Three Lectures on Neutral Functional Differential Equations ⋆

Vladimir Răsvan *

* *Department of Automatic Control, University of Craiova, A.I.Cuza, 13 Craiova, RO-200585 Romania (e-mail: vrasvan@automation.ucv.ro).*

Abstract: The main idea of this cycle is that mixed initial boundary value problems for partial differential equations of hyperbolic type in two dimensions modeling lossless propagation are a valuable source of functional differential equations, in particular of neutral type. Starting from the simplest examples there are discussed such topics as basic theory (including various explanations for "what could actually define a neutral equation"), stability and forced oscillations. Rather than giving strictly rigorous proofs, the good motivations and final results are given priority. It is author's strong belief that well formulated applied problems are able to supply interesting, appealing while not always easy to solve problems.

Keywords: Neutral functional differential equations, Propagation systems, Qualitative theory

3. LECTURE THREE. FOR AND AGAINST LIAPUNOV FUNCTION(AL)S

The title of this lecture is an adaptation of the title of a paper of A. Halanay published some 35 years ago - Halanay (1971). At that time the equivalence between the "good" properties of a certain Liapunov function and a certain frequency domain inequality due to V. M. Popov was a quite "young" result and the application of both methods to such problems as absolute stability but especially forced oscillations in nonlinear systems with sector restricted nonlinearities was still establishing itself. On the other hand the case of time delay and distributed parameter systems again with sector restricted nonlinearities was tackled almost exclusively *via* the frequency domain inequality for at least three reasons: a) the use of the Liapunov Krasovskii functionals was not very popular (perhaps due to the fact that the LMI technique was not very "handy" since the corresponding software had not yet been elaborated); b) a Liapunov function(al) had to be "guessed" while a frequency domain inequality could be obtained in a more systematic way; c) as pointed out by several experts of the field, even in the distributed parameter case, the state space might be infinite dimensional but the number of the input/output terminals was necessarily finite.

In this context the cited paper Halanay (1971) was not a polemic one but it just tried to mention the competition of the two approaches and to advise which of them was more suitable for one problem or another.

The situation is now changed. Let us mention some of the achievements of the third of century that passed since then. From the theoretical point of view there exists now the above mentioned equivalence within an abstract framework; the counterpart is that there is little experience in applying it to e.g. time delay systems. There exists a large experience of using Liapunov Krasovskii functionals but they have not the most general form prescribed by the above mentioned equivalence.

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The third fact is that with the advent of the LMI powerful computation toolbox, certain roles of the Liapunov function(al)s and frequency domain inequalities are interchanged: if previously the fulfilment of a certain frequency domain inequality would mean feasibility of some LMI, now it is more likely to associate a LMI to some frequency domain inequality and to check the fulfilment of this inequality *via* the LMI feasibility. In the following we shall be discussing some applications of time/frequency domain methods i.e. Liapunov Krasovskii functionals/frequency domain approaches.

3.1 Linear systems stability

Our basic system will be again system (1) below

$$\begin{aligned}\dot{x}_1 &= A_0x_1(t) + A_1x_2(t-\tau)\\ x_2(t) &= A_2x_1(t) + A_3x_2(t-\tau)\end{aligned} \qquad (1)$$

For this system we have already mentioned - Răsvan (2009a) the possibilities and the drawbacks of the complex domain approach *via* the characteristic equation. Based on a second order example we were able to introduce delay independent and delay dependent stability. Worth mentioning that these notions have been discussed firstly within the time domain approach based on Liapunov Krasovskii functionals. We think interesting to give here a brief historical sketch of the approach. It seems that the first very simple examples of quadratic Liapunov functionals for linear time delay systems were given in the book of Krasovskii (1959). In the neutral case we know the conference paper of Infante (1971) which was concerned with the circuit incorporating a LC line and a nonlinear circuit device - the tunnel diode. On the other hand the quadratic Liapunov Krasovskii functional in a rather general form occurred when estimating a certain quadratic integral index or for the state feedback control synthesis in order to minimize a quadratic cost functional. In order to make the things more clear let us remember that in the finite dimensional case the exponential stability conditions for $\dot{x} = Ax$ are equivalent to the existence of a solution to the symmetric Liapunov matrix equation

$$A^TP + PA = -I$$

the matrix $P > 0$ thus defining a positive definite quadratic Liapunov function $V(x) = x^T Px$ whose derivative along system's solutions is $W(x) = -|x|^2$. This is nothing more than the perfect equivalence of the two approaches - time domain and complex (frequency) domain. This result has a semi-group counterpart due to Datko (1970). But from the abstract scheme to the specific applications there are several steps to be taken. In the retarded case early results are due to Repin (1965) who clearly showed that the Liapunov operator equation turned to be a coupled system of algebraic equations, ODE and PDE. The line has been followed by Infante and Castelan (1978) but the application remained quite difficult. The advancement of the LMI simplified the job but the stability criteria remained very "conservative" i.e. containing sufficient conditions being quite far from the necessary ones. The more recent results of Kharitonov and Zhabko (2003) have partially relaxed this "conservativeness".

It is worth insisting here on the motivation of the results of V. L. Kharitonov as seen in a classical control system context. A control system with constant reference signal is described from the point of view of the control error by the following system in deviations

$$\dot{x} = Ax \ , \ \varepsilon = c^T x \tag{2}$$

For this system, assumed exponentially stable, the control quality is measured e.g by the following Integral of the Square Error (ISE) criterion

$$ISE = \int_0^\infty \varepsilon^2(t)dt = \int_0^\infty x^T(t)cc^T x(t)dt \tag{3}$$

which can be computed by solving a Liapunov matrix equation

$$A^T P + PA = -cc^T$$

for which exponential stability is assumed *a priori* and the RHS is given. The solution $P > 0$ exists provided (c^T, A) is observable (an this is the case in practice). Sometimes ISE is replaced by another integral, a quadratic form of the error and its derivatives up to $n-1$, where n is the order of the system (these new integral criteria were proposed by A. A. Fel'dbaum some 50 years ago). Or the construction of Kharitonov and Zhabko (2003) starts exactly from an exponentially stable time delay system with an *a priori* assumed derivative of the quadratic Liapunov functional along system's solutions.

For neutral FDE as well as for system (1) there exist several extensions of the Liapunov approach. More precisely it is again the LMI technique ensuring sufficient conditions of stability. Following Niculescu (2001) we find that the most popular Liapunov functional candidate for (1) is

$$V(x,\phi) = x^* Px + \int_{-\tau}^{0} \phi(\theta)^* S\phi(\theta)d\theta \tag{4}$$

with $P > 0$ and $S > 0$ being constant matrices. The LMI for the derivative function reads as

$$\begin{pmatrix} A_0^*P + PA_0 + A_2^*SA_2 & PA_1 + A_2^*SA_3 \\ A_1^*P + A_3^*SA_2 & A_3^*SA_3 - S \end{pmatrix} < 0 \tag{5}$$

which may lead to a degenerate Liapunov functional since $A_2^T SA_2 \geq 0$ even if $S > 0$; therefore $P \geq 0$ if A_0 is a Hurwitz matrix unless an additional observability assumption is made; if not this may be still handled provided the approach of Aizerman and Gantmakher (1963) based on "degenerate" Liapunov functions is used together with some structural properties of the system. Let us remark that the assumption for A_0 to be a Hurwitz matrix clearly sends to the delay independent stability case.

3.2 Stability and forced oscillations of systems with sector restricted nonlinearities

The mathematical object of this section will be the nonlinear system

$$\begin{cases} \dot{x}^1(t) = A_0x^1(t) + A_1x^2(t-\tau) - b_1\varphi(c_o^*x^1(t)) + f^1(t) \\ x^2(t) = A_2x^1(t) + A_3x^2(t-\tau) - b_2\varphi(c_o^*x^1(t)) + f^2(t) \end{cases} \tag{6}$$

with $\varphi(\sigma)$ satisfying a sector restriction of the form

$$\underline{\varphi} \leq \frac{\varphi(\sigma)}{\sigma} \leq \overline{\varphi} \tag{7}$$

with the inequalities being possibly strict. For this system two problems with engineering significance will be considered

3.2.1 *A stability problem - the problem of the absolute stability*. This problem is stated for the autonomous system (6) i.e. with $f^i(t) \equiv 0$ and is as follows: *find conditions on linear subsystem's coefficients* (A_i, b_i, c_o) *in order that the zero solution of the autonomous system (6) should be globally asymptotically stable for all nonlinear functions restricted to the sector (7).*

It is quite well known that in solving this problem two approaches coexist and compete: the method of the Liapunov functional leading to some Linear Operator Inequalities which in special cases become computer feasible Linear Matrix Inequalities and the method of Popov-like frequency domain inequalities. it is now well established that the two approaches are perfectly equivalent theoretically: in the finite dimensional case this follows from the Yakubovich - Kalman - Popov lemma while in the infinite dimensional case there exist extensions due to D. Wexler, then to Yakubovich and Likhtarnikov further to R. Curtain and her co-workers in the most general case of the Pritchard- Salamon systems (the reader is sent to the survey Răsvan (2002) for more details). From the point of view of the applications the competition still exists; to illustrate this we state both types of such results as follows

Theorem 1. (Răsvan (1973)) Consider system (6) under the following assumptions: i) the linear system (1) is exponentially stable; ii) the nonlinear function φ is subject to the sector condition (7) with $\underline{\varphi} = 0$; iii) $f^i(t) \equiv 0$; iv) there exists some $\beta \geq 0$ such that the Popov-like frequency domain inequality is fulfilled

$$\frac{1}{\overline{\varphi}} + \Re e(1 + \imath\omega\beta)\gamma(\imath\omega) > 0, \ \forall\omega \in \mathbb{R} \tag{8}$$

where $\gamma(s)$ is the transfer function of the linear part of (6) namely

$$\gamma(s) = (c_o^* \ \ 0)H_o(s)^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{9}$$

with $H_o(s)$ defined in Răsvan (2009b). Then system (6) has the zero solution which is globally asymptotically stable for all nonlinear functions satisfying assumption ii).

Concerning the proof of this result which is given in Răsvan (2002) it is worth mentioning that *its prerequisites are exactly exponential stability of the linear part and an appropriate formula of variations of constants for system (6)* with $\varphi(\sigma) \equiv 0$. Remark also the structural resemblance of (6) and the structurally perturbed system from the theory of the stability radii - see Răsvan (2009b). Consequently the transfer function is the same; this fact motivated analysis of the stability radii in the absolute stability context (Halanay and Răsvan (1997), Răsvan (2000)).

If the Liapunov approach is taken, a straightforward result is the following

Theorem 2. Consider system (6) under the assumptions i) - iii) of Theorem 1 and assume that iv') there exist positive definite matrices P and S and some parameter $\beta \geq 0$ in order that the following linear matrix inequality is fulfilled

$$\begin{pmatrix} H_{11} & h_{12} & H_{13} \\ h_{12}^* & \chi_{22} & h_{23}^* \\ H_{13}^* & h_{23} & H_{33} \end{pmatrix} < 0 \,, \tag{10}$$

where we denoted

$$\begin{aligned} H_{11} &= A_0^* P + P A_0 + A_2^* S A_2, \\ h_{12} &= -Pb_1 - A_2^* S b_1 + \frac{1}{2}(\beta A_0^* + I)c_o \\ H_{13} &= PA_1 + A_2^* S A_3 \\ \chi_{22} &= -\frac{1}{2}\beta(c_o^* b_1 + b_1^* c_o) - \overline{\varphi} + b_2^* S b_2 \\ h_{23} &= -A_3^* S b_2,\ H_{33} = A_3^* S A_3 - S \end{aligned} \tag{11}$$

Then the conclusion of Theorem 1 holds.

At this points some comments are necessary and also useful. The first comment concerns feasibility of (10). In the early stage of the absolute stability problem this LMI feasibility was tackled *via* analytical methods. The Yakubovich Kalman Popov lemma reduced this feasibility to some frequency domain inequality of Popov type. Moreover this inequality may provide the most general Liapunov function(al) of the form "quadratic form + integral of the nonlinearity". Using engineer's tools the frequency condition appeared as easier to check (e.g. graphically, as the Nyquist condition in the linear feedback case). With the advancement of the software tools this connection has been reversed: now a frequency domain inequality is checked *via* some LMI. Worth mentioning also that sometimes it is easier even to check analytically the conditions required for a wisely chosen Liapunov function(al) than those for a frequency domain inequality (see e.g. Răsvan and Niculescu (2002).

The second comment concerns the Liapunov functional

$$\begin{aligned} V(x^1, x^2(\cdot)) = (x^1)^* P x^1 + \int_{-\tau}^{0} x^2(\theta)^* S x^2(\theta) d\theta + \\ +\beta \int_0^{c_o^* x^1} \varphi(\sigma) d\sigma \end{aligned} \tag{12}$$

obtained from (4) by adding the integral of the nonlinear function. This is not the most general Liapunov functional that may be associated to our problem; consequently the frequency domain inequality associated to it *via* YKP lemma is stronger than the general condition (8): its fulfillment implies fulfillment of (8).

The third comment concerns usefulness of the two approaches for two kinds of problems: as a proof tool and as a computational tool. At present the method of Liapunov - *via* the LMI techniques - is better suited for computation. The frequency domain inequality may give most general conditions in basic theorems but there are still areas (e.g. instability) where the method of Liapunov appears as better suited even as a proof tool. Concerning this dialectics of the "for and against Liapunov functions" see Halanay (1971), Răsvan (2002). Further this aspect will be again present.

3.2.1 *A problem of forced oscillations; the almost linear behavior*. This problem is stated for the "complete" (forced) system (6) and from the engineering point of view it is a problem of signal processing: if $f^i(t) \equiv const$ we have sources of constant signals, if $f^i(t)$ are periodic this corresponds to a.c. sources in electrical engineering and if $f^i(t)$ are almost periodic this corresponds to modulated signals. It is but well known that in the linear case the system displays a steady state i.e. a solution defined on the whole real axis $\mathbb{R}$ and this steady state is of the same type as the input signal (constant, periodic or almost periodic); moreover, if the autonomous system is exponentially stable this steady state is exponentially stable. The following theorem will display *a similar behavior for system* (6)

Theorem 3. (Halanay and Răsvan (1977)) Consider system (6) under the following assumptions: i) the linear system (1) is exponentially stable; ii) the nonlinear function φ is globally Lipschitz i.e.

$$0 \leq \frac{\varphi(\sigma_1) - \varphi(\sigma_2)}{\sigma_1 - \sigma_2} \leq L \ , \sigma_1 \neq \sigma_2 \tag{13}$$

iii) $|f^i(t)| \leq M$; the Popov like frequency domain inequality (8) holds for $\beta = 0$ i.e. the circle-like frequency domain inequality is valid

$$\frac{1}{L} + \Re e\ \gamma(\imath\omega) > 0 \ , \ \forall \omega \in \mathbb{R}_+ \tag{14}$$

where $\gamma(s)$ is the same as in Theorem 1. Then system (6) has a unique bounded on $\mathbb{R}$ solution which is exponentially stable; if f^i are constant, T-periodic or almost periodic then this solution is also constant, T-periodic or almost periodic respectively.

Theorem 4. (Răsvan and Niculescu (2002)) Assume that i)-iii) of Theorem 3 hold and, additionally, there exist positive definite matrices P and S in order that LMI (10) holds with $\beta = 0$ and $\overline{\varphi} = L$ i.e. with h_{12} and χ_{22} as follows

$$\begin{aligned} h_{12} &= -Pb_1 - A_2^* S b_1 + \frac{1}{2} c_o \\ \chi_{22} &= -L + b_2^* S b_2 \end{aligned} \tag{15}$$

Then the conclusion of Theorem 3 follows.

Summarizing the results contained in the theorems of this section we obtained the following: under the above assumptions system (6) has in the autonomous case a unique equilibrium which is globally asymptotically stable and, if forced by a constant, periodic or almost periodic exogeneous signal, displays a globally exponentially stable steady state which is of the type of that signal i.e. constant, periodic or almost periodic respectively. This is what we call *almost linear behavior*(Barbălat and Halanay (1974), Răsvan (2001)).

3.3 Self-sustained oscillations in the sense of Yakubovich. Instability and dissipativity

The object of this section will be again the autonomous (with $f^i(t) \equiv 0$) system (6). Starting from the idea that globally stable periodic solutions are a very special and very seldom phenomenon, V. A. Yakubovich and his co-workers introduced a special kind of relaxed oscillatory behavior for which existence results may be obtained easier than for the periodic one Gueorguevskii *et al* (1972), Yakubovich (1973), Yakubovich (1977), Tomberg and Yakubovich (1989).

A solution of the autonomous system (6) is called $[-\alpha,\beta]$-*oscillation* with respect to the output $v(t)$ for $t \to \infty$ if the solution is bounded for $t > 0$ and the output - a linear functional on system's state space - has the following properties: i) it changes sign infinitely many times for $t > 0$; ii) it belongs infinitely many times either to $(-\infty,-\alpha)$ and $[-\alpha,\beta]$ or to $[-\alpha,\beta]$ and $(\beta,+\infty)$ as $t \to \infty$. If additionally there exists some $T > 0$ such that the time of confinement of the output in any interval is not larger than T, the oscillation is called *non-dilating*. If the above properties hold for $t \to -\infty$ the solution is called $[-\alpha,\beta]$-*oscillation* with respect to the output $v(t)$ for $t \to -\infty$; such a solution also may be *non-dilating*. Let $\mathscr{M}$ be a set of $[-\alpha,\beta]$-oscillations: if there exists some t_* such that for all $t > t_*$ $(t < t_*)$ the non-dilation property holds for the same T, then $\mathscr{M}$ is called *the set of uniformly non-dilating* $[-\alpha,\beta]$-*oscillations*. Any $[-\alpha,\beta]$-oscillation for some $\alpha \geq 0$, $\beta \geq 0$, $\alpha+\beta > 0$ is called *oscillation* or *oscillatory solution*. An oscillation is called *bilateral* if it is an oscillation both for $t \to +\infty$ and $t \to -\infty$ while for different pairs α,β. A system whose almost all solutions are oscillations is called *oscillatory*.

Since the problem for time delay and propagation systems is still open, it is worth pointing out the pre-requisites and the main features that may be useful. As in the case of the stable limit cycles in the state plane (in fact the Poincaré-Bendixson theorem ensures that any Yakubovich oscillation in dimension 2 approaches asymptotically or it is a limit cycle itself), if there is a unique equilibrium at the origin, this equilibrium has to be *exponentially unstable*. The instability result is obtained in a standard way if Liapunov function(al)s are to be used Yakubovich (1970); its counterpart in the frequency domain inequalities framework may be found in Popov (1974) where an exponential dichotomy result is obtained for systems described by integral equations. Less obvious is how to recognize an unstable system (for the stable one it is quite clear throughout the paper).

Next, as in the case of the stable limit cycles in the state plane, the large deviations have to enter eventually some bounded domain containing the equilibrium. This is the so-called *dissipativity* (in the sense of N. Levinson) and it can be established also *via* the method of Liapunov (Yoshizawa (1966), Yoshizawa (1975)); for systems with sector restricted nonlinearities the results are due to Barbălat and Halanay (1971). There is no result concerning dissipativity using the frequency domain inequality without the Liapunov function(al)s at least as proof instruments.

The abstract version of YKP lemma is a serious inhibitory factor in applying the Liapunov method e.g. for system (6); in fact we should construct the entire Hilbert space framework required in this case. The frequency domain inequality reduces to the standard one (which may be obtained also by the integral equations approach) but the Liapunov functional is the most general quadratic form on the chosen Hilbert space e.g. $\mathbb{R}^{n_1} \times L^2(-\tau,0;\mathbb{R}^{n_2})$. Nevertheless if we accept sufficient conditions then commonly used Liapunov Krasovskii functionals may be satisfactory. The specific results in specific applications can give hints on the sharpness of various approaches.

3.4 Conclusions

This cycle was intended to cover in fact two topics. The first one was concerned with the general features of the Functional Differential Equations of neutral type. It was shown that a special class of FDE, in fact a system of coupled delay differential and difference (algebraic) equations, occurs in a natural way when the method of d'Alembert is applied to the Initial Boundary Value Problems for some hyperbolic PDE in the plane (two variables). At their turn these Initial Boundary Value Problems are describing power control systems (both thermal and hydraulic), electrical circuits with LC lines, nuclear reactor dynamics. There were shown the specific features of the FDE of neutral type and the fact that the systems generated by PDE of hyperbolic type meet these features being thus "affiliated" to the class of neutral FDE gives a good motivation for these equations which are obtained in a natural way starting from broad classes of applications.

The second topic dealt with what we finally called to be *linear* and *almost linear* behavior. There were considered linear or nonlinear systems of the type described above (the nonlinear systems incorporating sector restricted nonlinearities) to which there were associated quadratic Liapunov Krasovskii functionals and/or Popov like frequency domain inequalities.These mathematical objects are involved in the analysis of the following problems: basic theory, asymptotic stability,forced and self sustained oscillations (in the sense of Yakubovich). We consider answer to these problems as pre-requisites for the almost linear behavior of the nonlinear systems: a single, globally asymptotically (or exponentially) stable equilibrium and forced oscillatory behavior of the same type (periodic, almost periodic) as the forcing signal, the oscillatory solution being exponentially stable.

It is felt that studying this kind of systems and the above enumerated problems is still rewarding and may contribute also to the development of theoretical instruments (e.g. Liapunov techniques, control synthesis).

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