in [4] to the case of the stable time-delay plants, starting from the idea that the pole placement in the complex plane predominantly determines the control system dynamic performances, but the number of zeros and their placement in the left half s-plane also influence the system performances [3].

The design of the controller by a suitable placement of the closed-loop system poles is based on three premises:

a) The plant is stable, non-derivative (without null zeros) and with large time-delay ;

b) The open-loop system is of simple integrator type (with one pole in origin) to have zero steady-state error to a step input;

c) The closed-loop control system has a minimum possible time-delay (equal to the plant time-delay, as in the Smith predictor case) and a suitable pole polynomial to have the best control performances for a given magnitude ratio.

In the first procedure, which provides a finite and monotonic control system response to a step input, the last condition c) is fulfilled by choosing all of the closed-loop system poles to be real, negative and equal. Theoretically, by a suitable choice of the pole polynomial, the step response time of the closed-loop system can be, except the time-delay, as small as desired. However, this is not possible in practice due to the plant model uncertainty, which imposes restrictions on the magnitude of the controller output to a step reference or disturbance. For instant, in the case of a plant of proportional type, the pole polynomial must be chosen such that the step response lag time of the control system is at most 2...5 times less than the step response lag time of the plant. In practical applications, the magnitude coefficient (ratio) of the controller output must be bounded to a given value (less than 20 for a proportional plant and than 10 for an integrator plant) to assure a smooth control, to diminish the noise amplification, to reduce the wear and tear of the plant, to decrease the fuel and energy consumption.

For a control system with proportional plant and semi-proper controller, the magnitude ratio of the control variable c is defined in [4] as the ratio $c(0+)/c(\infty)$ between the initial value and the final value of this variable to a step reference. For a control system with simple integrator plant and semi-proper controller, the magnitude ratio of the control variable c is defined as the ratio $c(0+)/r_0$ between the initial value of the control variable c to a step reference and the reference value r_0 (under the assumption that these variables are expressed in percent).

2. FIRST DESIGN PROCEDURE

The main ideas of the design procedure are given by the following three theorems.

Theorem 1. *Consider a time-delay plant given by*

$$G_P(s) = G(s)e^{-Ts}, \tag{1}$$

where

$$G(s) = \frac{q(s)}{p(s)} = \frac{q_k s^k + q_{k-1}s^{k-1} + \ldots + q_0}{p_n s^n + p_{n-1}s^{n-1} + \ldots + p_0},$$

is a stable, non-derivative and minimum-phase rational function having the polynomials $p(s)$ *and* $r(s)$ *coprime,* $k < n$, $p_n \neq 0$, $q_k \neq 0$.

For any $n-k$ *degree polynomial*

$$P(s) = (T_1 s+1)(T_2 s+1)\ldots(T_{n-k}s+1) \tag{2}$$

with all $T_i > 0$, *choosing the stable semi-proper controller*

$$G_C(s) = \frac{1}{G(s)[P(s) - e^{-Ts}]}, \tag{3}$$

the open-loop system is of simple integrator type and the closed–loop system has the transfer function

$$G_0(s) = \frac{e^{-Ts}}{P(s)}. \tag{4}$$

Proof. First we prove that the controller $G_C(s)$ is semi-proper and stable. The controller is semi-proper because

$$\lim_{s\to\infty} G_C(s) = \lim_{s\to\infty} \frac{1}{G(s)P(s)} = \frac{q_k}{p_n T_1 T_2 \ldots T_{n-k}} \neq 0.$$

Since $G(s)$ is of minimum-phase and non-derivative (that is, $G(s)$ has all its roots in the left half s-plane), the controller is stable if the equation

$$P(s) - e^{-Ts} = 0$$

has all its nonzero roots in the left half s-plane. Let $s = x + \mathrm{j}y \neq 0$ be a nonzero root of this equation. From

$$P(x+\mathrm{j}y) = e^{-T(x+iy)},$$

we get

$$\prod_{i=1}^{n-k}[(T_i x+1)^2 + T_i^2 y^2] = e^{-2Tx}.$$

Clearly, this equality doesn't hold for $x \geq 0$, since

$$(T_i x+1)^2 + T_i^2 y^2 > 1$$

for all indices i, and hence

$$\prod_{i=1}^{n-k}[(T_i x+1)^2 + T_i^2 y^2] > 1 \geq e^{-2Tx}.$$

Therefore, the real part x of the root $s = x + \mathrm{j}y \neq 0$ is negative. Thus, the controller (3) is semi-proper and stable.

The open-loop (direct) system is of simple integrator type, because its transfer function has the expression

$$G_d(s) = G_C(s)G_P(s) = \frac{e^{-Ts}}{P(s) - e^{-Ts}}, \tag{5}$$

and hence

$$\lim_{s\to 0} sG_d(s) = \frac{1}{T_1 + T_2 + \ldots + T_{n-k} + T} \neq 0.$$

The closed-loop system has the transfer function

$$G_0(s) = \frac{G_d}{1+G_d} = \frac{e^{-Ts}}{P(s)}.$$

Remark 1°. If the plant is of proportional type, that is $p_0 \neq 0$ and $q_0 \neq 0$, then the controller is simple integrator, since

$$\lim_{s\to 0} sG_C(s) = \frac{p_0}{(T_1 + T_2 + \ldots + T_{n-k})q_0} \neq 0.$$

If the plant is of simple integrator type, that is $p_0 = 0$, $p_1 \neq 0$ and $q_0 \neq 0$, then the controller is of proportional type, since

$$\lim_{s\to 0} G_C(s) = \frac{p_1}{(T_1 + T_2 + \ldots + T_{n-k} + T)q_0} \neq 0.$$

Remark 2°. Let $G_1(s)$ be the transfer function from reference r to controller output c. Since

$$G_1(s) = \frac{G_0(s)}{G_P(s)} = \frac{1}{P(s)G(s)} \tag{6}$$

does not depend on the time-delay T, it follows that the control system response $c(t)$ to a unit step reference does not depend on T, too; that means that $c(t)$ is the same as in the case of a plant without time-delay.

Remark 3°. To implement the controller (3) one can use the block-scheme in Figure 1, where

$$C_1(s) = \frac{1}{G(s)P(s)}, \tag{7}$$

$$C_2(s) = G_0(s) = \frac{e^{-Ts}}{P(s)}. \tag{8}$$

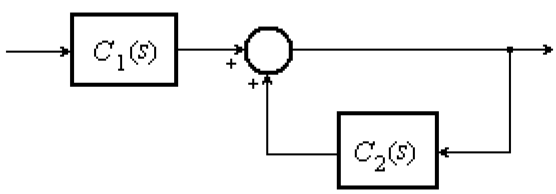

Fig. 1. The controller implementation scheme.

Theorem 2. *Consider the plant* $G_P(s)$ *in Theorem 1 of proportional type* $(p_0 \neq 0, q_0 \neq 0)$. *For a given magnitude ratio* M *, the step response time of the control system has the smallest value by choosing*

$$T_1 = T_2 = \mathbf{L} = T_{n-k} = \sqrt[n-k]{\frac{q_0 p_n}{M p_0 q_k}} \,. \qquad (9)$$

Proof. For a stable control system with proportional plant, the magnitude ratio is the ratio between the initial value and the final value of the controller output c to a unit step reference [4]

$$M = \frac{c(0+)}{c(\infty)} \,.$$

Let $G_1(s)$ be the transfer function from reference to controller output. Since

$$G_1(s) = \frac{G_0(s)}{G_P(s)} = \frac{1}{P(s)G(s)} \,,$$

using the initial and final value theorems, we get

$$c(0+) = \lim_{s\to\infty} G_1(s) = \frac{1}{\lim_{s\to\infty} P(s)G(s)} \,,$$

and

$$c(\infty) = \lim_{s\to 0} G_1(s) = \frac{1}{P(0)G(0)} = \frac{1}{G(0)}$$

This implies that

$$M = \frac{G(0)}{\lim_{s\to\infty} P(s)G(s)} \,, \qquad (10)$$

and hence

$$M = \frac{q_0 p_n}{p_0 q_k T_1 T_2 \,\mathbf{L}\, T_{n-k}} \,,$$

or

$$T_1 T_2 \,\mathbf{L}\, T_{n-k} = \frac{q_0 p_n}{M p_0 q_k} \,.$$

For a given magnitude ratio M, from the arithmetic mean-geometric mean inequality

$$\frac{T_1 + T_2 + \mathbf{L} + T_{n-k}}{n-k} \ge \sqrt[n-k]{T_1 T_2 \,\mathbf{L}\, T_{n-k}} \,,$$

it follows that the sum $T_1 + T_2 + \mathbf{L} + T_{n-k}$ attains its minimal value when

$$T_1 = T_2 = \mathbf{L} = T_{n-k} = \sqrt[n-k]{\frac{q_0 p_n}{M p_0 q_k}} \,.$$

On the other hand, since

$$(T_1 + T_2 + \mathbf{L} + T_{n-k})s + 1$$

is the best first order approximation of $P(s)$ by Padé approximation, we can approximate the closed-loop transfer function (4) by

$$G_0(s) \approx \frac{e^{-Ts}}{(T_1 + T_2 + \mathbf{L} + T_{n-k})s + 1} \,.$$

Therefore, the closed-loop response time to a step reference has the smallest value if the equivalent time constant

$$T_1 + T_2 + \mathbf{L} + T_{n-k}$$

is minimal; that is, when (9) holds.

Theorem 3. *Consider the plant $G_P(s)$ in Theorem 1 of simple integrator type ($p_0 = 0, p_1 \ne 0, q_0 \ne 0$). For a given magnitude ratio M, the step response time of the control system has the smallest value by choosing*

$$T_1 = T_2 = \mathbf{L} = T_{n-k} = \sqrt[n-k]{\frac{p_n}{M q_k}} \,. \qquad (11)$$

Proof. For a stable control system with simple integrator plant, the magnitude ratio is equal to the initial value of the controller output c to a unit step reference

$$M = c(0+) \,.$$

In practice, the magnitude ratio M must be bounded to a value less than 10

As shown in the proof of Theorem 2, we have

$$M = \frac{1}{\lim_{s\to\infty} P(s)G(s)} \,. \qquad (12)$$

From (12), we get

$$M == \frac{p_n}{q_k T_1 T_2 \,\mathbf{L}\, T_{n-k}} \,,$$

or

$$T_1 T_2 \,\mathbf{L}\, T_{n-k} = \frac{p_n}{M q_k} \,.$$

Therefore, for the same reasons as in the proof of Theorem 2, choosing

$$T_1 = T_2 = \mathbf{L} = T_{n-k} = \sqrt[n-k]{\frac{p_n}{M q_k}}$$

provides the smallest closed-loop response time to a step reference.

To illustrate this proposed design procedure we present two applications for a proportional and integrator plant, respectively.

Example 1. Let us consider the proportional plant

$$G_P(s) = \frac{2(2s+1)e^{-10s}}{(3s+1)^2(6s+1)(8s+1)} \,.$$

For $M = 1$, from (9), (2), (7) and (8) we get

$$T_1 = T_2 = T_3 = 6 \,, \quad P(s) = (6s+1)^3 \,,$$

$$C_1(s)=\frac{(3s+1)^2(8s+1)}{2(2s+1)(6s+1)^2},$$

$$C_2(s)=G_0(s)=\frac{e^{-10s}}{(6s+1)^3}.$$

Similarly, for $M=8$, we get

$$T_1=T_2=T_3=3, \quad P(s)=(3s+1)^3,$$

$$C_1(s)=\frac{(3s+1)(6s+1)(8s+1)}{2(2s+1)(3s+1)^2},$$

$$C_2(s)=G_0(s)=\frac{e^{-10s}}{(3s+1)^3}.$$

In Figures 2 and 3 are shown the unit step response $y_P(t)$ of the proportional plant and the unit step responses $c(t)$ and $y(t)$ of the closed-loop system for $M=1$ and $M=8$, respectively (c - control variable, y - controlled variable).

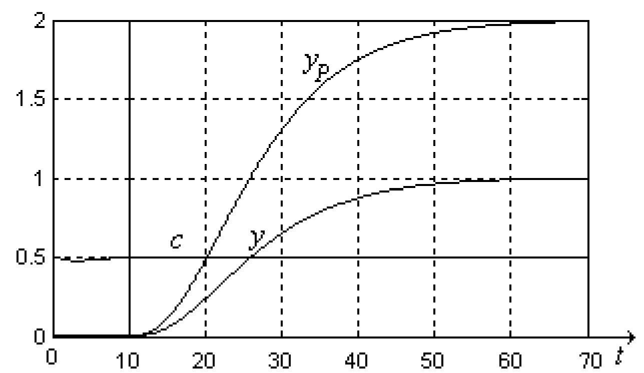

Fig. 2. Unit step responses of the plant (y_P) and closed-loop system (c and y) for $M=1$.

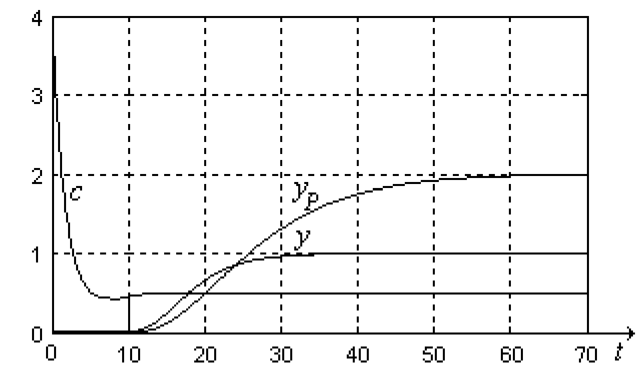

Fig. 3. Unit step responses of the plant (y_P) and closed-loop system (c and y) for $M=8$.

The robustness of the control algorithm in the presence of uncertainty concerning the estimated value of the plant time-delay is illustrated in Figure 4 and 5, where three values of the time-delay in the controller block (8) are considered: $T=8$ (response y_1), $T=10$ (response y) and $T=12$ (response y_2). One can see that the control system designed for a larger value of the magnitude ratio M is a little less robust with respect to the estimated plant time-delay.

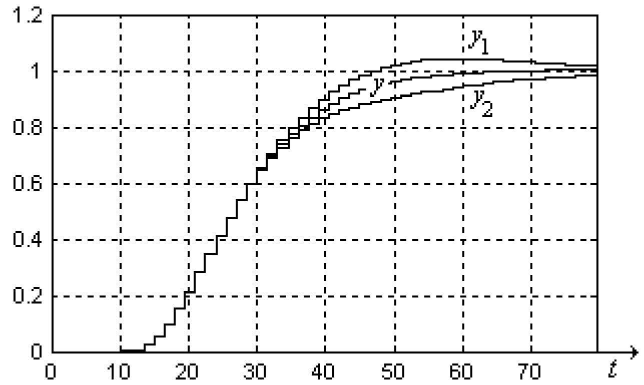

Fig. 4. Unit step responses of the closed-loop system for $M=1$ and three different values of the time-delay in the controller block (8).

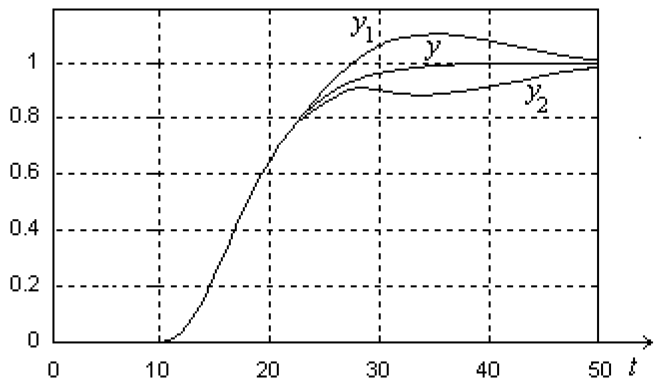

Fig. 5. Unit step responses of the closed-loop system for $M=8$ and three different values of the time-delay in the controller block (8).

Example 2. Let us consider the integrator plant with the transfer function

$$G_P(s)=\frac{(s+1)e^{-10s}}{s(2s+1)(4s+1)(8s+1)}.$$

For $M=1$, from (11), (2), (7) and (8) we get

$$T_1=T_2=T_3=4, \quad P(s)=(4s+1)^3,$$

$$C_1(s)=\frac{s(2s+1)(8s+1)}{(s+1)(4s+1)^2},$$

$$C_2(s)=G_0(s)=\frac{e^{-10s}}{(4s+1)^3}.$$

For $M=8$, we get

$$T_1=T_2=T_3=2, \quad P(s)=(2s+1)^3,$$

$$C_1(s)=\frac{s(4s+1)(8s+1)}{(s+1)(2s+1)^2},$$

$$C_2(s) = G_0(s) = \frac{e^{-10s}}{(2s+1)^3}.$$

In Figures 6 and 7 are shown the unit step response $y_P(t)$ of the integrator plant and the unit step responses $c(t)$ and $y(t)$ of the closed-loop system for $M=1$ and $M=8$.

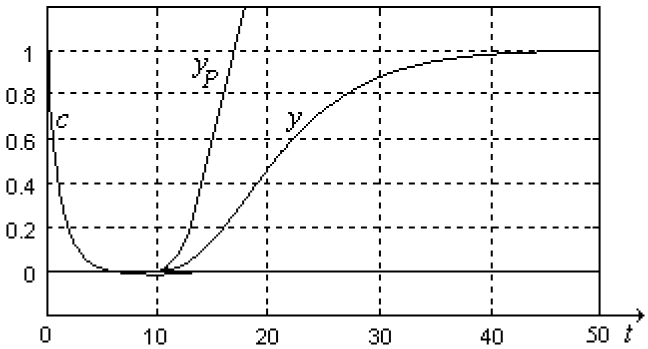

Fig. 6. Unit step responses of the closed-loop system with integrator plant for $M=1$.

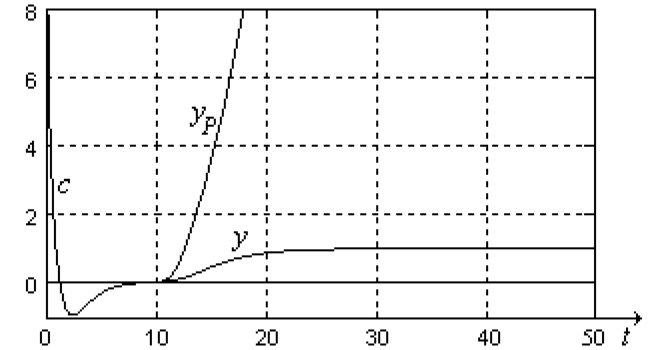

Fig. 7. Unit step responses of the closed-loop system with integrator plant for $M=8$.

3. SECOND DESIGN PROCEDURE

Theorem 4. *Consider a time-delay plant given by*

$$G_P(s) = G(s)e^{-Ts}, \tag{13}$$

where

$$G(s) = \frac{q(s)}{p(s)} = \frac{q_k s^k + q_{k-1}s^{k-1} + \ldots + q_0}{p_n s^n + q_{n-1}s^{n-1} + \ldots + p_0},$$

is a stable, non-derivative and minimum-phase rational function having the polynomials $p(s)$ *and* $r(s)$ *coprime,* $k<n$, $p_n \neq 0$, $q_k \neq 0$.

For any $n-k+1$ *degree polynomial*

$$\mathcal{P}(s) = (T_1 s+1)(T_2 s+1)\ldots(T_{n-k+1}s+1) \tag{14}$$

with all $T_i > 0$, *choosing the semi-proper controller*

$$G_C(s) = \frac{As+1}{G(s)[\mathcal{P}(s) - (As+1)e^{-Ts}]}, \tag{15}$$

where $A = T_1 + T_2 + \ldots + T_{n-k+1}$, *the open-loop system is of simple integrator type and the closed–loop system has the transfer function*

$$G_0(s) = \frac{(As+1)e^{-Ts}}{\mathcal{P}(s)}. \tag{16}$$

Proof. The controller is semi-proper because

$$\lim_{s\to\infty} G_C(s) = \lim_{s\to\infty} \frac{As+1}{G(s)\mathcal{P}(s)} = \frac{q_k A}{p_n T_1 T_2 \ldots T_{n-k+1}} \neq 0.$$

The open-loop system is of simple integrator type, because from its transfer function

$$G_d(s) = G_C(s)G_P(s) = \frac{(As+1)e^{-Ts}}{\mathcal{P}(s) - (As+1)e^{-Ts}}, \tag{17}$$

we get

$$\lim_{s\to 0} sG_d(s) = \frac{1}{T}.$$

The closed-loop system has the transfer function

$$G_0(s) = \frac{G_d}{1+G_d} = \frac{(As+1)e^{-Ts}}{\mathcal{P}(s)}.$$

Remark 4°. In the case of a proportional plant, the controller is of simple integrator type, since

$$\lim_{s\to 0} sG_C(s) = \frac{p_0}{q_0 T}.$$

It is easy to check that the controller becomes of double integrator type if the plant time-delay is zero.

If the plant is of simple integrator type, that is $p_0 = 0$, $p_1 \neq 0$ and $q_0 \neq 0$, then the controller is of proportional type, since

$$\lim_{s\to 0} G_C(s) = \frac{p_1}{q_0 T}.$$

For an integrator plant without time-delay, the controller becomes of simple integrator type.

Remark 5°. To implement the controller (15) one can use the block-scheme in Figure 1, where

$$C_1(s) = \frac{As+1}{G(s)\mathcal{P}(s)}, \tag{18}$$

$$C_2(s) = G_0(s) = \frac{(As+1)e^{-Ts}}{\mathcal{P}(s)}. \tag{19}$$

Remark 6°. We can write the closed-loop system transfer function (16) in the form

$$G_0(s)=\frac{(T_1+T_2+\mathbf{L}\,T_{n-k+1})s+1}{(T_1s+1)(T_2s+1)\,\mathbf{L}\,(T_{n-k+1}s+1)}e^{-Ts}. \qquad (20)$$

The step response of the closed-loop system is non-monotonic, but the overshoot and the response lag time are small if we chose a dominant time constant significantly larger than the other time constants [3]. Choosing

$$T_1>>T_2=T_3=\mathbf{L}=T_{n-k+1}, \qquad (21)$$

we get

$$G_0(s)=\frac{(j+n-k)T_2s+1}{(jT_2s+1)(T_2s+1)^{n-k}}e^{-Ts}, \qquad (22)$$

where $j=T_1/T_2>1$. In this case, the step response overshoot $s_{n-k}(j)$ is a strictly decreasing function satisfying

$$\lim_{j\to\infty} s_{n-k}(j)=0.$$

For a control system with proportional plant and semi-proper controller, the magnitude ratio of the control variable c is given by

$$M=\frac{c(0+)}{c(\infty)}=G(0)\lim_{s\to\infty}\frac{As+1}{P(s)G(s)},$$

that is

$$M=\frac{q_0p_n(T_1+T_2+\mathbf{L}+T_{n-k+1})}{p_0q_kT_1T_2\,\mathbf{L}\,T_{n-k+1}},$$

or

$$M=\frac{(j+n-k)q_0p_n}{jp_0q_kT_2^{\,n-k}}. \qquad (23)$$

Thus, the controller design procedure is as follows:

- determine $j=T_1/T_2$ to have an overshoot less than or equal to the imposed one;

- determine T_2 from (23) to have the imposed value of the magnitude ratio M;

- determine $T_1=jT_2$, $P(s)=(T_1s+1)(T_2s+1)^{n-k}$ and then $G_C(s)$ with (18) and (19), where $A=T_1+(n-k)T_2$.

Remark 7°. Using the proposed procedures one can design a two degree of freedom controller in order to have a large magnitude ratio related to a step disturbance additive to the controlled variable and a smaller magnitude ratio related to a step reference. To do this, we can use a suitable first-order filter for the reference signal.

Example 3. For the proportional plant

$$G_P(s)=\frac{2(3s+1)e^{-10s}}{(2s+1)(6s+1)(8s+1)},$$

we propose to design a controller such that $s\le 5.5\,\%$ and $M=60/7$, where s is the overshoot of the closed-loop system, and M is the magnitude ratio of the control variable to a step reference.
By (24), we have

$$G_0(s)=\frac{(j+2)T_2s+1}{(jT_2s+1)(T_2s+1)^2}e^{-10s}.$$

Since $s=5.46\,\%$ for $j=28$, using this value of j, from (25) we get $T_2=2$. Then,

$$T_1=jT_2=56,\quad P(s)=(56s+1)(2s+1)^2,$$

$$A=T_1+2T_2=60,$$

$$C_1(s)=\frac{(60s+1)(6s+1)(8s+1)}{2(3s+1)(56s+1)(2s+1)},$$

$$C_2(s)=G_0(s)=\frac{(60s+1)e^{-10s}}{(56s+1)(2s+1)^2}.$$

In Figure 8 are shown the closed-loop responses $c(t)$ and $y(t)$ to a unit step reference.

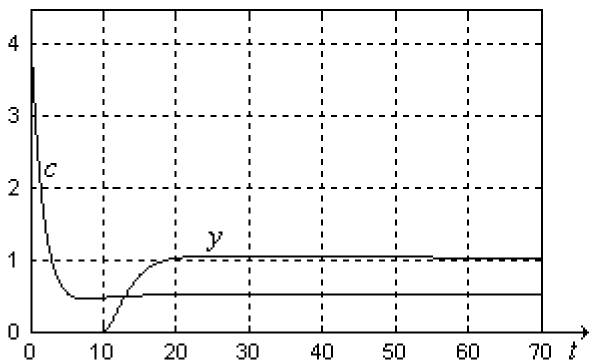

Fig. 8. Unit step responses of the closed-loop system with proportional plant ($s=5.46\,\%$, $M=60/7$).

4. CONCLUSIONS

Two procedures concerning the controller design for a time-delay control system have been proposed.
In accordance to Theorem 1, the designed open-loop system is of simple integrator type, the controller is semi-proper and stable, and the closed-loop system is of monotonic type and has a time-delay equal to the one of the plant, no zeros and negative equal poles. The poles are chosen to have a small response time and a given magnitude ratio to step reference.
By the second design procedure, the designed open-loop system is also of simple integrator type, the controller is semi-proper, and the closed-loop system has a time-delay equal to the one of the plant, one negative zero and negative poles from which one is dominant and the other are equal.

The presented numerical examples for proportional and integrator plants show the effectiveness of the proposed control procedures. As a rule, the control system designed for a larger value of the magnitude ratio has a smaller step response time, but is a little less robust with respect to the estimated plant time-delay.

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