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Delay-Independent Stabilization of Linear Discrete-Time Systems with Uncertain Delay

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Abstract: We propose two methods for testing the delay-independent stability of discrete-time systems with unknown delay. Both have LMI form. The first method is based on a polynomial Lyapunov equation that is equivalent to the stability of a parametric matrix. The second method tests the strict positivity of a bivariate matrix polynomial. Both methods are shown to give better results than previous methods. Also, we extend the polynomial Lyapunov function approach to finding a delay-independent stabilizing static output feedback, using an iterative algorithm in which each iteration consists of solving LMIs. The behavior of the algorithm is illustrated with extensive results.

Keywords: discrete-time systems, uncertain delay, robust stability, Lyapunov function

1. THE PROBLEM

Discrete-time systems with unknown delay have drawn renewed attention as, besides their applications in modeling by discretization of spatially distributed systems, they are appropriate in the study of networked systems that are genuinely digital and of increasing importance.

We consider the discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-\tau) + Bu(k), \\ y(k) &= Cx(k) + C_d x(k-\tau) \end{aligned} \tag{1}$$

with given matrices $A, A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C, C_d \in \mathbb{R}^{p\times n}$. The delay τ has an unknown positive integer value. We aim to solve two problems.

1. Determine if the system (1) is delay-independent stable, i.e. stable $\forall \tau \in \mathbb{Z}^+$.

2. Find the static output feedback $u(k) = Fy(k)$ such that the closed loop system

$$x(k+1) = \tilde{A}x(k) + \tilde{A}_d x(k-\tau), \tag{2}$$

where

$$\tilde{A} = A + BFC, \quad \tilde{A}_d = A_d + BFC_d, \tag{3}$$

is delay-independent stable.

Simple sufficient delay-independent stability conditions were given by Mori et al. (1982), Kaszkurewicz and Bhaya (1993). Chen and Latchman (1995) proposed frequency tests based on the computation of the spectral radius of a frequency dependent matrix. The stability tests that appear to work best have LMI form and were proposed by Mahmoud (2000) (the test is equivalent to the Riccati equations of Verriest and Ivanov (1995)) and Fridman and Shaked (2005). We propose LMIs for testing delay-independent stability, using a polynomial Lyapunov function approach, showing that the approach of Mahmoud (2000) (which appears to give the same numerical results as that of Fridman and Shaked (2005)) cannot give better results even if the degree of the polynomial is zero.

The stabilization problem has been less studied in itself, but rather as a part of other control problems. The particular case of static output feedback (SOF) stabilization appears in He et al. (2008) and few other papers, in the context of time-varying delay, with bounded delays. The delay-independent SOF stabilization method proposed here is significantly simpler and can be used as a first attempt in the stabilization of systems with time-varying delays. In case of failure, one can try more specific and costlier methods.

The paper is structured as follows. Section 2 presents our polynomial Lyapunov function approach to delay-independent stability testing, together with a method based on sum-of-squares polynomials in two variables. Both tests have LMI form. Comparison between them and with previous methods are also given. Section 3 is dedicated to SOF stabilization and section 4 presents experimental results obtained by extensive runs.

2. DELAY-INDEPENDENT STABILITY TESTS

The system (1) is delay-independent stable if and only if

$$\det(zI - A - z^{-\tau}A_d) \neq 0, \quad \forall |z| \geq 1, \ \forall \tau \in \mathbb{Z}^+. \tag{4}$$

Since this condition must hold for all values $\tau \in \mathbb{Z}^+$, we introduce a new independent variable $\zeta = z^{-\tau}$. This idea

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was first proposed by Kamen (1980). Then, condition (4) is equivalent to

$$\det(zI - A - \zeta A_d) \neq 0, \quad \forall |z| \geq 1, \ \forall |\zeta| \leq 1. \tag{5}$$

Otherwise said, the matrix $G(\zeta) = A + \zeta A_d$ must be Schur for all $|\zeta| \leq 1$.

2.1 Polynomial Lyapunov function approach

Since the spectral radius of $G(\zeta)$ is maximum for $|\zeta| = 1$, condition (5) is equivalent, as shown by Bliman (2004), to the existence of a polynomial Lyapunov function $P(\zeta) \succ 0$ such that

$$P(\zeta) - G(\zeta)^H P(\zeta) G(\zeta) \succ 0, \quad \forall |\zeta| = 1. \tag{6}$$

Denoting

$$R(\zeta) = \begin{bmatrix} P(\zeta) & (A^T + A_d^T \zeta^{-1}) P(\zeta^{-1}) \\ P(\zeta)(A + A_d \zeta) & P(\zeta) \end{bmatrix}, \tag{7}$$

this is equivalent to

$$R(\zeta) \succ 0, \quad \forall |\zeta| = 1. \tag{8}$$

In general, the degree of $P(\zeta)$ cannot be determined beforehand. For a practical test we bound the degree of $P(\zeta)$ and thus obtain a sufficient stability condition.

Test 1 (PLF(κ)—polynomial Lyapunov function of degree κ). Taking

$$P(\zeta) = P_0 + \sum_{i=1}^{\kappa} (P_i \zeta^i + P_i^T \zeta^{-i}), \tag{9}$$

the inequality (8) is a positivity condition on the trigonometric polynomial $R(\zeta)$, whose matrix coefficients depend linearly on those of $P(\zeta)$. Hence (8) can be expressed as an LMI. Denoting $R_i \in \mathbb{R}^{2n \times 2n}$, $i = 0 : \kappa + 1$, the coefficients of the polynomial from (7), it results that ($P_{\kappa+1} = 0$, $P_{-1} = P_1^T$)

$$R_i = \begin{bmatrix} P_i & A^T P_i + A_d^T P_{i+1} \\ P_i A_i + P_{i-1} A_d & P_i \end{bmatrix}. \tag{10}$$

The inequality (8) holds if and only if (see Dumitrescu (2007)) there exists a symmetric positive definite matrix $Q \in \mathbb{R}^{2(\kappa+2)n \times 2(\kappa+2)n}$, split in blocks of size $2n \times 2n$ by $Q = [Q_{i\ell}]_{i,\ell=0:\kappa+1}$, such that

$$R_i = \sum_{\ell=i}^{\kappa+1} Q_{\ell,\ell-i}. \tag{11}$$

We say that PLF(κ) holds if such a matrix Q can be found and hence the system is stable. It is clear that if PLF(κ_0) holds for some κ_0, then it holds for all $\kappa \geq \kappa_0$. ■

Example. If $\kappa = 0$, then $P(\zeta) = P_0$ and the test amounts to finding $Q \in \mathbb{R}^{4n \times 4n}$, $Q = [Q_{i\ell}]_{i,\ell=0:1} \succ 0$, such that

$$\begin{aligned} Q_{00} + Q_{11} &= \begin{bmatrix} P_0 & A^T P_0 \\ P_0 A & P_0 \end{bmatrix}, \\ Q_{10} &= \begin{bmatrix} 0 & 0 \\ P_0 A_d & 0 \end{bmatrix}. \end{aligned} \tag{12}$$

Comparison. The test from (Mahmoud, 2000, Rem.1) says that the system (1) is stable if there exist $X, W \succ 0$ such that the following LMI holds

$$\begin{bmatrix} X - A^T X A - W & A^T X A_d \\ A_d^T X A & W - A_d^T X A_d \end{bmatrix} \succ 0. \tag{13}$$

Theorem 1. If (13) holds, then the test PLF(0) holds. ■

Proof. By using Schur complements, the LMI (13) is equivalent to

$$\begin{bmatrix} X - W & 0 & A^T X \\ 0 & W & A_d^T X \\ XA & XA_d & X \end{bmatrix} \succ 0. \tag{14}$$

Splitting $Q_{22} = \begin{bmatrix} V & Y^T \\ Y & Z \end{bmatrix}$, the matrix Q satisfying (12) has the form

$$Q = \begin{bmatrix} P_0 - V & A^T P_0 - Y^T & 0 & A_d^T P_0 \\ P_0 A - Y & P_0 - Z & 0 & 0 \\ 0 & 0 & V & Y^T \\ P_0 A_d & 0 & Y & Z \end{bmatrix} \succ 0. \tag{15}$$

If (14) holds, by taking $V = 0$, $Y = 0$, $P_0 = X$, $Z = W$, it results (after eliminating the third block row and column) that (15) is a permuted version of (14). Since the matrix from (14) is strictly positive definite, one can find small perturbations $V \succ 0$ and Y such that (15) holds. ■

We conclude that the tests PLF(κ) are more comprehensive than the test of Mahmoud (2000).

2.2 Bivariate sum-of-squares approach

Dividing by z in (4), the condition still holds. Denoting $z^{-1} = z_1$, $z^{-\tau-1} = z_2$, condition (4) becomes equivalent to

$$\det(I - A z_1 - A_d z_2) \neq 0, \quad \forall |z_1| \leq 1, \ \forall |z_2| \leq 1. \tag{16}$$

Denote $H(z_1, z_2) = I - A z_1 - A_d z_2$. Using the DeCarlo-Strintzis stability test for multivariate systems, see Strintzis (1977), condition (16) holds if and only if i) $\det H(z_1, 1) \neq 0$ for $|z_1| \leq 1$, ii) $\det H(1, z_2) \neq 0$ for $|z_2| \leq 1$, iii) $\det H(z_1, z_2) \neq 0$ for all $|z_1| = 1$, $|z_2| = 1$. Transforming i) and ii) into spectral radius conditions and iii) into checking the positivity of

$$\begin{aligned} R_1(z_1, z_2) &= H(z_1, z_2) H(z_1^{-1}, z_2^{-1})^T \\ &= I + AA^T + A_d A_d^T - A z_1 - A^T z_1^{-1} - A_d z_2 \\ &\quad - A_d^T z_2^{-1} + A A_d^T z_1 z_2^{-1} + A_d A^T z_1^{-1} z_2 \end{aligned} \tag{17}$$

we obtain the following stability test (see a similar treatment of the 2-D Fornasini-Marchesini model in Dumitrescu (2008)).

Test 2 (BSOS—bivariate sum-of-squares) The system (1) is delay-independent stable if $\rho[(I - A)^{-1} A_d] < 1$, $\rho[(I - A_d)^{-1} A] < 1$ and there exists positive definite $Q \in \mathbb{R}^{4n \times 4n}$, split as $Q = [Q_{i\ell}]_{i,\ell=0:3}$, such that

$$\begin{aligned} I + AA^T + A_d A_d^T &= \sum_{i=1}^{4} Q_{i,i} \\ -A &= Q_{10} + Q_{32} \\ -A_d &= Q_{20} + Q_{31} \\ A_d A^T &= Q_{21} \\ 0 &= Q_{30} \end{aligned} \tag{18}$$

Table 1. Number of systems found stable by the studied stability tests

n	Mahmoud (2000)	PLF(0)	PLF(1)	BSOS
2	714	714	734	734
3	637	637	671	671
4	670	670	724	724
5	681	681	731	731
6	702	702	753	753

2.3 Experimental comparison

We have implemented the stability tests described above using the CVX library by Grant and Boyd (2008). For extensive testing, we have generated random systems by giving to A and A_d values of the form $u\Gamma$, where u is a scalar uniformly distributed in [0, 1] and Γ is a matrix whose elements are normally distributed with zero mean and unit variance. Table 1 reports the number of systems found stable by each method, in 1000 systems with n going from 2 to 6. The test of Mahmoud (2000) and PLF(0) have given the same results for each system; also the test of (Fridman and Shaked, 2005, Cor.1) had exactly the same behavior (this test has also LMI form, but we were unable to find a theoretical relation with Mahmoud (2000) and PLF(0)). Similarly, the tests PLF(1) and BSOS produced identical results; these tests always held when PLF(0) held. Of course, we cannot affirm that PLF(1) and BSOS have classified as stable all stable systems, but it is likely that these tests are practically near-necessary.

3. STABILIZATION VIA POLYNOMIAL LYAPUNOV EQUATION

When we seek a stabilizing static output feedback F, the matrix equalities (10) and (18) become nonlinear due to the dependence (3) on F of the system matrices. Hence, one cannot use directly the LMIs that give the tests PLF(κ) and BSOS. However, the polynomial Lyapunov approach can be used for finding a stabilizing F. Using again the Schur complement, relation (8) is equivalent to

$$\begin{bmatrix} P(\zeta) & \tilde{A}^T + \tilde{A}_d^T\zeta^{-1} \\ \tilde{A} + \tilde{A}_d\zeta & U(\zeta) \end{bmatrix} \succ 0, \quad \forall|\zeta| = 1, \qquad P(\zeta)U(\zeta) = I. \tag{19}$$

To simplify the problem, we approximate it by taking $U(\zeta)$ as a symmetric polynomial of the same degree as $P(\zeta)$. For illustration and since it is enough for practical purposes, we take $U(\zeta) = U_0 + U_1\zeta + U_1^T\zeta^{-1}$, i.e. a polynomial of degree 1.

Even so, (19) is not a convex problem, but can be solved by a simple relaxation technique. Assume that $P(\zeta)$ is given, i.e. P_0 and P_1 are known. Then, using a parameterization similar to (12), we can find the polynomial $U(\zeta)$ that best approximates (19) by solving

$$\begin{aligned} \min\ & \|P_0U_0 + P_1U_1^T + P_1^TU_1 - I \quad P_0U_1 + P_1U_0 \quad P_1U_1\|_F \\ \text{s.t.}\ & \begin{bmatrix} P_0 & (A+BFC)^T \\ A+BFC & U_0 \end{bmatrix} = Q_{00} + Q_{11} \\ & \begin{bmatrix} P_1 & 0 \\ A_d + BFC_d & U_1 \end{bmatrix} = Q_{10} \\ & \begin{bmatrix} Q_{00} & Q_{10}^T \\ Q_{10} & Q_{11} \end{bmatrix} \succ 0 \end{aligned} \tag{20}$$

This is an LMI in the variables F, U_0, U_1 and Q. The criterion is the Frobenius norm of the concatenated coefficients of the matrix polynomial $P(\zeta)U(\zeta) - I$, which ideally should be zero. Note that if U_0 and U_1 are given, then the best approximation $P(\zeta)$ can be found by solving the LMI (20).

Based on (20), we propose the following algorithm, named SOF_PLF(κ) (Static Output Feedback based on Polynomial Lyapunov Function of degree κ), with $\kappa \leq 1$. If $\kappa = 0$, then P_1 and U_1 are forced to zero in (20).

Algorithm SOF_PLF(κ).

1. Choose $P_0 \succ 0$, for example randomly. Take $P_1 = 0$. Choose a tolerance ε and a maximum number of iterations N.

2. With the current P_0 and P_1 fixed, solve (20) for U_0 and U_1.

3. With the current U_0 and U_1 fixed, solve (20) for P_0 and P_1.

4. If the criterion is less than ε, stop.

5. Repeat steps 2-4 at most N times.

6. Check if the system (2) is stable for the obtained F.

The algorithm converges, since the criterion decreases after each iteration. However, it is not guaranteed that $P(\zeta)U(\zeta) \approx I$. So, there is no guarantee that a stabilizing feedback is obtained, hence the need of step 6. In case of failure, the algorithm can be repeated with a different initialization.

4. EXPERIMENTAL RESULTS

We report here the results obtained with the algorithm proposed in the previous section on two examples. We have taken $\varepsilon = 10^{-3}$, $N = 20$. The initialization was $P_0 = \Gamma\Gamma^T$, with Γ a $n \times n$ matrix with normally distributed random elements. Each example was run with 10000 different initializations.

Example 1. The system is (He et al. (2008))

$$A = \begin{bmatrix} 0.9 & 0.5 \\ 0.8 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.3 & 0 \\ 0.8 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{21}$$

Since $m = 1, p = 2$, the feedback matrix is $F = [f_0\ f_1]$. The two elements of F found by the algorithm SOF_PLF(κ) are represented in Figure 1. It is visible that the "area" of solutions found by SOF_PLF(1) is larger and includes that given by SOF_PLF(0). This appears to be a relatively easy problem, since a stabilizing feedback was found for about 81.4% of initializations for SOF_PLF(0) and 80% of initializations for SOF_PLF(1). We may assume that there are systems for which SOF_PLF(1) is able to find a solution, while SOF_PLF(0) is not.

We note that in He et al. (2008) the matrix $F = [-0.3170\ -0.1519]$ was designed in a variable delay case; stability is reported for delays less than 12; in fact, even with this feedback matrix, the system is delay-independent

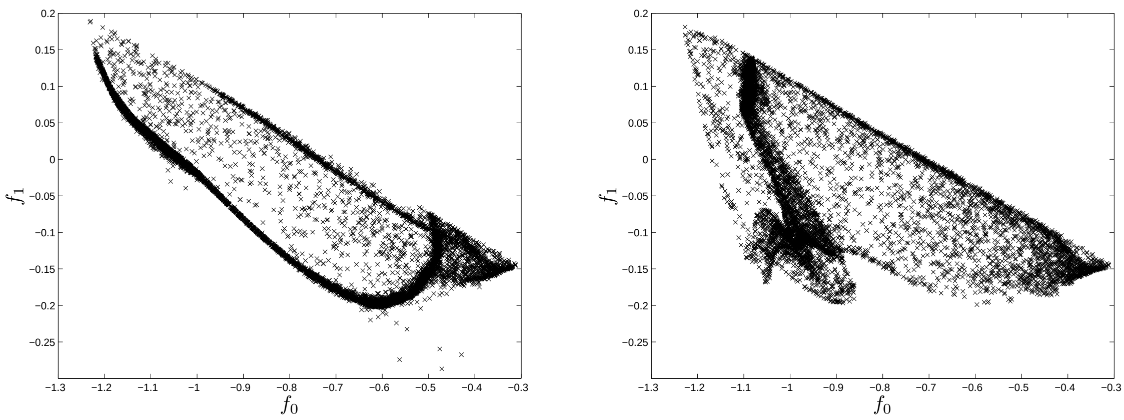

Fig. 1. Stabilizing feedback matrices found for Example 1, by SOF_PLF(0) (left) and SOF_PLF(1) (right).

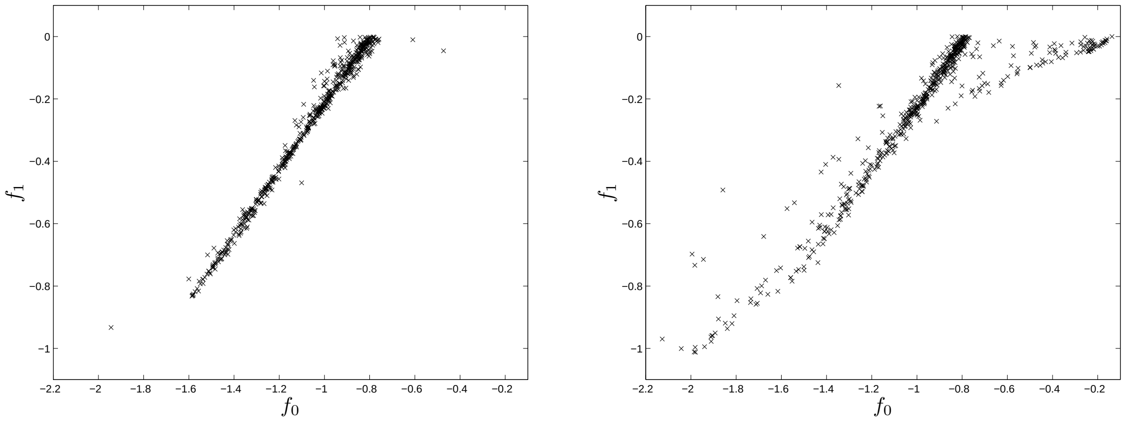

Fig. 2. Stabilizing feedback matrices found for Example 2, by SOF_PLF(0) (left) and SOF_PLF(1) (right).

stable. The position of this F in Figure 1 is at the right extremity of the "good" area.

Example 2. The system is

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{22}$$

So, in this case, the feedback affects only the matrix A. The obtained solutions are represented in Figure 2. Both SOF_PLF(0) and SOF_PLF(1) give a solution only for about 5.7% of the initializations. Again, SOF_PLF(1) give solutions covering a larger area than SOF_PLF(0).

5. CONCLUSIONS

We have proposed a polynomial Lyapunov function approach for testing the stability of a discrete-time system with unknown delay (test PLF(κ)) and extended it for computing a stabilizing static output feedback (algorithm SOF_PLF(κ)). Experimental tests have shown that, increasing the order κ of the polynomial Lyapunov function from 0 (constant matrix) to 1 (true trigonometric polynomial with matrix coefficients), the results are clearly better, in the sense that i) more stable systems are found by PLF(1) than by PLF(0) and ii) for a given system, more stabilizing feedback matrices are found by SOF_PLF(1) than by SOF_PLF(0). We have also shown that the test PLF(0) (which is more conservative than PLF(κ), for $\kappa > 0$) is less conservative than the test proposed by Mahmoud (2000).

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