Mean-square delay-distribution-dependent exponential synchronization of discrete-time Markov jump chaotic neural networks with random delay

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Abstract: This paper investigates the mean square delay-distribution-dependent exponential synchronization problem of Markovian jumping discrete-time chaotic neural networks with random delays. Introduced the probability distribution of the time delay, a random variable that satisfying Bernoulli distribution is formulated to produce a new system which includes the information of the probability distribution. Based on the Lyapunov-Krasovskii functional, the Jensen's inequality theory and linear matrix inequality (LMI) technique, delay-distribution-dependent sufficient criteria are established for the discussed Markovian jumping discrete-time chaotic neural networks with random delays to be exponentially synchronized in the mean square. The derived criteria are expressed in terms of linear matrix inequalities and are dependent on the sizes as well as probabilities distribution of delays. The feasibility and the effectiveness of the presented synchronization scheme are demonstrated by one example.

Keywords: Discrete-time neural networks, Markovian jump, Linear matrix inequality, Exponential synchronization.

1. INTRODUCTION

Since the pioneering works of Pecora and Carroll (Pecora and Carroll (1990)), the synchronization problem of chaotic systems has attracted the attention of many researchers and has been successively applied in many engineering fields such as secure communications (Wen, S. et al. (2013)), image encryption (Kalpana, M. et al. (2018)) and biomedical engineering (Tang, Y. et al. (2014)). In addition, neural networks as one of the most important dynamical system is ubiquitous in both nature and man-made system. It has been shown that neural networks with or without time-delays may exhibit chaotic behavior, where the time-delays can be classified as constant, time-varying, neutral delay, leakage delay and distributed delay (Gilli, M. (1993); Zou and Nossek (1993); Lu (2002); Lu and He (1993)). Therefore, synchronization analysis for chaotic neural networks with time-delay is an attractive subject of research in recent years and various results have been reported in (Khadra, Liu and Shen (2005); Xing, Peng and Wang (2010); Zhang, Lv and Li (2017); Abdurahman, Jiang and Teng (2016); Zhang, Shen and Wang (2013); Zhang, Lv and Li (2017); Cheng and Peng (2016); Pratap, et al. (2018); Yang, Cao and Qiu (2015); Liu, Yang and Chen (2011); Hu, et al. (2018). For instance, in (Xing, Peng and Wang (2010)), global exponential synchronization of a class of time-varying delayed chaotic neural networks is investigated based on M-matrix theory. In Zhang, Lv and Li (2017), the lag synchronization of chaotic neural networks with time-delay is discussed by employing the impulsive control theory. The mean square exponential synchronization problem of a class of stochastic neutral type chaotic neural networks with mixed delay is studied by using stochastic analysis and inequality technique in Liu, Yang and Chen (2011).

On the other hand, neural networks with Markov jump is a special class of hybird system, which can cause abrupt changes in their parameters or structures due to some phenomenon such as random failures of the components and sudden environment. Recently, dynamics analysis and synchronization problem of neural networks with Markov jump and time-delay have stirred initial research interests (Ma and Zheng (2018); Senthilraj, et al. (2016); Rakkiyappan, et al. (2014); Ma and Zheng (2015); Tong, et al. (2015)). In practice, the time delay in some neural networks often exists in a random form (Liu, Wang and Liu (2008)), and its probabilities can be measured by the statistical methods such as Bernoulli distribution, normal distribution, uniform distribution and Poisson distribution. It is uncomplicated to see that disturbances invariably exist, which can lead to instability and poor performances always real physical systems (Rakkiyappan, et al. (2014); Nagamani and Ramasamy (2016)). Consequently, how to cut down the effect of disturbances in the synchronization process for chaotic systems has become a significant issue. It's worth noting that the important problem of the mean square delay-distribution-dependent exponential synchronization of discrete-time Markov jump chaotic neural networks with random delays has not been completely considered, so this situation motivates to our present study.

As discussed above, this paper have mainly studied the mean-square delay-distribution-dependent exponential synchronization for a class of Markovian jumping discrete-time chaotic neural networks with random delays. The main contributions of this paper as follows: (i) By introducing a stochastic variable which satisfies Bernoulli distribution, the information of probabilistic time delay is equivalently transformed into the deterministic time delay with stochastic parameters. (ii) A suitable Lyapunov-Krasovskii functional is constructed with the full information of probabilistic time. (iii) By employing the Jensen's inequality theory, several delay-distribution-dependent sufficient conditions have been derived in terms of simplified LMI. (iv) one examples is given to illustrate the feasibility and the effectiveness of the theoretical results. The remainder of this paper is organized as follows. In Section 2, the master system and the slave system are introduced, some necessary assumptions, definition and lemmas are given. Our main results and their rigorous proofs are described in Section 3. In Section 4, one numerical simulation are given to illustrate the effectiveness of our results. In Section 5, conclusions are given.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n \times m$ real matrices. The superscript "T" denotes the transpose of matrix or vector. I denotes the identity matrix with compatible dimensions, "$*$" denotes the symmetric parts, and $\|\cdot\|$ refers to the Euclidean vector norm in $\mathbb{R}^n$. By $A > 0$, we mean that A is a real symmetric positive-definite matrix. If $\mathscr{A}$ is a symmetric matrix, $\lambda_{max}(\mathscr{A})$ or $\lambda_{min}(\mathscr{A})$ denotes the maximum eigenvalue of matrix $\mathscr{A}$ or the minimum eigenvalue of matrix $\mathscr{A}$. Moreover, we may fix a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, where $\mathscr{P}$ is the probability measure, has total mass 1. Finally, $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation operator with respect to the given probability measure $\mathscr{P}$.

2. SYSTEM DESCRIPTION AND PRELIMINARIES

Consider the following Markovian jumping discrete-time chaotic neural networks with time delay of the term:

$$\begin{cases} x(k+1) = C(\gamma_k)x(k) + A(\gamma_k)\tilde{f}(x(k)) \\ \qquad\qquad + B(\gamma_k)\tilde{g}(x(k-\tau(k))) + J, \\ x(i) = \varphi_1(i),\ i \in \mathbb{N}\big[-\tau_M, 0\big], \end{cases} \tag{1}$$

where $k \in \mathbb{N}$, $x(k) = \big[x_1(k), x_2(k), \ldots, x_n(k)\big]^{\mathrm{T}} \in \mathbb{R}^n$ is the neural state vector, and $x_j(k)$ is the state of the jth neuron at time k; $C(\gamma_k) = \mathrm{diag}\big(c_1(\gamma_k), c_2(\gamma_k), \ldots, c_n(\gamma_k)\big) \in \mathbb{R}^{n\times n}$ describes the rate which the each neuron will rest its potential to the resting state in solation when disconnected from the networks and external inputs; $A(\gamma_k) \in \mathbb{R}^{n\times n}$ and $B(\gamma_k) \in \mathbb{R}^{n\times n}$ are the connection non-delayed weight matrix and the discretely delayed connection weight matrix, respectively; $J = \big[J_1, J_2, \ldots, J_n\big]^{\mathrm{T}} \in \mathbb{R}^n$ is an external input vector; $\tilde{f}(x(k)) = \Big[\tilde{f}_1(x_1(k)), \tilde{f}_2(x_2(k)), \ldots, \tilde{f}_n(x_n(k))\Big]^{\mathrm{T}} \in \mathbb{R}^n$ and $\tilde{g}(x(k)) = [\tilde{g}_1(x_1(k)), \tilde{g}_2(x_2(k)), \ldots, \tilde{g}_n(x_n(k))]^{\mathrm{T}} \in \mathbb{R}^n$ are the neuron activation functions; discrete time-varying delay $\tau(k)$ satisfies

$$0 < \tau_m \le \tau(k) \le \tau_M,$$

where τ_m and τ_M are known positive integers; the parameter $\{\gamma_k\}$ $\big(k \in \{1, 2, \ldots, K\}, K$ may be finite of infinite $\big)$ is assumed to be a Markov chain taking value in a finite set $\mathcal{S}$={1, 2, ..., s} with transition probabilities

$$\Pr\big\{\gamma_{k+1} = q \mid \gamma_k = p\big\} = \pi_{pq},\ \forall p, q \in \mathcal{S},$$

where $0 \le \pi_{pq} \le 1$ and $\sum\limits_{q=1}^{s} \pi_{pq} = 1$; $\varphi(i)$ is the initial condition.

Assumption 1. (Liu, Wang and Liu (2009)) For any u, $v \in \mathbb{R}$, $u \neq v$, the continuous and bounded activation functions $\tilde{f}_j(\cdot)$ and $\tilde{g}_j(\cdot)$ satisfy

$$F_j^- \le \frac{\tilde{f}_j(u) - \tilde{f}_j(v)}{u - v} \le F_j^+,$$
$$G_j^- \le \frac{\tilde{g}_j(u) - \tilde{g}_j(v)}{u - v} \le G_j^+,\ j = 1,\ 2, \ldots,\ n,$$

where F_j^-, F_j^+, G_j^- and G_j^+ are known constants.

Assumption 2. (Nagamani and Ramasamy (2016)) In what follows, in order to employ the information of probability distribution of the discrete time-varying delay $\tau(k)$, we define two sets and mapping functions by

$$\Omega_1 = \big\{k \mid \tau(k) \in \big[\tau_m, \tau_0\big]\big\},\quad \Omega_2 = \big\{k \mid \tau(k) \in \big(\tau_0, \tau_M\big]\big\}$$

and

$$\tau_1(k) = \begin{cases} \tau(k), & \text{for } k \in \Omega_1 \\ \tilde{\tau}_1, & \text{for } k \in \Omega_2 \end{cases},$$
$$\tau_2(k) = \begin{cases} \tau(k), & \text{for } k \in \Omega_2 \\ \tilde{\tau}_2, & \text{for } k \in \Omega_1 \end{cases}$$

where $\tau_0 \in \big[\tau_m, \tau_M\big]$, $\tilde{\tau}_1 \in \big[\tau_m, \tau_0\big]$ and $\tilde{\tau}_2 \in \big[\tau_0, \tau_M\big]$. Obviously, $\Omega_1 \cup \Omega_2 = \mathbb{R}^+$, $\Omega_1 \cap \Omega_2 = \varnothing$ (empty set). According to the definitions of Ω_1 and Ω_2, it is easy to see that $k \in \Omega_1$ means the event $\tau(k) \in \big[\tau_m, \tau_0\big]$ occurs and $k \in \Omega_2$ means the event $\tau(k) \in \big[\tau_0, \tau_M\big]$ occurs. Therefore the stochastic variable $\theta(k)$ can be defined as

$$\theta(k) = \begin{cases} 1, & \text{for } k \in \Omega_1, \\ 0, & \text{for } k \in \Omega_2. \end{cases}$$

Assumption 3. (Nagamani and Ramasamy (2016)) $\theta(t)$ is a Bernoulli distributed sequence with

$$\mathrm{Prob}\big\{\theta(k) = 1\big\} = \mathbb{E}\big\{\theta(k)\big\} = \theta_0,$$
$$\mathrm{Prob}\big\{\theta(k) = 0\big\} = 1 - \mathbb{E}\big\{\theta(k)\big\} = 1 - \theta_0,$$

where $0 \le \theta_0 \le 1$ is a constant.

Remark 1. According to Assumption 3, we can show that

$$\mathbb{E}\big\{\theta(k) - \theta_0\big\} = 0,\ \mathbb{E}\left\{(\theta(k) - \theta_0)^2\right\} = \theta_0(1 - \theta_0).$$

Then, based on the analysis in Assumption 2 and 3, the Markovian jumping discrete-time chaotic neural network (1) can be rewritten as:

$$\begin{cases} x(k+1) = C(\gamma_k)x(k) + A(\gamma_k)\tilde{f}(x(k)) \\ \qquad\qquad + \theta(k)B(\gamma_k)\tilde{g}(x(k-\tau_1(k))) \\ \qquad\qquad + (1-\theta(k))B(\gamma_k)\tilde{g}(x(k-\tau_2(k))) + J, \\ x(i) = \varphi_1(i),\ i \in \mathbb{N}\big[-\tau_M, 0\big]. \end{cases} \tag{2}$$

The system (2) is considered as a master system, the corresponding controlled slave system is given by

$$\begin{cases} y(k+1) = C(\gamma_k)y(k) + A(\gamma_k)\tilde{f}(y(k)) \\ \qquad + \theta(k)B(\gamma_k)\tilde{g}(y(k-\tau_1(k))) \\ \qquad + (1-\theta(k))B(\gamma_k)\tilde{g}(y(k-\tau_2(k))) \\ \qquad + J + u(k) + \delta(k, e(k), \gamma_k)\omega(k), \\ y(i) = \varphi_2(i), \ i \in \mathbb{N}[-\tau_M, 0], \end{cases} \tag{3}$$

where $y(k) = [y_1(k), y_2(k), \ldots, y_n(k)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector; $\varphi_2(i) \in \mathbb{R}^n$ denotes the initial conditions; $u(k) = [u_1(k), u_2(k), \ldots, u_n(k)]^{\mathrm{T}} \in \mathbb{R}^n$ is the control input; $e(k) = (e_1(k), e_2(k), \ldots, e_n(k))^{\mathrm{T}} \in \mathbb{R}^n$ is the synchronization error vector; $\delta(\cdot,\cdot,\cdot) : \mathbb{N} \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^n$ is the noise intensity function vector; $\omega(k)$ is a scalar Wiener process on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with

$$\mathbb{E}\{\omega(k)\} = 0, \ \mathbb{E}\{\omega^2(k)\} = 1, \ \mathbb{E}\{\omega(i)\omega(j)\} = 0 \ (i \neq j).$$

Assumption 4. The noise intensity function $\delta(\cdot,\cdot,\cdot) : \mathbb{N} \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^n$ satisfies the Lipschitz condition and there exist positive definite matrix Σ_{γ_k} such that

$$\delta^{\mathrm{T}}(k, z, \gamma_k)\delta(k, z, \gamma_k) \leq z^{\mathrm{T}}\Sigma_{\gamma_k} z,$$

for any $z \in \mathbb{R}^n$.

To investigate the synchronization problem for (2) and (3), we define the synchronization error state as $e(k) = y(k) - x(k)$ and the control input $u(k)$ in the slave system is given by

$$u(k) = K(\gamma_k)e(k), \tag{4}$$

where $K(\gamma_k) \in \mathbb{R}^{n\times n}$ is controller gain matrices to be determined. Then, subtracting (2) from (3), which yields the synchronization error dynamical system as follows:

$$\begin{cases} e(k+1) = \big(C_p + K_p\big)e(k) + A_p f(e(k)) \\ \qquad + \theta(k)B_p g(e(k-\tau_1(k))) \\ \qquad + (1-\theta(k))B_p g(e(k-\tau_2(k))) \\ \qquad + \delta(k, e(k), p)\omega(k) \\ \quad = \tilde{e}(k) + \delta(k, e(k), p)\omega(k), \\ e(i) = \varphi_2(i) - \varphi_1(i) \equiv \varphi(i), \ i \in \mathbb{N}[-\tau_M, 0], \end{cases} \tag{5}$$

which is equivalent to

$$\begin{aligned} e(k+1) =& \big(C_p + K_p\big)e(k) + A_p f(e(k)) \\ &+ \theta_0 B_p g(e(k-\tau_1(k))) \\ &+ (1-\theta_0)B_p g(e(k-\tau_2(k))) \\ &+ (\theta(k)-\theta_0)B_p\big[g(e(k-\tau_1(k))) \\ &- g(e(k-\tau_2(k)))\big] + \delta(k, e(k), p)\omega(k) \\ =& \tilde{\eta}(k) + \delta(k, e(k), p)\omega(k), \end{aligned} \tag{6}$$

where $\gamma_k = p, p \in \mathcal{S}, \ f(e(k)) = \tilde{f}(x(k)+e(k)) - \tilde{f}(x(k))$, $g(e(k)) = \tilde{g}(x(k)+e(k)) - \tilde{g}(x(k))$.

To derive the main results, we introduce the following synchronization concept and Lemmas.

Definition 1. (Wang, et al. (2009)) The master system (2) and the slave system (3) are said to be globally exponentially synchronized in the mean square if the error dynamic system (5) is globally exponentially stable in mean square , i.e., there exist constants $\alpha > 0$ and $0 < \beta < 1$ such that

$$\mathbb{E}\left\{\|e(k)\|^2\right\} \leq \alpha\beta^k \sup_{-\tau_M \leq i \leq 0} \mathbb{E}\left\{\|\varphi(i)\|^2\right\}.$$

Lemma 1. (Chen and Fei (2014)) Let $Z \in \mathbb{R}^{n\times n}$ be a symmetric positive semidefinite matrix, two positive integers a and b satisfying $b \geq a$, and a vector function $\omega : [a, b] \longrightarrow \mathbb{R}^n$. Then, the following inequality holds

$$\left(\sum_{i=a}^{b}\omega_i\right)^{\mathrm{T}} Z \left(\sum_{i=a}^{b}\omega_i\right) \leq (b-a+1)\sum_{i=a}^{b}\omega_i^{\mathrm{T}} Z\omega_i.$$

Lemma 2. (Park, Ko and Jeong (2011)) For a given scalar $\alpha \in [0, 1]$, an $n \times n$ matrix $Z > 0$, and two vectors $\zeta_1, \zeta_2 \in \mathbb{R}^n$, define the function $\Theta(\alpha, Z)$ as

$$\Theta(\alpha, Z) = \frac{1}{\alpha}\zeta_1^{\mathrm{T}} Z\zeta_1 + \frac{1}{1-\alpha}\zeta_2^{\mathrm{T}} Z\zeta_2.$$

If there is a matrix $M \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} Z & M \\ M^{\mathrm{T}} & Z \end{bmatrix} > 0,$$

then the following inequality holds

$$\min_{\alpha\in(0,1)} \Theta(\alpha, Z) \geq \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z & M \\ M^{\mathrm{T}} & Z \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

3. MAIN RESULTS

In this section, by utilizing a new Lyapunov-Krasovskii functional, the mean square delay-distribution-dependent exponential stability criteria for the error dynamic system (5) will be derived. Then based on the stability criteria, the criteria of the globally exponentially synchronized in the mean square for neural networks (2) and (3) will be obtained.

Theorem 1. Under Assumptions 1–4, the synchronization error dynamical system (5) is globally exponentially stable in the mean square if there exist positive-definite matrices P_p $(p = 1, 2, \ldots, s)$, $Q_1,\ Q_2,\ Q_3,\ Z_1,\ Z_2,\ S_1,\ S_2,\ R_1,\ R_2 \in \mathbb{R}^{n\times n}$, diagonal positive-definite matrices $U_p,\ \Lambda_p \in \mathbb{R}^{n\times n}$, matrices $M,\ N \in \mathbb{R}^{n\times n}$, controller gain $K_p \in \mathbb{R}^{n\times n}$ and a positive scalar λ such that for any $p \in \mathcal{S}$, the following matrix inequalities hold:

$$\begin{bmatrix} Z_1 & M \\ * & Z_1 \end{bmatrix} > 0, \tag{7}$$

$$\begin{bmatrix} Z_2 & N \\ * & Z_2 \end{bmatrix} > 0, \tag{8}$$

$$\sum_{q=1}^{s}\pi_{pq}P_q + \tau_1^2 Z_1 + \tau_2^2 Z_2 \leq \lambda I, \tag{9}$$

$$\begin{bmatrix} \Xi_{11} & \Xi_{12}^{\mathrm{T}} & \Xi_{13}^{\mathrm{T}} & \Xi_{14}^{\mathrm{T}} \\ * & -\left(\sum_{q=1}^{s}\pi_{pq}P_q\right)^{-1} & 0 & 0 \\ * & * & \Xi_{33} & 0 \\ * & * & * & \Xi_{44} \end{bmatrix} < 0, \tag{10}$$

where $\Xi_{11} = \big(\Omega_{i,j}\big)_{10\times 10}$ with

$\Omega_{1,1} = -P_p + Q_1 + Q_2 + Q_3 + (\tau_1+1)S_1 + (\tau_2+1)S_2 - U_pF_1 - \Lambda_pG_1 + \lambda\Sigma_p,\ \Omega_{1,2} = \Omega_{1,3} = \Omega_{1,4} = \Omega_{1,5} = \Omega_{1,6} = 0,\ \Omega_{1,7} = U_pF_2,\ \Omega_{1,8} = \Lambda_pG_2,\ \Omega_{1,9} = \Omega_{1,10} = 0,$
$\Omega_{2,2} = -Q_1 - Z_1,\ \Omega_{2,3} = Z_1 - M,\ \Omega_{2,4} = Z_2,\ \Omega_{2,5} = \Omega_{2,6} = \Omega_{2,7} = \Omega_{2,8} = \Omega_{2,9} = \Omega_{2,10} = 0,\ \Omega_{3,3} = -S_1 - 2Z_2 + M^{\mathrm{T}} + M,\ \Omega_{3,4} = Z_1 - M,\ \Omega_{3,5} = \Omega_{3,6} = \Omega_{3,7} = \Omega_{3,8} = \Omega_{3,9} = \Omega_{3,10} = 0,\ \Omega_{4,4} = -Q_2 - Z_1 - Z_2,$
$\Omega_{4,5} = Z_2 - N,\ \Omega_{4,6} = Z_2,\ \Omega_{4,7} = \Omega_{4,8} = \Omega_{4,9} = \Omega_{4,10} = 0,\ \Omega_{5,5} = -S_2 - 2Z_2 + N^{\mathrm{T}} + N,\ \Omega_{5,6} = Z_2 - N,\ \Omega_{5,7} = \Omega_{5,8} = \Omega_{5,9} = \Omega_{5,10} = 0,\ \Omega_{6,6} = -Q_3 - Z_2,\ \Omega_{6,7} = \Omega_{6,8} = \Omega_{6,9} = \Omega_{6,10} = 0,\ \Omega_{7,7} = -U_p,$
$\Omega_{7,8} = \Omega_{7,9} = \Omega_{7,10} = 0,\ \Omega_{8,8} = -\Lambda_p + (\tau_1+1)R_1 + (\tau_2+1)R_2,\ \Omega_{8,9} = \Omega_{8,10} = 0,\ \Omega_{9,9} = -R_1,$
$\Omega_{9,10} = 0,\ \Omega_{10,10} = -R_2,\ \tau_1 = \tau_0 - \tau_m,\ \tau_2 = \tau_M - \tau_0,$
$\Xi_{12} = \begin{bmatrix} C_p + K_p & 0 & 0 & 0 & 0 & 0 & A_p & 0 & \theta_0B_p & (1-\theta_0)B_p \end{bmatrix},$
$\Xi_{13} = \begin{bmatrix} \tau_1\Omega & 0 & 0 & 0 & 0 & 0 & \tau_1A_p & 0 & \tau_1\theta_0B_p & \bar{\tau}_1B_p \\ \tau_2\Omega & 0 & 0 & 0 & 0 & 0 & \tau_2A_p & 0 & \tau_2\theta_0B_p & \bar{\tau}_2B_p \end{bmatrix},$
$\Xi_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_1\bar{\theta}_0B_p & -\tau_1\bar{\theta}_0B_p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_2\bar{\theta}_0B_p & -\tau_2\bar{\theta}_0B_p \end{bmatrix},$
$\Xi_{33} = \Xi_{44} = \mathrm{diag}\begin{bmatrix} -Z_1^{-1} & -Z_2^{-1} \end{bmatrix},\ \Omega = C_p + K_p - I,$
$\bar{\tau}_1 = \tau_1(1-\theta_0),\ \bar{\tau}_2 = \tau_2(1-\theta_0),\ \bar{\theta}_0 = \sqrt{\theta_0(1-\theta_0)}.$

Proof. In order to show the stability of error system (5), we construct a new Lyapunov-Krasovskii functional candidate as follows:

$$V(k, e(k), \gamma_k) = \sum_{\kappa=1}^{8} V_\kappa(k, e(k), \gamma_k), \tag{11}$$

where

$$\begin{aligned}
V_1(k, e(k), \gamma_k) =& e^{\mathrm{T}}(k)P(\gamma_k)e(k),\\
V_2(k, e(k), \gamma_k) =& \sum_{l=k-\tau_m}^{k-1} e^{\mathrm{T}}(l)Q_1e(l) + \sum_{l=k-\tau_0}^{k-1} e^{\mathrm{T}}(l)Q_2e(l)\\
&+ \sum_{l=k-\tau_M}^{k-1} e^{\mathrm{T}}(l)Q_3e(l),\\
V_3(k, e(k), \gamma_k) =& (\tau_0 - \tau_m)\sum_{\upsilon=-\tau_0}^{-\tau_m-1}\sum_{l=k+\upsilon}^{k-1} \eta^{\mathrm{T}}(l)Z_1\eta(l),\\
V_4(k, e(k), \gamma_k) =& (\tau_M - \tau_0)\sum_{\upsilon=-\tau_M}^{-\tau_0-1}\sum_{l=k+\upsilon}^{k-1} \eta^{\mathrm{T}}(l)Z_2\eta(l),\\
V_5(k, e(k), \gamma_k) =& \sum_{l=k-\tau_1(k)}^{k-1} e^{\mathrm{T}}(l)S_1e(l)\\
&+ \sum_{\upsilon=-\tau_0+1}^{-\tau_m}\sum_{l=k+\upsilon}^{k-1} e^{\mathrm{T}}(l)S_1e(l),\\
V_6(k, e(k), \gamma_k) =& \sum_{l=k-\tau_2(k)}^{k-1} e^{\mathrm{T}}(l)S_2e(l)\\
&+ \sum_{\upsilon=-\tau_M+1}^{-\tau_0}\sum_{l=k+\upsilon}^{k-1} e^{\mathrm{T}}(l)S_2e(l),\\
V_7(k, e(k), \gamma_k) =& \sum_{l=k-\tau_1(k)}^{k-1} g^{\mathrm{T}}(e(l))R_1g(e(l))\\
&+ \sum_{\upsilon=k-\tau_0+1}^{k-\tau_m}\sum_{l=\upsilon}^{k-1} g^{\mathrm{T}}(e(l))R_1g(e(l)),\\
V_8(k, e(k), \gamma_k) =& \sum_{l=k-\tau_2(k)}^{k-1} g^{\mathrm{T}}(e(l))R_2g(e(l))\\
&+ \sum_{\upsilon=k-\tau_M+1}^{k-\tau_0}\sum_{l=\upsilon}^{k-1} g^{\mathrm{T}}(e(l))R_2g(e(l)),\\
\eta(l) =& e(l+1) - e(l).
\end{aligned}$$

Letting

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\}\\
&= \mathbb{E}\{V(k+1, e(k+1), \gamma_{k+1} \mid e(k), \gamma_k = p) - V(k, e(k), p)\},\\
&\sum_{q=1}^{s} \Pr\{\gamma_{k+1} = q \mid \gamma_k = p\} P_q = \sum_{q=1}^{s} \pi_{pq}P_q.
\end{aligned}$$

Calculating the difference of $V(k, e(k), \gamma_k)$ along the solution of the error system (5), we can get that

$$\begin{aligned}
&\mathbb{E}\{\Delta V_1(k)\}\\
=&\mathbb{E}\left\{e^{\mathrm{T}}(k+1)\sum_{q=1}^{s}\pi_{pq}P_qe(k+1) - e^{\mathrm{T}}(k)P_pe(k)\right\}\\
=&\mathbb{E}\left\{\tilde{e}^{\mathrm{T}}(k)\sum_{q=1}^{s}\pi_{pq}P_q\tilde{e}(k) - e^{\mathrm{T}}(k)P_pe(k)\right\} +\\
&\delta^{\mathrm{T}}(k, e(k), p)\sum_{q=1}^{s}\pi_{pq}P_q\delta(k, e(k), p),
\end{aligned}\tag{12}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_2(k)\}\\
=&\mathbb{E}\{e^{\mathrm{T}}(k)(Q_1 + Q_2 + Q_3)e(k)\\
&- e^{\mathrm{T}}(k-\tau_m)Q_1e(k-\tau_m) - e^{\mathrm{T}}(k-\tau_0)Q_2e(k-\tau_0)\\
&-e^{\mathrm{T}}(k-\tau_M)Q_3e(k-\tau_M)\},
\end{aligned}\tag{13}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_3(k)\}\\
=&\mathbb{E}\Big\{(\tau_0-\tau_m)^2\eta^{\mathrm{T}}(k)Z_1\eta(k)\\
&-(\tau_0-\tau_m)\sum_{l=k-\tau_0}^{k-\tau_m-1}\eta^{\mathrm{T}}(l)Z_1\eta(l)\Big\}\\
=&\mathbb{E}\Big\{(\tau_0-\tau_m)^2(\tilde{\eta}(k)-e(k))^{\mathrm{T}}Z_1(\tilde{\eta}(k)-e(k))\\
&-(\tau_0-\tau_m)\sum_{l=k-\tau_0}^{k-\tau_m-1}\eta^{\mathrm{T}}(l)Z_1\eta(l)\Big\}\\
&+(\tau_0-\tau_m)^2\delta^{\mathrm{T}}(k, e(k), p)Z_1\delta(k, e(k), p),
\end{aligned}\tag{14}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_4(k)\}\\
=&\mathbb{E}\Big\{(\tau_M-\tau_0)^2\eta^{\mathrm{T}}(k)Z_2\eta(k)\\
&-(\tau_M-\tau_0)\sum_{l=k-\tau_M}^{k-\tau_0-1}\eta^{\mathrm{T}}(l)Z_2\eta(l)\Big\}\\
=&\mathbb{E}\Big\{(\tau_M-\tau_0)^2(\tilde{\eta}(k)-e(k))^{\mathrm{T}}Z_2(\tilde{\eta}(k)-e(k))
\end{aligned}$$

$$-(\tau_M-\tau_0)\sum_{l=k-\tau_M}^{k-\tau_0-1}\eta^{\mathrm{T}}(l)Z_2\eta(l)\Bigg\}$$
$$+(\tau_M-\tau_0)^2\delta^{\mathrm{T}}(k,e(k),p)\,Z_2\delta(k,e(k),p), \qquad (15)$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_5(k)\}\\
=&\mathbb{E}\Bigg\{e^{\mathrm{T}}(k)S_1e(k)+\sum_{l=k+1-\tau_1(k+1)}^{k-1}e^{\mathrm{T}}(l)S_1e(l)\\
&-e^{\mathrm{T}}(k-\tau_1(k))S_1e(k-\tau_1(k))+\sum_{l=k+1-\tau_1(k)}^{k-1}e^{\mathrm{T}}(l)S_1e(l)\\
&+\big(\tau_0-\tau_m\big)e^{\mathrm{T}}(k)S_1e(k)-\sum_{l=k+1-\tau_0}^{k-\tau_m}e^{\mathrm{T}}(l)S_1e(l)\Bigg\}\\
\le&\mathbb{E}\left\{\big(\tau_0-\tau_m+1\big)e^{\mathrm{T}}(k)S_1e(k)\right.\\
&\left.-e^{\mathrm{T}}(k-\tau_1(k))S_1e(k-\tau_1(k))\right\}, \qquad (16)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_6(k)\}\\
\le&\mathbb{E}\left\{\big(\tau_M-\tau_0+1\big)e^{\mathrm{T}}(k)S_2e(k)\right.\\
&\left.-e^{\mathrm{T}}(k-\tau_2(k))S_2e(k-\tau_2(k))\right\}, \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_7(k)\}\\
\le&\mathbb{E}\left\{\big(\tau_0-\tau_m+1\big)g^{\mathrm{T}}(e(k))R_1g(e(k))\right.\\
&\left.-g^{\mathrm{T}}(e(k-\tau_1(k)))R_1g(e(k-\tau_1(k)))\right\}, \qquad (18)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_8(k)\}\\
\le&\mathbb{E}\left\{\big(\tau_M-\tau_0+1\big)g^{\mathrm{T}}(e(k))R_2g(e(k))\right.\\
&\left.-g^{\mathrm{T}}(e(k-\tau_2(k)))R_2g(e(k-\tau_2(k)))\right\}, \qquad (19)
\end{aligned}$$

By (6), the first terms in (14) and (15) can be obtained that

$$\begin{aligned}
&\big(\tilde\eta(k)-e(k)\big)^{\mathrm{T}}Z_1\big(\tilde\eta(k)-e(k)\big)\\
=&\xi^{\mathrm{T}}(k)\mathscr{A}_1^{\mathrm{T}}Z_1\mathscr{A}_1\xi(k)+\big(\theta(k)-\theta_0\big)^2\xi^{\mathrm{T}}(k)\mathscr{A}_2^{\mathrm{T}}Z_1\mathscr{A}_2\xi(k)+\\
&2\big(\theta(k)-\theta_0\big)\xi^{\mathrm{T}}(k)\mathscr{A}_1^{\mathrm{T}}Z_1\mathscr{A}_2\xi(k), \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&\big(\tilde\eta(k)-e(k)\big)^{\mathrm{T}}Z_2\big(\tilde\eta(k)-e(k)\big)\\
=&\xi^{\mathrm{T}}(t)\mathscr{A}_1^{\mathrm{T}}Z_2\mathscr{A}_1\xi(t)+\big(\theta(t)-\theta_0\big)^2\xi^{\mathrm{T}}(t)\mathscr{A}_2^{\mathrm{T}}Z_3\mathscr{A}_2\xi(t)+\\
&2\big(\theta(t)-\theta_0\big)\xi^{\mathrm{T}}(t)\mathscr{A}_1^{\mathrm{T}}Z_2\mathscr{A}_2\xi(t), \qquad (21)
\end{aligned}$$

where

$$\begin{aligned}
\xi(k)=\big[&e^{\mathrm{T}}(k),e^{\mathrm{T}}\big(k-\tau_m\big),e^{\mathrm{T}}(k-\tau_1(k)),e^{\mathrm{T}}\big(k-\tau_0\big),\\
&e^{\mathrm{T}}(k-\tau_2(k)),\ e^{\mathrm{T}}\big(t-\tau_M\big),f^{\mathrm{T}}(e(k)),g^{\mathrm{T}}(e(k)),\\
&g^{\mathrm{T}}(e(k-\tau_1(k))),g^{\mathrm{T}}(e(k-\tau_2(k)))\big]^{\mathrm{T}},
\end{aligned}$$
$$\mathscr{A}_1=[\,C_p+K_p-I\ 0\ 0\ 0\ 0\ 0\ A_p\ 0\ \theta_0B_p\ (1-\theta_0)B_p\,],$$
$$\mathscr{A}_2=[\,0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ B_p\ -B_p\,].$$

Using Lemma 1, we deal with the second term in (14)

$$\begin{aligned}
&-(\tau_0-\tau_m)\sum_{l=k-\tau_0}^{k-\tau_m-1}\eta^{\mathrm{T}}(l)Z_1\eta(l)\\
=&-(\tau_0-\tau_m)\sum_{l=k-\tau_0}^{k-\tau_1(k)-1}\eta^{\mathrm{T}}(l)Z_1\eta(l)\\
&-(\tau_0-\tau_m)\sum_{l=k-\tau_1(k)}^{k-\tau_m-1}\eta^{\mathrm{T}}(l)Z_1\eta(l)\\
\le&-\frac{\tau_0-\tau_m}{\tau_0-\tau_1(k)}\left(\sum_{l=k-\tau_0}^{k-\tau_1(k)-1}\eta(l)\right)^{\mathrm{T}}Z_1\left(\sum_{l=k-\tau_0}^{k-\tau_1(k)-1}\eta(l)\right)\\
&-\frac{\tau_0-\tau_m}{\tau_1(k)-\tau_m}\left(\sum_{l=k-\tau_1(k)}^{k-\tau_m-1}\eta(l)\right)^{\mathrm{T}}Z_1\left(\sum_{l=k-\tau_1(k)}^{k-\tau_m-1}\eta(l)\right).
\end{aligned}$$

According to Lemma 2, there exists matrix M such that $\begin{bmatrix}Z_1 & M\\ M^{\mathrm{T}} & Z_1\end{bmatrix}>0$, then we can have

$$\begin{aligned}
&-(\tau_0-\tau_m)\sum_{l=k-\tau_0}^{k-\tau_m-1}\eta^{\mathrm{T}}(l)Z_1\eta(l)\\
\le&\begin{bmatrix}e(k-\tau_m)\\ e(k-\tau_1(k))\\ e(k-\tau_0)\end{bmatrix}^{\mathrm{T}}\Sigma_1\begin{bmatrix}e(k-\tau_m)\\ e(k-\tau_1(k))\\ e(k-\tau_0)\end{bmatrix}, \qquad (22)
\end{aligned}$$

where

$$\Sigma_1=\begin{bmatrix}-Z_1 & Z_1-M & Z_1\\ * & -2Z_1+M^{\mathrm{T}}+M & Z_1-M\\ * & * & -Z_1\end{bmatrix}.$$

Similarly, there exists matrix N such that $\begin{bmatrix}Z_2 & N\\ N^{\mathrm{T}} & Z_2\end{bmatrix}>0$, then one can deal with the second term in (15)

$$\begin{aligned}
&-(\tau_M-\tau_0)\sum_{l=k-\tau_M}^{k-\tau_0-1}\eta^{\mathrm{T}}(l)Z_2\eta(l)\\
\le&\begin{bmatrix}e(k-\tau_0)\\ e(k-\tau_2(k))\\ e(k-\tau_M)\end{bmatrix}^{\mathrm{T}}\Sigma_2\begin{bmatrix}e(k-\tau_0)\\ e(k-\tau_2(k))\\ e(k-\tau_M)\end{bmatrix}, \qquad (23)
\end{aligned}$$

where

$$\Sigma_2=\begin{bmatrix}-Z_2 & Z_2-N & Z_2\\ * & -2Z_2+N^{\mathrm{T}}+N & Z_2-N\\ * & * & -Z_2\end{bmatrix}.$$

Based on Assumption 1, we can acquire the following inequalities:

$$\big(f_j(e_j(t))-F_j^{+}e_j(t)\big)\big(f_j(e_j(t))-F_j^{-}e_j(t)\big)\le 0,$$
$$\big(g_j(e_j(t))-G_j^{+}e_j(t)\big)\big(g_j(e_j(t))-G_j^{-}e_j(t)\big)\le 0,$$

where $j=1,2,\ldots,n$. Then, there exist matrices $U_p=\mathrm{diag}(u_{p1},u_{p2},\ldots,u_{pn})>0$, $\Lambda_p=\mathrm{diag}(\lambda_{p1},\lambda_{p2},\ldots,\lambda_{pn})>0$ for any $p\in\mathcal{S}$ such that

$$\begin{aligned}
&\sum_{j=1}^{n}u_{pj}\begin{bmatrix}e(t)\\ f(e(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}F_j^{+}F_j^{-}\upsilon_j\upsilon_j^{\mathrm{T}} & -\frac{F_j^{+}+F_j^{-}}{2}\upsilon_j\upsilon_j^{\mathrm{T}}\\ -\frac{F_j^{+}+F_j^{-}}{2}\upsilon_j\upsilon_j^{\mathrm{T}} & \upsilon_j\upsilon_j^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}e(t)\\ f(e(t))\end{bmatrix}\\
=&\begin{bmatrix}e(t)\\ f(e(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}U_pF_1 & -U_pF_2\\ -U_pF_2 & U_p\end{bmatrix}\begin{bmatrix}e(t)\\ f(e(t))\end{bmatrix}\le 0, \qquad (24)
\end{aligned}$$

$$\sum_{j=1}^{n}\lambda_{pj}\begin{bmatrix} e(t)\\ g(e(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} G_j^{+}G_j^{-}\upsilon_j\upsilon_j^{\mathrm{T}} & -\frac{G_j^{+}+G_j^{-}}{2}\upsilon_j\upsilon_j^{\mathrm{T}}\\ -\frac{G_j^{+}+G_j^{-}}{2}\upsilon_j\upsilon_j^{\mathrm{T}} & \upsilon_j\upsilon_j^{\mathrm{T}}\end{bmatrix}\begin{bmatrix} e(t)\\ g(e(t))\end{bmatrix}$$
$$=\begin{bmatrix} e(t)\\ g(e(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \Lambda_pG_1 & -\Lambda_pG_2\\ -\Lambda_pG_2 & \Lambda_p\end{bmatrix}\begin{bmatrix} e(t)\\ g(e(t))\end{bmatrix}\le 0, \tag{25}$$

where

$$F_1=\mathrm{diag}\left[F_1^{+}F_1^{-},F_2^{+}F_2^{-},\dots,F_n^{+}F_n^{-}\right],$$
$$F_2=\mathrm{diag}\left[\frac{F_1^{+}+F_1^{-}}{2},\frac{F_2^{+}+F_2^{-}}{2},\dots,\frac{F_n^{+}+F_n^{-}}{2}\right],$$
$$G_1=\mathrm{diag}\left[G_1^{+}G_1^{-},G_2^{+}G_2^{-},\dots,G_n^{+}G_n^{-}\right],$$
$$G_2=\mathrm{diag}\left[\frac{G_1^{+}+G_1^{-}}{2},\frac{G_2^{+}+G_2^{-}}{2},\dots,\frac{G_n^{+}+G_n^{-}}{2}\right],$$

and υ_j denotes the unit column vector having "1" element on its $j-$th row and zeros elsewhere.

On the other hand, it follows from Assumption 4 and the condition (9) that

$$\delta^{\mathrm{T}}\left(k,e(k),p\right)\Sigma_3\delta\left(k,e(k),p\right)\le\lambda e^{\mathrm{T}}(k)\Sigma_pe(k), \tag{26}$$

where

$$\Sigma_3=\sum_{q=1}^{s}\pi_{pq}P_q+\left(\tau_0-\tau_m\right)^2Z_1+\left(\tau_M-\tau_0\right)^2Z_2.$$

Combining (12)-(26), we deduce

$$\mathbb{E}\left\{\Delta V(k)\right\}=\mathbb{E}\left\{\xi^{\mathrm{T}}(k)\Xi\xi(k)\right\}, \tag{27}$$

where

$$\begin{aligned}\Xi=&\Xi_{11}+\Xi_{12}^{\mathrm{T}}\sum_{q=1}^{s}\pi_{pq}P_q\Xi_{12}\\ &+\left(\tau_0-\tau_m\right)^2\mathscr{A}_1^{\mathrm{T}}Z_1\mathscr{A}_1+\left(\tau_M-\tau_0\right)^2\mathscr{A}_1^{\mathrm{T}}Z_2\mathscr{A}_1\\ &+\theta_0\left(1-\theta_0\right)\Big[\left(\tau_0-\tau_m\right)^2\mathscr{A}_2^{\mathrm{T}}Z_1\mathscr{A}_2\\ &+\left(\tau_M-\tau_0\right)^2\mathscr{A}_2^{\mathrm{T}}Z_2\mathscr{A}_2\Big].\end{aligned}$$

Utilizing Schur complement formula and Eq. (10), the following results are yielded:

$$\begin{aligned}\Xi=&\Xi_{11}+\Xi_{12}^{\mathrm{T}}\sum_{q=1}^{s}\pi_{pq}P_q\Xi_{12}\\ &+\left(\tau_0-\tau_m\right)^2\mathscr{A}_1^{\mathrm{T}}Z_1\mathscr{A}_1+\left(\tau_M-\tau_0\right)^2\mathscr{A}_1^{\mathrm{T}}Z_2\mathscr{A}_1\\ &+\theta_0\left(1-\theta_0\right)\Big[\left(\tau_0-\tau_m\right)^2\mathscr{A}_2^{\mathrm{T}}Z_1\mathscr{A}_2\\ &+\left(\tau_M-\tau_0\right)^2\mathscr{A}_2^{\mathrm{T}}Z_2\mathscr{A}_2\Big]<0.\end{aligned}$$

Next, we will investigate the exponential stability in the mean square of error system (5). We can obtain from (7)–(10) that there exists a sufficiently small scalar $\varepsilon>0$ such that

$$\begin{aligned}&\mathbb{E}\left\{\Delta V(k)\right\}\\ =&-\varepsilon\Bigg(\mathbb{E}\left\{\|e(k)\|^2\right\}+\sum_{l=k-\tau_m}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\sum_{l=k-\tau_0}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}+\sum_{l=k-\tau_M}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\sum_{\upsilon=-\tau_0}^{-\tau_m-1}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|\eta(l)\|^2\right\}+\sum_{\upsilon=-\tau_M}^{-\tau_0-1}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|\eta(l)\|^2\right\}\\ &+\sum_{l=k-\tau_1(k)}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}+\sum_{\upsilon=-\tau_0+1}^{-\tau_m}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\sum_{l=k-\tau_2(k)}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}+\sum_{\upsilon=-\tau_M+1}^{-\tau_0}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\sum_{l=k-\tau_1(k)}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}+\sum_{\upsilon=k-\tau_0+1}^{k-\tau_m}\sum_{l=\upsilon}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\\ &+\sum_{l=k-\tau_2(k)}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\\ &+\sum_{\upsilon=k-\tau_M+1}^{k-\tau_0}\sum_{l=\upsilon}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\Bigg).\end{aligned} \tag{28}$$

In addition, we can conclude from the definition of function $V(k,e(k),\gamma_k)$ that

$$\begin{aligned}&\mathbb{E}\left\{V(k)\right\}\\ \le&\lambda_{\max}(P_p)\mathbb{E}\left\{\|e(k)\|^2\right\}\\ &+\lambda_{\max}(Q_1)\sum_{l=k-\tau_m}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\lambda_{\max}(Q_2)\sum_{l=k-\tau_0}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\lambda_{\max}(Q_3)\sum_{l=k-\tau_M}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\left(\tau_0-\tau_m\right)\lambda_{\max}(Z_1)\sum_{\upsilon=-\tau_0}^{-\tau_m-1}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|\eta(l)\|^2\right\}\\ &+\left(\tau_M-\tau_0\right)\lambda_{\max}(Z_2)\sum_{\upsilon=-\tau_M}^{-\tau_0-1}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|\eta(l)\|^2\right\}\\ &+\lambda_{\max}(S_1)\Bigg(\sum_{l=k-\tau_1(k)}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\sum_{\upsilon=-\tau_0+1}^{-\tau_m}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\Bigg)\\ &+\lambda_{\max}(S_2)\Bigg(\sum_{l=k-\tau_2(k)}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\ &+\sum_{\upsilon=-\tau_M+1}^{-\tau_0}\sum_{l=k+\upsilon}^{k-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\Bigg)\\ &+\lambda_{\max}(R_1)\Bigg(\sum_{l=k-\tau_1(k)}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\upsilon=k-\tau_0+1}^{k-\tau_m}\sum_{l=\upsilon}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\Bigg)\\
&+\lambda_{\max}(R_2)\left(\sum_{l=k-\tau_2(k)}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\right.\\
&\left.+\sum_{\upsilon=k-\tau_M+1}^{k-\tau_0}\sum_{l=\upsilon}^{k-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\right). \qquad (29)
\end{aligned}
$$

Meanwhile, for any scalar $\mu > 1$ satisfying

$$
\begin{aligned}
\varepsilon > \frac{\mu-1}{\mu}\big\{&\lambda_{\max}(P_p),\ \lambda_{\max}(Q_1),\ \lambda_{\max}(Q_2),\ \lambda_{\max}(Q_3),\\
&(\tau_0-\tau_m)\lambda_{\max}(Z_1),\ (\tau_M-\tau_0)\lambda_{\max}(Z_2),\\
&\lambda_{\max}(S_1),\ \lambda_{\max}(S_2),\ \lambda_{\max}(R_1),\ \lambda_{\max}(R_2)\big\}.
\end{aligned}
$$

Thus, from (28) and (29), it follows that

$$
\begin{aligned}
&\mathbb{E}\left\{\mu^k V(k)-V(0)\right\}\\
=&\mathbb{E}\left\{\sum_{\kappa=0}^{k-1}\left[\mu^{\kappa+1}V(\kappa+1)-\mu^{\kappa}V(\kappa)\right]\right\}\\
=&\sum_{\kappa=0}^{k-1}\mathbb{E}\left\{\mu^{\kappa+1}\Delta V(\kappa+1)-\mu^{\kappa}V(\kappa)\right\}\\
\le&\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(P_p)\big]\sum_{\kappa=0}^{k-1}\mu^{\kappa}\mathbb{E}\left\{\|e(\kappa)\|^2\right\}\\
&+\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(Q_1)\big]\sum_{\kappa=0}^{k-1}\mu^{\kappa}\sum_{l=\kappa-\tau_m}^{\kappa-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\
&+\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(Q_2)\big]\sum_{\kappa=0}^{k-1}\mu^{\kappa}\sum_{l=\kappa-\tau_0}^{\kappa-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\
&+\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(Q_3)\big]\sum_{\kappa=0}^{k-1}\mu^{\kappa}\sum_{l=\kappa-\tau_M}^{\kappa-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\\
&+\big[-\mu\varepsilon+(\mu-1)(\tau_0-\tau_m)\lambda_{\max}(Z_1)\big]\\
&\times\sum_{\kappa=0}^{k-1}\mu^{\kappa}\sum_{\upsilon=-\tau_0}^{-\tau_m-1}\sum_{l=\kappa+\upsilon}^{\kappa-1}\mathbb{E}\left\{\|\eta(l)\|^2\right\}\\
&+\big[-\mu\varepsilon+(\mu-1)(\tau_M-\tau_0)\lambda_{\max}(Z_2)\big]\\
&\times\sum_{\kappa=0}^{k-1}\mu^{\kappa}\sum_{\upsilon=-\tau_M}^{-\tau_0-1}\sum_{l=\kappa+\upsilon}^{\kappa-1}\mathbb{E}\left\{\|\eta(l)\|^2\right\}\\
&+\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(S_1)\big]\\
&\times\sum_{\kappa=0}^{k-1}\mu^{\kappa}\left(\sum_{l=\kappa-\tau(\kappa)}^{\kappa-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\right.\\
&\left.+\sum_{\upsilon=\kappa-\tau_0+1}^{\kappa-\tau_m}\sum_{l=\upsilon}^{\kappa-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\right)\\
&+\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(S_2)\big]\\
&\times\sum_{\kappa=0}^{k-1}\mu^{\kappa}\left(\sum_{l=\kappa-\tau(\kappa)}^{\kappa-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\right.\\
&\left.+\sum_{\upsilon=\kappa-\tau_M+1}^{\kappa-\tau_0}\sum_{l=\upsilon}^{\kappa-1}\mathbb{E}\left\{\|e(l)\|^2\right\}\right)\\
&+\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(R_1)\big]\\
&\times\sum_{\kappa=0}^{k-1}\mu^{\kappa}\left(\sum_{l=\kappa-\tau(\kappa)}^{\kappa-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\right.\\
&\left.+\sum_{\upsilon=\kappa-\tau_0+1}^{\kappa-\tau_m}\sum_{l=\upsilon}^{\kappa-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\right)\\
&+\big[-\mu\varepsilon+(\mu-1)\lambda_{\max}(R_2)\big]\\
&\times\sum_{\kappa=0}^{k-1}\mu^{\kappa}\left(\sum_{l=\kappa-\tau(\kappa)}^{\kappa-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\right.\\
&\left.+\sum_{\upsilon=\kappa-\tau_M+1}^{\kappa-\tau_0}\sum_{l=\upsilon}^{\kappa-1}\mathbb{E}\left\{\|g(e(l))\|^2\right\}\right)\le 0. \qquad (30)
\end{aligned}
$$

Let

$$
\zeta=\max_{1\le j\le n}\left\{|F_j^-|^2,\ |F_j^+|^2,\ |G_j^-|^2,\ |G_j^+|^2\right\}. \qquad (31)
$$

From (11), (30) and (31), it follows that

$$
\lambda_{\min}(P_p)\mu^k\mathbb{E}\left\{\|e(k)\|^2\right\}\le\mu^k\mathbb{E}\left\{V(k)\right\}\le\mathbb{E}\left\{V(0)\right\}, \qquad (32)
$$

where

$$
\begin{aligned}
&\mathbb{E}\{V(0)\}\\
\le&\{\lambda_{\max}(P_p)+\tau_m\lambda_{\max}(Q_1)+\tau_0\lambda_{\max}(Q_2)\\
&+\tau_M\lambda_{\max}(Q_3)+\left[\frac{(\tau_0-\tau_m)^2(\tau_0+\tau_m+1)}{2}\lambda_{\max}(Z_1)\right.\\
&\left.+\frac{(\tau_M-\tau_0)^2(\tau_M+\tau_0+1)}{2}\lambda_{\max}(Z_2)\right]\mathbb{E}\left\{\Pi^{\mathrm{T}}\Pi\right\}\\
&+\left[\tau_0+(\tau_0-\tau_m)\frac{(\tau_0+\tau_m+1)}{2}\right]\lambda_{\max}(S_1)\\
&+\left[\tau_M+\frac{(\tau_M-\tau_0)(\tau_M+\tau_0+1)}{2}\right]\lambda_{\max}(S_2)\\
&+\left[\tau_0+\frac{(\tau_0-\tau_m)(\tau_0+\tau_m+1)}{2}\right]\zeta\lambda_{\max}(R_1)\\
&+\left.\left[\tau_M+(\tau_M+\tau_0+1)\frac{(\tau_M-\tau_0)}{2}\right]\zeta\lambda_{\max}(R_2)\right\}\\
&\times\sup_{-\tau_M\le i\le 0}\mathbb{E}\left\{\|\varphi(i)\|^2\right\},
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbb{E}\left\{\Pi^{\mathrm{T}}\Pi\right\}\le&\|C_p+K_p-I\|+\zeta\left[\|A_p\|+\|B_p\|\right]\\
&+\lambda_{\max}(\Sigma_p).
\end{aligned}
$$

Thus, it follows from (32) that

$$
\mathbb{E}\left\{\|e(k)\|^2\right\}\le\alpha\beta^k\sup_{-\rho\le i\le 0}\mathbb{E}\left\{\|\varphi(i)\|^2\right\},
$$

where $\alpha=\frac{\gamma}{\lambda_{\min}(P_p)}$, $\beta=\frac{1}{\mu}$ and

$$
\begin{aligned}
\gamma =&\lambda_{\max}(P_p)+\tau_m\lambda_{\max}(Q_1)+\tau_0\lambda_{\max}(Q_2)\\
&+\tau_M\lambda_{\max}(Q_3)+\left[\frac{(\tau_0-\tau_m)^2(\tau_0+\tau_m+1)}{2}\lambda_{\max}(Z_1)\right.\\
&\left.+\frac{(\tau_M-\tau_0)^2(\tau_M+\tau_0+1)}{2}\lambda_{\max}(Z_2)\right]\mathbb{E}\left\{\Pi^{\mathrm{T}}\Pi\right\}\\
&+\left[\tau_0+(\tau_0-\tau_m)\frac{(\tau_0+\tau_m+1)}{2}\right]\lambda_{\max}(S_1)\\
&+\left[\tau_M+\frac{(\tau_M-\tau_0)(\tau_M+\tau_0+1)}{2}\right]\lambda_{\max}(S_2)\\
&+\left[\tau_0+\frac{(\tau_0-\tau_m)(\tau_0+\tau_m+1)}{2}\right]\zeta\lambda_{\max}(R_1)\\
&+\left[\tau_M+(\tau_M+\tau_0+1)\frac{(\tau_M-\tau_0)}{2}\right]\zeta\lambda_{\max}(R_2).
\end{aligned}
$$

Therefore, by Definition 1, it is shown that the synchronization error dynamical system (5) is globally exponentially stable in the mean square. This completes the proof.

On the basis of Theorem 1, we can derive the synchronization criteria in the form of strict LMIs by designing the feedback controller (4) such that the master system (2) and the slave system (3) are globally exponentially synchronized in the mean square.

Theorem 2. Under Assumptions 1–4, the master system (2) and the slave system (3) are globally exponentially synchronized in the mean square if there exist positive-definite matrices P_p $(p=1,2,\dots,s)$, $Q_1,\ Q_2,\ Q_3,\ Z_1,\ Z_2,\ S_1,\ S_2,\ R_1,\ R_2\in\mathbb{R}^{n\times n}$, diagonal positive-definite matrices $U_p,\ \Lambda_p\in\mathbb{R}^{n\times n}$, matrices $M,\ N,\ X,\ Y_p\in\mathbb{R}^{n\times n}$ and a positive scalar λ such that (7), (8), (9) and the following LMIs hold for any $p\in\mathcal{S}$:

$$
\begin{bmatrix}
\Xi_{11} & \overline{\Xi}_{12}^{\mathrm{T}} & \overline{\Xi}_{13}^{\mathrm{T}} & \overline{\Xi}_{14}^{\mathrm{T}}\\
* & \sum_{q=1}^{s}\pi_{pq}P_q-X^{\mathrm{T}}-X & 0 & 0\\
* & * & \overline{\Xi}_{33} & 0\\
* & * & * & \overline{\Xi}_{44}
\end{bmatrix}<0, \tag{33}
$$

where τ_1, τ_2, $\overline{\tau}_1$, $\overline{\tau}_2$, $\overline{\sigma}$ and Ξ_{11} follow the same definitions as those in Theorem 1, and

$$
\begin{aligned}
\overline{\Xi}_{12}&=\left[\,XC_p+Y_p\ 0\ 0\ 0\ 0\ 0\ XA_p\ 0\ \theta_0XB_p\ (1-\theta_0)XB_p\,\right],\\
\overline{\Xi}_{13}&=\begin{bmatrix}\tau_1\ 0\ 0\ 0\ 0\ 0\ \tau_1XA_p\ 0\ \tau_1\theta_0XB_p\ \overline{\tau}_1XB_p\\ \tau_2\ 0\ 0\ 0\ 0\ 0\ \tau_2XA_p\ 0\ \tau_2\theta_0XB_p\ \overline{\tau}_2XB_p\end{bmatrix},\\
\overline{\Xi}_{14}&=\begin{bmatrix}0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \tau_1\bar{\theta}_0XB_p\ -\tau_1\bar{\theta}_0XB_p\\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \tau_2\bar{\theta}_0XB_p\ -\tau_2\bar{\theta}_0XB_p\end{bmatrix},\\
\overline{\Xi}_{33}&=\overline{\Xi}_{44}=\mathrm{diag}\left[Z_1-X^{\mathrm{T}}-X\ \ Z_2-X^{\mathrm{T}}-X\right],\\
\bar{\Omega}&=XC_p+Y_p-X.
\end{aligned}
$$

Moreover, the feedback controller gain matrix can be designed by $K_p=X^{-1}Y_p$.

Proof. Defining the following new matrix variables as

$$Y_p=XK_p,$$

and pre-and, post-multiplying Eq. (10) by

$$
\begin{gathered}
\mathrm{diag}\left\{I,I,I,I,I,I,I,I,I,I,X,X,X,X,X\right\},\\
\mathrm{diag}\left\{I,I,I,I,I,I,I,I,I,I,X^{\mathrm{T}},X^{\mathrm{T}},X^{\mathrm{T}},X^{\mathrm{T}},X^{\mathrm{T}}\right\},
\end{gathered}
$$

respectively, we obtain

$$
\begin{bmatrix}
\Xi_{11} & \overline{\Xi}_{12}^{\mathrm{T}} & \overline{\Xi}_{13}^{\mathrm{T}} & \overline{\Xi}_{14}^{\mathrm{T}}\\
* & X\left(\sum_{q=1}^{s}\pi_{pq}P_q\right)^{-1}X^{\mathrm{T}} & 0 & 0\\
* & * & \overline{\Xi}_{33} & 0\\
* & * & * & \overline{\Xi}_{44}
\end{bmatrix}<0, \tag{34}
$$

Notice that $\sum_{q=1}^{s}\pi_{pq}P_q>0$ and $Z_i>0,\ (i=1,2)$, we can obtain that

$$
\begin{gathered}
\left(\sum_{q=1}^{s}\pi_{pq}P_q-X\right)^{\mathrm{T}}\left(\sum_{q=1}^{s}\pi_{pq}P_q\right)^{-1}\left(\sum_{q=1}^{s}\pi_{pq}P_q-X\right)\geq 0,\\
(Z_i-X)^{\mathrm{T}}Z_i^{-1}(Z_i-X)\geq 0,
\end{gathered}
$$

which imply

$$
\begin{aligned}
-X\left(\sum_{q=1}^{s}\pi_{pq}P_q\right)^{-1}X^{\mathrm{T}}&\leq\sum_{q=1}^{s}\pi_{pq}P_q-X-X^{\mathrm{T}},\\
-XZ_i^{-1}X^{\mathrm{T}}&\leq Z_i-X-X^{\mathrm{T}},
\end{aligned}\tag{35}
$$

from which one can see that if (33) is satisfied, then (34) is also satisfied. Hence, we can obtain from Theorem 1 that the master system (2) and the slave system (3) are globally exponentially synchronized in the mean square, and then the proof is completed.

Remark 2. If there is no stochastic term and $\theta(k)\equiv 1(\theta_0=1)$, the synchronization error system (5) is reduced to the following form

$$
\begin{aligned}
e(k+1)=&(C_p+K_p)e(k)+A_pf(e(k))\\
&+B_pg(e(k-\tau(k)))+\delta(k,e(k),p)\omega(k)\\
=&\tilde{e}(k)+\delta(k,e(k),p)\omega(k).
\end{aligned}\tag{36}
$$

Construct the following Lyapunov-Krasovskii functional for synchronization error system (36) as

$$V(k,e(k),\gamma_k)=\sum_{\kappa=1}^{5}V_\kappa(k,e(k),\gamma_k),$$

where

$$
\begin{aligned}
V_1(k,e(k),\gamma_k)&=e^{\mathrm{T}}(k)P(\gamma_k)e(k),\\
V_2(k,e(k),\gamma_k)&=\sum_{l=k-\tau_m}^{k-1}e^{\mathrm{T}}(l)Q_1e(l)\\
&\quad+\sum_{l=k-\tau_M}^{k-1}e^{\mathrm{T}}(l)Q_2e(l),\\
V_3(k,e(k),\gamma_k)&=(\tau_M-\tau_m)\sum_{\upsilon=-\tau_M}^{-\tau_m-1}\sum_{l=k+\upsilon}^{k-1}\eta^{\mathrm{T}}(l)Z\eta(l),\\
V_4(k,e(k),\gamma_k)&=\sum_{l=k-\tau(k)}^{k-1}e^{\mathrm{T}}(l)Se(l)\\
&\quad+\sum_{\upsilon=-\tau_M+1}^{-\tau_m}\sum_{l=k+\upsilon}^{k-1}e^{\mathrm{T}}(l)Se(l),\\
V_5(k,e(k),\gamma_k)&=\sum_{l=k-\tau(k)}^{k-1}g^{\mathrm{T}}(e(l))Rg(e(l))
\end{aligned}
$$

$$+\sum_{v=k-\tau_M+1}^{k-\tau_m}\sum_{l=v}^{k-1} g^{\mathrm{T}}(e(l))Rg(e(l)),$$
$$\eta(l) = e(l+1) - e(l).$$

By utilizing the similar proof methods of Theorem 1 and Theorem 2, we can acquire the following corollary.

Corollary 1. Under Assumptions 1–4, the master system (2) and the slave system (3) are globally exponentially synchronized in the mean square if there exist positive-definite matrices P_p $(p = 1, 2, \ldots, s)$, Q_1, Q_2, Z, S, $R \in \mathbb{R}^{n\times n}$, diagonal positive-definite matrices U_p, $\Lambda_p \in \mathbb{R}^{n\times n}$, matrices M, X, $Y_p \in \mathbb{R}^{n\times n}$ and a positive scalar λ such that the following LMIs hold for any $p \in \mathcal{S}$:

$$\begin{bmatrix} Z & M \\ * & Z \end{bmatrix} > 0, \tag{37}$$

$$\sum_{q=1}^{s} \pi_{pq}P_q + \tilde{\tau}^2 Z \le \lambda I, \tag{38}$$

$$\begin{bmatrix} \Phi_{11} & \Phi_{12}^{\mathrm{T}} & \Phi_{13}^{\mathrm{T}} \\ * & \sum_{q=1}^{s} \pi_{pq}P_q - X^{\mathrm{T}} - X & 0 \\ * & * & Z - X^{\mathrm{T}} - X \end{bmatrix} < 0, \tag{39}$$

where $\Phi_{11} = \left(\Theta_{i,j}\right)_{7\times 7}$ with

$\Theta_{1,1} = -P_p + Q_1 + Q_2 + \left(\tilde{\tau}+1\right)S - U_pF_1 - \Lambda_pG_1 + \lambda\Sigma_p,\ \Theta_{1,2} = \Theta_{1,3} = \Theta_{1,4} = 0,\ \Theta_{1,5} = U_pF_2,$
$\Theta_{1,6} = \Lambda_pG_2,\ \Theta_{1,7} = 0,\ \Theta_{2,2} = -Q_1 - Z,$
$\Theta_{2,3} = Z - M,\ \Theta_{2,4} = Z,\ \Theta_{2,5} = \Theta_{2,6} = \Theta_{2,7} = 0,$
$\Theta_{3,3} = -S - 2Z + M^{\mathrm{T}} + M,\ \Theta_{3,4} = Z - M,$
$\Theta_{3,5} = \Theta_{3,6} = \Theta_{3,7} = 0,\ \Theta_{4,4} = -Z,\ \Theta_{4,5} = \Theta_{4,6} = \Theta_{4,7} = 0,\ \Theta_{5,5} = -U_p,\ \Theta_{5,6} = \Theta_{5,7} = 0,\ \Theta_{6,6} = -\Lambda_p + \left(\tilde{\tau}+1\right)R,\ \Theta_{6,7} = 0,\ \Theta_{7,7} = -R,\ \tilde{\tau} = \tau_M - \tau_m,$
$\Phi_{12} = \left[\, XC_p + Y_p\ \ 0\ \ 0\ \ 0\ \ XA_p\ \ 0\ \ XB_p \,\right],$
$\Phi_{13} = \left[\, \tilde{\tau}\left(XC_p + Y_p - X\right)\ \ 0\ \ 0\ \ 0\ \ \tilde{\tau}XA_p\ \ 0\ \ \tilde{\tau}XB_p \,\right].$

Moreover, the feedback controller gain matrices can be designed by $K_p = X^{-1}Y_p$.

4. NUMERICAL EXAMPLE

In this section, one example is presented to show the effectiveness of the proposed results. Consider the parameters of the Markovian jumping neural networks (2) and (3) as follows:

$$C_1 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.21 & -0.012 \\ -0.51 & 0.32 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.16 & -0.01 \\ -0.02 & -0.24 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.2 & -0.01 \\ -0.5 & 0.45 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.15 & -0.01 \\ -0.04 & -0.42 \end{bmatrix}.$$

Respectively, the neuron activation functions and the noise intensity function vectors are given as

$$\tilde{f}(x(k)) = \tilde{g}(x(k)) = \begin{bmatrix} \tanh(x_1(k)) \\ \tanh(x_2(k)) \end{bmatrix},$$
$$\delta(k, e(k), 1) = \begin{bmatrix} \sqrt{0.1}e_1(k) & 0 \\ 0 & \sqrt{0.1}e_2(k) \end{bmatrix},$$
$$\delta(k, e(k), 2) = \begin{bmatrix} \sqrt{0.2}e_1(k) & 0 \\ 0 & \sqrt{0.2}e_2(k) \end{bmatrix}.$$

Clearly, the neuron activation functions and the noise intensity function vector satisfy the Assumption 1 and 4 with the following parameters:

$$F_1 = G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ F_2 = G_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$\Sigma_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

The transition probability matrix is assumed to be $\pi = \begin{bmatrix} 0.35 & 0.65 \\ 0.45 & 0.55 \end{bmatrix}$. The time delay is taken as $\tau(k) = \frac{e^{0.1k}}{0.1*(1+e^{0.1k})}$. It can be verified that $\tau_m = 5$, $\tau_0 = 8$, $\tau_M = 10$ and let $\theta_0 = 0.35$. By solving the LMIs in Theorem 2, the controller gain matrices K_1 and K_2 are designed as follows:

$$K_1 = \begin{bmatrix} -0.0360 & 0.1301 \\ 0.0042 & -0.1044 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.0298 & 0.1309 \\ -0.0057 & -0.1709 \end{bmatrix}.$$

Fig. 1 depicts the master system (2) has a chaotic attractor with initial values $x(0) = [-0.5\ 0.4]^{\mathrm{T}}$. In the absence of control input $u(t)$, Fig. 2 shows the slave system (3) has a chaotic attractor with initial values $y(0) = [-3\ 3]^{\mathrm{T}}$. Based on the above controller gain matrices K_1 and K_2, the responses of the state $x(t)$ and $y(t)$, and the error signal $e(t)$ are shown Figs. 3, 4 and 5, respectively. From simulation results Figs. 3, 4 and 5, we can see that the master system (2) and the slave system (3) are exponentially synchronized in the mean square.

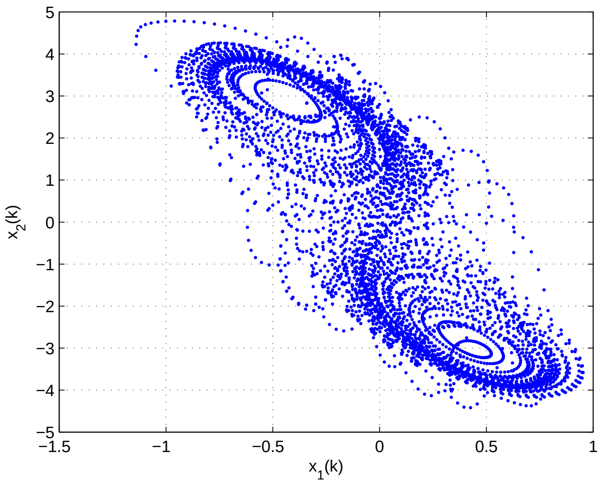

Fig. 1. Chaotic attractor of the master system (2).

5. CONCLUSIONS

In this paper, we have dealt with the mean-square delay-distribution-dependent exponential synchronization for a class of Markovian jumping discrete-time chaotic neural networks with random delays. By constructing an appropriate Lyapunov-Krasovskii functional and utilizing the Jensen's inequality, several delay-distribution-dependent sufficient conditions have been derived. Furthermore, the derived criteria have been expressed in terms of linear matrix inequalities, which can be easily solved by MATLAB LMI control toolbox. One simulation example has

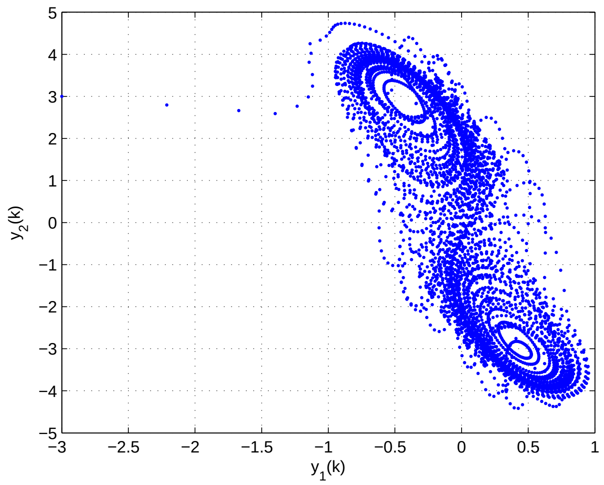

Fig. 2. Chaotic attractor of the slave system (2).

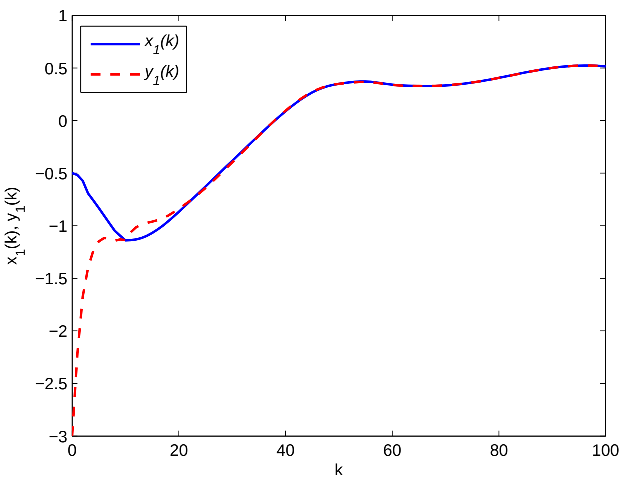

Fig. 3. State trajectories of $k - x_1(k) - y_1(k)$.

illustrated the feasibility and effectiveness of the presented synchronization scheme.

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REFERENCES

Pecora, L. M. and Carroll, T. L. (1990). Synchronization in chaotic systems. *Phys. Rev. Lett.*, 64(8), pp. 821–824.

Wen, S. et al. (2013). Global exponential synchronization of memristor-based recurrent neural networks with time-varying delays. *Neural Netw. Off. J. Int. Neural Netw. Soc.*, 48, pp. 195–203.

Kalpana, M. et al. (2018). Synchronization of chaotic-type delayed neural networks and its application. *Nonlinear Dyn.*, 93, pp. 543–555.

Tang, Y. et al. (2014). Synchronization in complex networks and its application a survey of recent advances and challenges. *Ann. Rev. Control*, 38(2), pp. 184–198.

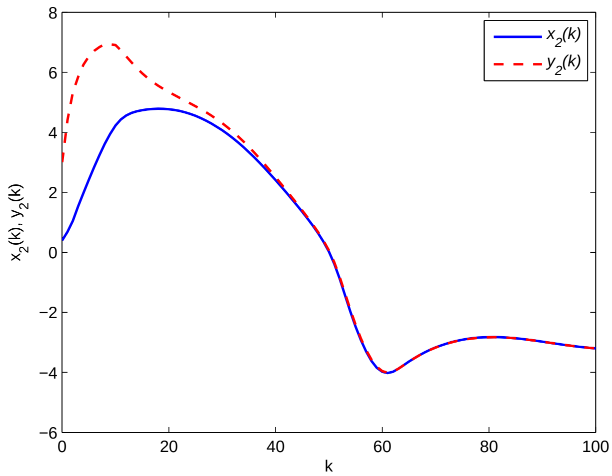

Fig. 4. State trajectories of $k - x_2(k) - y_2(k)$.

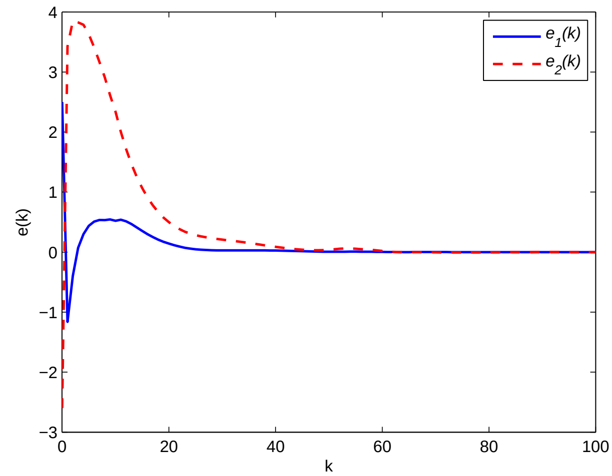

Fig. 5. The error state of $k - e(k)$.

Gilli, M. (1993). Strange attractors in delayed cellular neural networks. *IEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, 40(11), pp. 849–853.

Zou, F. and Nossek, J. A. (1993). Bifurcation and chaos in cellular neural networks. *IEE Trans. Circuits Syst. I: Fundam. Theor. Appl.*, 40, pp. 166–173.

Lu, H. (2002). Chaotic attractors in delayed neural networks. *Phys. Lett. A*, 298, pp. 109–116.

Lu, H. T. and He, Z. Y. (1993). Chaotic behavior in first-order autonomous continous-time system with time delay. *IEEE Trans. Circuit Syst.*, 43, pp. 700–702.

Khadra, A., Liu, X. Z. and Shen, X. (2005). Impulsively synchronizing chaotic systems with delay and applications to secure communication. *Automatica*, 41(9), pp. 1491–1502.

Xing, Z., Peng, J. and Wang, K.(2010). Exponential synchronization of chaotic neural networks with time delays: a M-matrix approach. *Nonlinear Dyn.*, 62, pp. 867–874.

Zhang, Z. M. et al. (2017). Exponential synchronization of chaotic neural networks with time-varying delay via intermittent output feedback approach. *Applied Math-*

ematics and Computation, 314, pp. 121–132.

Abdurahman, A., Jiang, H.J. and Teng, Z. D. (2016). Exponential lag synchronization for memristor-based neural networks with mixed time delays via hybrid switching control. *Journal of the Franklin Institute*, 353, pp. 2859–2880.

Zhang, G. D., Shen, Y. and Wang, L. M. (2013). Global anti-synchronization of a class of chaotic memristive neural networks with time-varying delays. *Neural Networks*, 46, pp. 1–8.

Zhang, X. Y., Lv, X. X. and Li, X. D. (2017). Sampled-data-based lag synchronization of chaotic delayed neural networks with impulsive control. *Nonlinear Dyn.*, 90, pp. 2199–2207.

Cheng, R. R. and Peng, M. S. (2016). Adaptive synchronization for complex networks with probabilistic time-varying delays. *Journal of the Franklin Institute*, 353, pp. 5099–5120.

Pratap, A. et al. (2018). Further synchronization in finite time analysis for time-varying delayed fractional order memristive competitive neural networks with leakage delay. *Neurocomputing*, 317, pp. 110–126.

Yang, X. S., Cao, J.D. and Qiu, J. L. (2015). Pth moment exponential stochastic synchronization of coupled memristor-based neural networks with mixed delays via delayed impulsive control. *Neural Networks*, 65, pp. 80–91.

Liu, Z. X., Yang, H. W. and Chen, F. W. (2011). Mean square exponential synchronization of stochastic neutral type chaotic neural networks with mixed delay. *International Journal of Mathematical and Computational Sciences*, 5(8), pp. 1298–1304.

Hu, B. X. et al. (2018). Global μ-synchronization of impulsive complex-valued neural networks with leakage delay and mixed time-varying delays. *Neurocomputing*, 307, pp. 106–116.

Ma, Y. C. and Zheng, Y. Q. (2018). Delay-dependent stochastic stability for discrete singular neural networks with Markovian jump and mixed time-delays. *Nural Comput. & Applic.*, 29, pp. 111–122.

Senthilraj, S. et al. (2016). Delay-interval-dependent passivity analysis of stochastic neural networks with Markovian jumping parameters and time delay in the leakage term. *Nonlinear Analysis: Hybird Systems*, 22, pp. 262–275.

Rakkiyappan, R. et al. (2014). Exponential state estimation of Markovian jumping genetic regulatory networks with mode-dependent probabilistic time-varying delays. *Mathematical Biosciences*, 251, pp. 30–53.

Ma, Y. C. and Zheng, Y. Q. (2015). Synchronization of continuous-time Markovian jumping singular complex networks with mixed mode-dependent time delays. *Neurocomputing*, 156, pp. 52–59.

Tong, D. B. et al. (2015). Exponential synchronization for stochastic neural networks with multi-delayed and Markovian switching via adaptive feedback control. *Commun. Nonlinear Sci. Numer. Simulat.*, 29, pp. 359–371.

Liu, Y., Wang, Z. and Liu, X. (2008). State estimation for discrete-time Markovian jumping neural networks with mixed mode-dependent delays. *Phys. Lett. A*, 372(48), pp. 7147–7155.

Liu, Y., Wang, Z. and Liu, X. (2009). Asymptotic stability for neural networks with mixed time-delays : the discrete-time case. *Neural Netw.*, 22 , pp. 67–74.

Nagamani, G. and Ramasamy, S. (2016). Stochastic deissativity and passivity analysis for discrete-time neural networks with probabilistic time-varying delays in leakage term. *Applied Mathematics and Computation*, 289, pp. 237–257.

Wang, Z. et al. (2009). Robust H_∞ control for a class of nonlinear discrete time-delay stochastic systems with missing measurements. *Automatica*, 45, pp. 684–691.

Chen, Y. and Fei, S. (2014). Exponential stabilization for discrete-time time-delay systems with actuator saturtion. *In : Proceeding of the 33rd Chinese Control Conference*, pp. 6130–6135.

Park, P., Ko, J, and Jeong, C. (2011). Reciprocally convex approach to stability of systems with time-varying delays. *Atomatica*, 47, pp. 235–238.