SOME ASPECTS CONCERNING THE MODELLING OF CHLORINE RESIDUALS IN DRINKING WATER DISTRIBUTION NETWORKS

Emil Petre, Dan Selisteanu

Department of Automatic Control, University of Craiova
A.I. Cuza Str. No. 13, RO-200585 Craiova, Romania, E-mail: epetre@automation.ucv.ro

Abstract: *This paper deals with the modelling of chlorine residuals in drinking water distribution networks (DWDNs) used for transportation of treated drinkable water from the water treating plants to consumers. To kill the microorganisms that can cause the waterborne ills, disinfection is usually the final treatment stage in the drinking water plants. The most common disinfectant used in DWDNs is chlorine because it is inexpensive and effectively annihilates a variety of disease-causing organisms. But disinfectant concentration in the water may decay during the transportation, and bacteria growth cannot be controlled if disinfectant concentration is lower than a certain level. As a result, the bacteria and waterborne pathogens may grow. Microorganisms can also grow up on the pipe and tank walls as not all organisms are killed in the water treatment plants. As a consequence, distribution network water quality control is a very important problem. A solution is the regulation of chlorine residual concentration within a prescribed set of bounds. Since the DWDNs are large scale systems with high uncertainties and time varying delays, in order to obtain useful models for control, in this paper, some aspects of modelling of chlorine residuals in these multiple-inputs, multiple-outputs complex systems are presented. The equations of chlorine residual concentration dynamics in DWDNs are obtained based on the mass conservation and reaction kinetics using the main aspects governing chlorine concentration in DWDNs: transportation, mixing and decay caused by chlorine reactions.*

Keywords: *Nonlinear systems, Large scale systems, Time Delay Systems, Modelling.*

1. INTRODUCTION

It is well known that water is one of the essential elements of life being an important resource both for industrial applications and domestic usage. Drinking water is usually take from ground sources (rivers, lakes), or underground sources (wells, springs). To reduce the risk of human exposure to pathogens, drinking water is treated in the water treating plants to filter out unwanted substances using physical and chemical methods thereby making it safe, clean and healthy for consume. From the water treating plants the treated drinkable water is transported to consumer taps by drinking water distribution networks (DWDNs). DWDNs are complex, large-scale systems composed of storage tanks, pumps, valves and user taps

connected together by pipes of various diameters made from variety of materials.

To kill the microorganisms that can cause the waterborne ills, disinfection is usually the final treatment stage in the drinking water plants. The most common disinfectant used in DWDNs is chlorine because it is inexpensive and effectively annihilates a variety of disease-causing organisms [1], [2], [8]. But disinfectant concentration in the water may decay during the transportation, and bacteria growth cannot be controlled if disinfectant concentration is lower than a certain level. As a result, the bacteria and waterborne pathogens may grow. Micro-organisms can also grow up on the pipe and tank walls as not all organisms are killed in the water treatment plants. The surviving bacteria can grow to harmful levels without further disinfection during water transportation in the DWDN. In order to reduce the number of these disease-causing organisms to a level that is harmless to human beings, chlorine is added at certain places in DWDNs [1], [2].

Although the magnitude of the waterborne diseases has reduced significantly when the chlorine to treat drinking water does not decrease a minimum level, at the same time during the water treatment process, chlorine (especially at large level) can react with organic compounds present in the bulk water or pipe walls to form Disinfection By-Products (DSP), some of which are suspected carcinogens [8], [11]. This leads to a limitation on the chlorine dosage in DWDNs and stricter monitoring of the chlorine residual concentration in the water network. For example, the U.S. EPA (Environmental Protection Agency of United States) established that the minimum chlorine residual that must be present at points of water consumption is 0.2 mg/l [8], [11], while the maximum residual chlorine in the DWDN is 4 mg/l. As a consequence, distribution network water quality control is an important problem, and the regulation of chlorine residual concentration within a prescribed set of bounds is part of the solution. At the same time, the accurate and reliable control of chlorine residuals within a DWDN is a new and complex problem [4], [5], [11].

For a large water distribution system it is difficult to maintain a uniform chlorine distribution by adding only chlorine to the output flows from the water treatment plant. Contradictory chlorine demand may occur at the zones close to the water reservoir and zones at the edge of the DWDNs, which leads to the application of chlorine booster stations in the DWDNs. However, suitable control technologies that can handle the special features of the chlorine concentration control in DWDNs (large time-varying time-delays associated with the transport time of water from one point to another and, in particular, from a point of chlorine input to a point of water use, operation and output constraints and uncertainties in the system) are still under research [11], [13]. The quality control in DWDNs is at present an extremely important application field that still waits for a practically sound solution [11], [13].

In order to obtain useful models for control, in this paper some aspects of modelling of chlorine residuals in these complex systems are presented. The equations of chlorine residual concentration dynamics in DWDNs are obtained based on the mass conservation and reaction kinetics. There are three main aspects governing chlorine concentration in DWDNs: transportation, mixing and decay caused by chlorine reactions [2], [5]. Calculating the time delays is complicated not only due to the topology of the DWDN, but also because consumer water use rates are varying and unknown. Therefore, nominal values of aggregate water demand are often known or assumed, any variations from these values being treated as disturbances [3], 11]. The operational control of a DWDN is usually repeated over certain period. Typically, the control horizon used in practice is 24 hours.

Tacking into account these aspects, in this paper the input-output relationship between chlorine concentrations at an injection node (input) and at a monitored node (output) is modeled as a linear discrete-time system with unknown parameters. The discrete-time formulation is suitable for handling the transport delay, which is inherently associated with the delivery of water [1], [2], [3], [11].

The rest of this paper is organized as follows. An implicit model of chlorine residuals in DWDNs are presented in Section 2. Section 3 describes the explicit model of chlorine residuals in DWDNs without tanks. The modelling of chlorine residuals in water network with storage facilities is presented in Section 4. Concluding remarks complete this paper.

2. IMPLICIT MODEL OF CHLORINE RESIDUALS IN DWDNs

Water is the carrier of the chlorine in the distribution networks. The characteristics of the chlorine transportation and mixing are determined directly by the water flows, velocities and detention times in the reservoirs. The hydraulics has significant effect on the chlorine concentration modelling, while the chlorine injection's effect on the hydraulic can be neglected [2].

To obtain the mathematical model of chlorine residual concentration in a DWDN, firstly we analyse the chlorine distribution in a pipe (Fig.1). The chlorine will travel down the length of a pipe with the same average velocity as the carrier fluid while at the same time reacting at certain rate. Neglecting the diffusion effects, the main transport mechanisms of chlorine in a pipe can be described as [5], [10]:

$$\frac{\partial c(x,t)}{\partial t} + v\frac{\partial c(x,t)}{\partial x} = r(c(x,t)) \tag{1}$$

where $c(x,t)$ is the chlorine concentration at position x and time t, $r(\cdot)$ stands for the reaction kinetics, v is the flow velocity in the pipe assumed to be constant during the hydraulic time step and x is the distance from the upstream node.

The chlorine reaction kinetics is still being investigated at present in order to understand the chemical and biological processes more accurately [5]. Usually the reactions are divided into two stages: rapid reaction and slow reaction. The former decay occurs at the first a few minutes when the chlorine is added into the water. The slow reaction occurs subsequently. In the modelling of chlorine residuals in DWDNs it is assumed that the rapid decay has been completed before the chlorine reaches the user taps. Rapid reactions are usually considered in the water treatment plant only [5]. In this paper, only slow decay process is discussed and rapid decay is assumed complete. First-order reaction kinetics is used at present in most

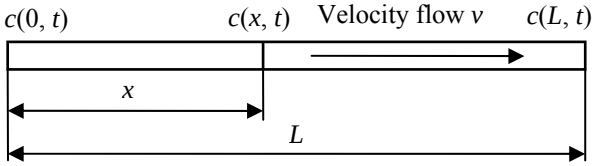

Fig. 1. Chlorine distribution in a pipe

models and DWDNs simulation packages [12], which is described by the following equation:

$$\frac{dc(t)}{dt} = -kc(t) \tag{2}$$

where $k > 0$ is the decay coefficient. It is widely accepted that chlorine in distribution systems will either decay due to reactions with compounds contained within the bulk water (bulk decay) or due to reactions at the pipe wall materials (wall decay). The simplest method of modelling bulk and wall decay is to define the overall decay constant as the sum of a bulk and wall decay constant:

$$k = k_b + k_w \tag{3}$$

where k_b and k_w are the bulk decay constant and wall decay constant, respectively. The rate of the reactions is also affected by the rate at which chlorine can be transported from the bulk flow to the pipe wall as [12]:

$$\frac{\partial c(x,t)}{\partial t} + v\frac{\partial c(x,t)}{\partial x} = -k_b c(x,t) - \frac{k_f}{r_h}(c - c_w)$$

$$= -kc(x,t), \text{ with } k = k_b + \frac{k_f k_w}{r_h(k_f + k_w)} \tag{4}$$

where k_f is the mass transfer coefficient, r_h is the hydraulic radius of pipe, c_w is the chlorine concentration at the pipe wall and k_w is the wall decay constant. For any pipe in the drinking water distribution systems the chlorine concentration can be expressed as:

$$\frac{\partial c_{ij}(x,t)}{\partial t} + v\frac{\partial c_{ij}(x,t)}{\partial x} = -kc_{ij}(x,t)$$

where $c_{ij}(x,t)$ is the chlorine concentration across pipe ij at position x with respect to node i at time t, and k is the overall decay constant.

The analytical solution of (4) can be obtained as in [5]:

$$c(x,t) = \begin{cases} c(x - vt, 0)\cdot\exp(-kt), & \text{for } x - vt \geq 0 \\ c\left(0, t - \dfrac{x}{t}\right)\cdot\exp\left(-k\dfrac{x}{t}\right), & \text{for } x - vt < 0 \end{cases} \tag{5}$$

At junctions receiving inflow from two or more pipes, the mixing of fluid is assumed to be complete and instantaneous. Thus, the concentration of chlorine in water leaving the junction is simply the flow-weighted sum of the concentration from the inflowing pipes. For node i it can be written as [5]:

$$c_i(t) = \frac{\sum_{k \in IN_i} c_{ki}(L_{ki}, t) q_{ki}(t) - \sum_{k \in OUT_i} c_i(t) q_{ik}(t)}{\sum_{k \in IN_i} q_{ki}(t) - \sum_{k \in OUT_i} q_{ik}(t)} \quad (6)$$

where IN_i is the node set of the pipes in which the flow is getting into node *i*, OUT_i is the node set of the pipes in which the flow is leaving node *i*, $c_{ki}(L_{ki}, t)$ represents chlorine concentration in the sink side of the pipe, and $c_{ki}(0, t) = c_i(t)$ is the chlorine concentration at the start node of an outflow pipe.

Assume that the contents of storage facilities (tanks and reservoirs) are completely mixed. Under these conditions the concentration in the tank is a blend of the current contents and that of any entering water. At the same time, the internal concentration could be changing due to reactions. The mixing and reactions can be written as [5]:

$$\frac{\partial (V_s(t) c_s(t))}{\partial t} = \sum_{i \in IN_s} q_{is} c_{is}(L_{is}, t) - \sum_{i \in OUT_s} q_{si} c_s(t) + r_s(c_s(t)) \quad (7)$$

where $V_s(t)$ and $c_s(t)$ are the volume and the concentration, respecively, in storage at time *t*, IN_s is the node set of the pipes in which flow is getting into the tanks, OUT_s is the node set of the outflow pipes, and $r_s(\cdot)$ is the function describing the chlorine reactions.

Although these models are sufficient for predicting chlorine residuals within a DWDN, however these involve all the nodes and storage facilities in the water networks and result in a distributed model. There is no explicit relationship between the input and output chlorine concentration [11], [14]. Because these models provide only an implicit input-output relationship, they are not convenient for controller design. Then, for chlorine transport in water networks, it is necessary to formulate an explicit input-output model useful also for controller design.

3. EXPLICIT MODEL OF CHLORINE RESIDUALS IN DWDNs

Based on the analytical solution of (4) in (5) the chlorine transportation time in a pipe can be calculated by tracking the water flow from the sink node to the upstream node assuming that the flow velocity is piece-wise constant over the entire considered time horizon. This backward tracking algorithm is called *path analysis* of chlorine transportation [5], [14]. The structure of paths from an injection node to a monitored node is depicted in Fig. 2. Starting from the monitored node, backward trace is performed until reaching the upstream node of the pipe where the flow may mix with inflow from a number of the upstream pipes. Each inflow in the upstream pipe constitutes one new path and the backward tracking continues in the upstream pipe separately through different new paths. New path is added at each junction node. Knowing all hydraulic information such as flows and velocities, the path analysis starts at the monitored nodes, runs recursively, and ends when the tracking reaches one chlorine injection node. Finally, the paths connecting the injection node and monitored node are obtained and delay times in the paths can be simultaneously calculated.

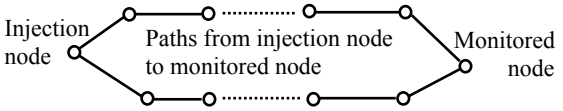

Fig. 2. Path set from an injection to a monitored node

3.1 Modelling of chlorine residuals in pipes

Using the result in (5) the chlorine concentration at the sink node of a pipe can be described by the flowing input-output equation [5], [11]:

$$y(t) = \beta(t) u(t - d(t)) \quad (8)$$

where the input $u(t)$ is the chlorine concentration at upstream node of the pipe at time *t*, the output $y(t)$ is the chlorine concentration at the sink node, $d(t)$ is the *detention time* of chlorine transportation from the input to the output node, and $\beta(t) = e^{-kd(t)}$ is the *decay factor* where $k > 0$ is the reaction constant defined in (4). The flow velocity in the pipe is piece-wise constant but the detention time is continuously time varying as the chlorine transportation in the pipe may cover more than one hydraulic time step, resulting in an overall time-varying flow velocity and time-delay.

The continuous variations in time delay make the model complicated in control design because

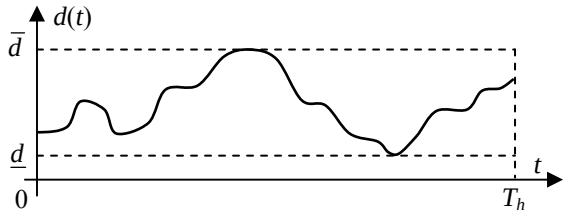

Fig. 3. Discretization of continuous time delay

exact function of $d(t)$ is unknown ($d(t)$ can be calculated only numerically using path analysis algorithm [14]). Such variations also make the model parameter estimation difficult for the same reason [7]. Fig. 3 presents the variation of time delay $d(t)$ over time horizon $[0, T_h]$, where $\bar{d}$ and $\underline{d}$ denote the maximum delay and minimum delay, respectively, and T_h is the modelling time horizon.

The range of the delay variation over $[0, T_h]$ can be obtained by numerically calculating the delay time. For this, the delay range can be discretized into a series of delay numbers:

$$I_{ij} = \{n_{\min}, n_{\min}+1, \mathrm{K}, n_{\max}-1, n_{\max}\},$$
$$n_{\min} = round(\underline{d}/T),\ n_{\max} = round(\bar{d}/T) \quad (9)$$

where I_{ij} is the delay number in pipe *ij* over time $[0, T_h]$, T is the discretization time step, $round(\cdot)$ is the function that takes the integer that is closest to a real number. So, the variable continuous delay will be replaced approximately by these delay numbers [5], [11]. Using (9), the model (8) yields:

$$y(t) = \sum_{i \in I_{ij}} a_i(t) u(t-i) + \varepsilon(t), \text{ for } t \in [0, T_h] \quad (10)$$

where $a_i(t)$ are the parameters associated with the delay numbers, $a_i(t) = e^{-kd(iT)}$, $\varepsilon(t)$ is the model structure error caused by discretization. Although time index t is the same as in the continuous model (8), in (10) it should be understood that denotes discrete time, that is $t = iT, i = 0, 1, 2, \mathrm{K}$.

It must be noted that in the discrete model (10) there are only time-varying parameters and there is no time-varying delay. In (10), variations in the detention time, caused by the variations of the flow rate of water in the pipe, imply only variations in the coefficients $a_i(t)$. Usually time-varying parameters are easier to be tackled in control design or parameter estimation [7], [11] than handling continuously varying time delay in (8).

The model structure error $\varepsilon(t)$ can be controlled by using small discrete time step T whose value depends on the hydraulic dynamics. For simplicity, in the next sections, $\varepsilon(t)$ will not appear in deriving chlorine residual model, and it will be presented only in the final obtained model structure.

3.2 Mixing at junction node

Two pipes in the drinking water distribution systems can be viewed as connecting in series or parallel. The model of pipes connected in series has the same form as in (8) and can be obtained by summing up the detention time in each pipe and multiplying the *decay factors* in each pipe.

When the pipes are connected in parallel the flows reach the junction node will be mixed there. If the mixing is assumed instantaneous, then [5]:

$$y(t) = \sum_{j \in IN_i} \omega_{ji}(t) \beta_{ji}(t) u(t - d_{ji}(t)) \quad (11)$$

with

$$\omega_{ji}(t) = \frac{|q_{ji}(t)|}{\sum_{k \in IN_i} |q_{ki}(t)|}, \quad (12)$$

where IN_i is the upstream node set of the pipes connected together, d_{ji} and β_{ji} are the detention time and decay factor in pipe *ji* respectively, and ω_{ji} is the mixing ratio for pipe *ji*. Both of these parameters are time varying.

3.3 Explicit model of chlorine residuals in DWDNs without tanks

To obtain a input-output model of a DWDN without tanks, consider a set of chlorine paths in a very simple DWDN without tanks as in Fig. 4. The source node of chlorine injection is node A and monitored node is G. Let P_{sj} denote the path set from source node A to node *j*. The input $u(t)$ and the output $y(t)$ denote the chlorine concentration at the injection node and at the monitored node, respectively. Using the equations obtained in the previous two subsections, the chlorine concentration at a node

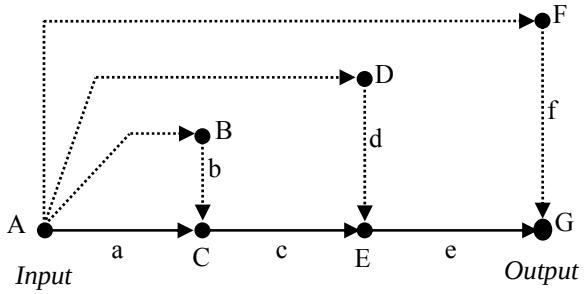

Fig. 4. Illustration of chlorine paths in a simple DWDN

j, denoted by $c_j(t)$, can be expressed by the following equation [5]:

$$c_j(t) = \sum_{p \in P_{sj}} a_{sjp}(t) u(t - d_{sjp}(t)) \tag{13}$$

where

$$a_{sjp}(t) = \prod_{l \in p} \frac{\left| q_{up(l),dn(l)}(t - d_l(t - d_{down(l)}(t))) \right|}{\sum_{k \in IN_{dn(l)}} \left| q_{up(l),dn(l)}(t - d_l(t - d_{down(l)}(t))) \right|} \times$$

$$\times \exp[-k_l d_l(t - d_{down(l)}(t))] \tag{14}$$

$$d_{sjp}(t) = \sum_{l \in p} d_l(t - d_{down(l)}(t)) \tag{15}$$

$$d_{down(l)}(t) = \sum_{n \in down(l)} d_n(t - d_{down(n)}(t)), \quad l \in p \tag{16}$$

where $a_{sjp}(t)$ is the chlorine contribution from injection node s to node j at time t through path p, defined as *impact coefficient*, $d_{sjp}(t)$ is the detention time from source node s to node j via path p, $d_l(t)$ is the detention time in pipe l in path p. Defined by (14), $a_{sjp}(t)$ is a function of chlorine reactions and mixing effects at nodes across path p, where, $up(l)$ and $dn(l)$ denote the upstream node and downstream node of pipe l respectively, $down(l)$ denotes downstream pipe set in path p (pipe l is excluded), $d_{down(l)}(t)$ is the detention time in downstream of pipe l in path p that is calculated recursively from the destination node by backtracking to the start node as described in (16).

Example 1. The calculation of the impact coefficient and detention time at time t in path A-C-E-G from Fig. 4 is given by [5]:

$$d_1 = d_e(t)$$
$$d_2 = d_c(t - d_1)$$
$$d_3 = d_a(t - d_2 - d_1)$$
$$d_{AGp}(t) = d_1 + d_2 + d_3$$

$$a_{AGp}(t) = \frac{\left| q_{EG}(t) \right| \exp(-k_e d_1)}{\left| q_{EG}(t) \right| \times \left| q_{FG}(t) \right|} \times$$

$$\times \frac{\left| q_{CE}(t - d_1) \right| \exp(-k_c d_2)}{\left| q_{CE}(t - d_1) \right| + \left| q_{DE}(t - d_1) \right|} \times$$

$$\times \frac{\left| q_{CE}(t - d_1 - d_2) \right| \exp(-k_a d_3)}{\left| q_{AC}(t - d_1 - d_2) \right| + \left| q_{BC}(t - d_1 - d_2) \right|} \tag{17}$$

where detention times d_1, d_2 and d_3 are recursively calculated and back-tracked from the destination node G till reaching source node A.

Finally, the input-output model for an arbitrary water network (with any number of pipes in series and parallel) without storage facilities can be obtained recursively based on equation (13):

$$y(t) = \sum_{p \in P(t)} \beta_p(t) u(t - d_p(t)) \tag{18}$$

where $P(t)$ is the path set from injection node to monitored node ($P(t)$ can be time varying), $d_p(t)$ is the detention time in path p and $\beta_p(t)$ is the *impact coefficient* with respect to path p. By analogy to the single-pipe case in Section 3.1, each detention time $d_p(t)$ in (18) can be discretized and a discrete input-output model can be obtained as:

$$y(t) = \sum_{i \in I(t)} a_i(t) u(t - i), \text{ for } t \in [0, T_h] \tag{19}$$

where $I(t)$ is defined as $\bigcup_{p \in P(t)} I^p(t)$, $I^p(t)$ being the discrete delay number with respect to path p on time horizon $[0, T_h]$. The time index t in $I(t)$ only hints that some subset of $I^p(t)$ is inactive or active due to path changes over certain time period. The set is piece-wise invariant over certain time-slots according to the path set changes.

Until now, only single-input and single-output (SISO) situation was considered. But, in general, chlorine concentration model for a DWDN should be multiple-input multiple-output (MIMO) model. Since in a DWDN the interactions between the inputs and outputs are inevitable and there are not interactions between outputs, to obtain a viable model for MIMO water networks, assume that the system can be simplified as a number of MISO systems. So, each MISO model structure can be modeled and identified by repeating the procedure

individually for each output. Then, for the first output of the MIMO model one obtains:

$$y_1(t) = \sum_{m=1}^{n_M} \sum_{i \in I_{1,m(t)}} a_{1,m,i}(t) u_m(t-i), \text{ for } t \in [0, T_h] \tag{20}$$

where n_M is the number of inputs, other parameter definitions being similar as before and $a_{1,m,i}$ is the i^{th} *impact coefficient* of the first output under the m^{th} input.

4. MODELLING CHLORINE RESIDUALS IN WATER DISTRIBUTED NETWORKS WITH STORAGE FACILITIES

In DWDNs, treated-water storage tanks are commonly used to satisfy demand fluctuations, to provide storage for unexpected abrupt demand for emergencies like fire fighting and to equalize operating pressures. But, the existence of the tanks in a DWDN makes the model of chlorine residuals more complicated because of the long residence times in these tanks [2].

In practical operation of DWDNs there are two types of tanks: (i) *continuous tanks* - the tanks for which the water flow can get into the tank and run out of the tank simultaneously; (ii) *switching tanks* - the tanks that can be operated only in a repeated sequential filling and draining cycles.

Two assumptions are made in the modelling of chlorine residual concentration dynamics in storage tanks: tank contents are completely mixed instantaneously; the kinetics of chlorine reactions to substance in tank water is first-order with respect to the chlorine concentration in tanks.

4.1 Chlorine residual modelling in storage tanks

Under the above assumptions, by using water mass conservation and chlorine mass conservation within the tank following differential equations can be obtained [6], [9]:

$$\frac{dV(t)}{dt} = q_f(t) - q_d(t) \tag{21}$$

$$\frac{dV(t)c(t)}{dt} = q_f(t)u(t) - q_d(t)c(t) - kV(t)c(t) \tag{22}$$

where $V(t)$ is the water volume in the tank at time t, $q_f(t)$ and $q_d(t)$ are the water flow filling into the tank and draining out of the tank, respectively, $c(t)$ is the chlorine concentration in the tank, $u(t)$ is the chlorine concentration of the water flowing into the tank, and $k > 0$ is the chlorine decay constant in the tank. Model (21) and (22) is a general chlorine concentration model that can work for both *continuous tank* and *switching tank*. For the *switching tank* the following condition is valuable:

$$q_f(t) \times q_d(t) = 0 \tag{23}$$

such that the inflow and outflow do not appear simultaneously.

The differential equations in (21) and (22) can be approximated by backward or forward difference equation. Assume $V(t)$ to be constant, which result in $q_d(t) = q_f(t)$. Then, the *backward difference* $\dot{c}_t \cong \big(c(t) - c(t-T)\big)/T$, where T is the discrete time step, used in (22) yields:

$$c(t) = \frac{V(t)}{(1+Tk)V(t) + Tq_f(t)} c(t-1) + \frac{Tq_f(t)}{(1+Tk)V(t) + Tq_f(t)} u(t) \tag{24}$$

The *forward difference* $\dot{c}_t \cong \big(c(t+T) - c(t)\big)/T$ used in (22) yields:

$$c(t) = \frac{(1-Tk)V(t-1) - Tq_f(t-1)}{V(t-1)} c(t-1) + \frac{Tq_f(t-1)}{V(t-1)} u(t-1) \tag{25}$$

For notation simplicity, in (24) and (25) the time index t is not changed after discretization. It is understood that, in these relations, t represents integer numbers. Usually, in practice backward approximation is applied, giving:

$$y(t) = \alpha(t) y(t-1) + \beta(t) u(t) \tag{26}$$

where

$$\alpha(t) = \frac{V(t)}{(1+Tk)V(t) + Tq_f(t)} \tag{27}$$

$$\beta(t) = \frac{Tq_f(t)}{(1+Tk)V(t) + Tq_f(t)} \tag{28}$$

The water volume $V(t)$ in the tank is usually far larger than the inflow volume $Tq_f(t)$ during the

discrete time interval T, that is $V(t) >> Tq_f(t)$. Hence, from (27) it can be obtained that $\alpha(t) \cong 1/(1+Tk)$. This is similar to the discrete alternative of the analytical solution of equation (2), $c(t) = e^{-kT} c(t-1)$ by backward difference approximation:

$$c(t) = \frac{1}{1+Tk} c(t-1) \tag{29}$$

The coefficient $\beta(t)$ is very small and this implies that the control action is very weak from the input to the tank. The controllability will be very weak if the water demand at a monitored node is provided mainly by the tank [11]. Therefore, hydraulic dynamics should be taken into account when selecting the positions of the injection nodes.

4.2 Chlorine residual modeling in water networks with continuous tanks

The modelling of the chlorine residuals in a DWDN depends on the location of the tanks in DWDN and topological relations of the tanks to the monitored nodes. If there are multiple tanks in the network, then the tank effects on the monitored nodes can be modeled as multiple sub-systems of tanks combined in series or in parallel. In general, each tank located at the chlorine transportation path from the injection node to the monitored node will introduce one more dynamic into the system. The order of the system depends on the number of the tanks involved in [2].

To illustrate the derivation of the model, we consider a simple water network with one tank [5], [11], as in Fig. 5, where $P1$ is the path from the injection node to the monitored node, $P2$ is the path from the injection node to the tank, and $P3$ is the path from the tank to the monitored node. For continuous tank the flow $P2$ and $P3$ are continuous, but for switching type tank $P2$ and $P3$ cannot exist simultaneously.

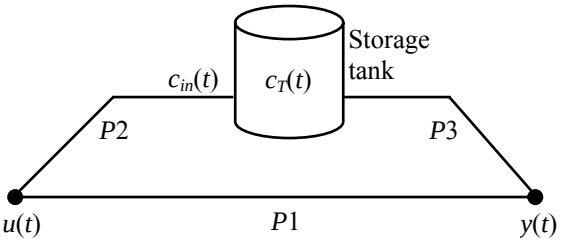

Fig. 5. A simple water network with one tank

Firstly, the continuous tank is considered. Let $u(t)$ and $y(t)$ be the chlorine concentration at the input node and the output node, respectively, and $c_{in}(t)$ denote the chlorine concentration in the water flow getting into the tank, and $c_T(t)$ denote the chlorine concentration in the tank, which is the chlorine concentration in the outflow of the tank as well. According to the models developed in the above sections the following equations can be obtained [5], [11]:

$$c_{in}(t) = \sum_{i \in I_{P2}} a2_i(t) u(t-i) \tag{30}$$

$$c_T(t) = \alpha(t) c_T(t-1) + \beta(t) c_{in}(t) \tag{31}$$

$$\begin{aligned} y(t) &= \omega(t) \sum_{i \in I_{P1}} \overline{a}1_i(t) u(t-i) \\ &+ (1-\omega(t)) \sum_{i \in I_{P3}} \overline{a}3_i(t) c_T(t-i) \\ &= \sum_{i \in I_{P1}} a1_i(t) u(t-i) + \sum_{i \in I_{P3}} a3_i(t) c_T(t-i) \end{aligned} \tag{32}$$

where I_{P1}, I_{P2} and I_{P3} are the delay number set in the paths $P1$, $P2$ and $P3$, $\alpha(t)$ and $\beta(t)$ are defined in (27) and (28), $a1_i, a2_i$ and $a3_i$ are *impact coefficients* corresponding to paths $P1$, $P2$ and $P3$, respectively, and $\omega(t)$ is flow mixing ratio defined in (12). Substituting (30) into (31) and using the resulting (31) at different time instant one obtains:

$$\begin{aligned} c_T(t) &= \alpha(t) c_T(t-1) + \sum_{i \in I_{P2}} \beta(t) a2_i(t) u(t-i) \\ c_T(t-1) &= \alpha(t-1) c_T(t-2) \\ &+ \sum_{i \in I_{P2}} \beta(t-1) a2_i(t-1) u(t-1-i) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_T(t-d_M^3+1) &= \alpha(t-d_M^3+1) c_T(t-d_M^3) \\ &+ \sum_{i \in I_{P2}} \beta(t-d_M^3+1) a2_i(t-d_M^3+1) u(t-d_M^3+1-i) \end{aligned} \tag{33}$$

where d_M^3 denotes the maximum delay number in path $P3$, which is the maximum number in I_{P3}. Now, by using the above equations recursively, we express $c_T(t-k)$, $0 \le k \le d_M^3$, in terms of $c_T(t-d_M^3)$, yielding [5]:

$$\begin{aligned} c_T(t-k) &= \prod_{i=k+1}^{d_M^3} \alpha(t-i+1) c_T(t-d_M^3) \\ &+ \sum_{i \in I_{P2}} \beta(t-k) a2_i(t-k) u(t-k-i) + \alpha(t-k) \times \\ &\times \sum_{i \in I_{P2}} \beta(t-k-1) a2_i(t-k-1) u(t-k-1-i) \\ &+ \Lambda + \alpha(t-d_M^3+2) \cdots \alpha(t-k) \times \end{aligned}$$

$$\sum_{i\in I_{P2}} \beta(t-d_M^3+1)a2_i(t-d_M^3+1)u(t-d_M^3+1-i) \tag{34}$$

By introducing new coefficients (34) can be rewritten as:

$$c_T(t-k) = \prod_{l=k+1}^{d_M^3} \alpha(t-l+1)c_T(t-d_M^3) + \sum_{l\in \bigcup_{j=k}^{d_M^3}(I_{P2+j})} a_1'(t)u(t-1) \tag{35}$$

where $I_{P2}+j$ defines a new number set in which each element is added by number j.

Substituting (35) into (32), to replace $c_T(t-i)$ items, yields [5]:

$$y(t) = \sum_{i\in I_{P1}} a1_i(t)u(t-i) + \sum_{i\in I_{P3}} a3_i(t)\times \left[\prod_{l=i+1}^{d_M^3} \alpha(t-l+1)c_T(t-d_M^3) + \sum_{l\in \bigcup_{j=k}^{d_M^3}(I_{P2+j})} a_1'(t)u(t-1) \right] = \alpha''(t)c_T(t-d_M^3) + \sum_{i\in I'} a_i''(t)u(t-i) \tag{36}$$

where

$$\alpha''(t) = \sum_{i\in I_{P3}} a3_i(t) \prod_{l=i+1}^{d_M^3} \alpha(t-l+1) \tag{37}$$

$$I' = \left\{ \bigcup_{i\in I_{P3}} \left[\bigcup_{j=1}^{d_M^3-1} (I_{p2}+j) \right] \right\} \cup I_{P1} \tag{38}$$

Next, we shall cancel $c_T(t-d_M^3)$ in equation (36). Firstly, (36) can be rewritten as:

$$y(t-1) = \alpha''(t-1)c_T(t-1-d_M^3) + \sum_{i\in I'} a_i''(t-1)u(t-1-i) \tag{39}$$

From its definition it can be seen that $a''(t)\neq 0$, so that from (37) one obtains:

$$c_T(t-1-d_M^3) = \frac{y(t-1) - \sum_{i\in I'} a_i''(t-1)u(t-1-i)}{\alpha''(t-1)} \tag{40}$$

Now we rewrite (33) as:

$$c_T(t-d_M^3) = \alpha(t-d_M^3)c_T(t-d_M^3-1) + \sum_{i\in I_{P2}} \beta(t-d_M^3)a2_i(t-d_M^3)u(t-d_M^3-i) \tag{41}$$

and substituting the obtained $c_T(t-1-d_M^3)$ into the above equation yields:

$$c_T(t-d_M^3) = \frac{\alpha(t-d_M^3)}{\alpha''(t-1)}\times \left[y(t-1) - \sum_{i\in I'} a_i''(t-1)u(t-1-i) \right] + \sum_{i\in I_{P2}} \beta(t-d_M^3)a2_i(t-d_M^3)u(t-d_M^3-i) \tag{42}$$

Finally, substituting (42) into (36) it can be derived that [2], [5]:

$$y(t) = b(t)y(t-1) + \sum_{i\in I} a_i(t)u(t-1) \tag{43}$$

where:

$$b(t) = \frac{\alpha''(t)\alpha(t-d_M^3)}{\alpha''(t-1)}$$

$a_i(t)$ are new introduced coefficients

$$I = I' \cup (I'+1) \cup (I_{P2}+d_M^3) \tag{44}$$

The input-output relationship established in (43) is in an ARMA (auto-regressive moving average) format. From the definition of the delay number set I' given by (38) and (44), it can be concluded that the general detention time in path $P2-P3$ is enlarged due to the existence of the tank.

4.3 Chlorine residual modeling in water networks with switching tanks

If the tank is of a switching type, condition $a''(t)\neq 0$ does not hold at any time instant [2], [5]. During the filling cycle, flow in $P3$ is zero and $(1-\omega(t))=0$, hence $a''(t)\neq 0$ (see (32) and (36)). So, (39) does not hold anymore. The backtracking in (35) must go further till reaching the last draining cycle in order to apply the same principle to eliminate c_T terms in the model. The switching sequence of tank operation is illustrated in Fig. 6, where two draining cycles and one filling cycle are presented, where j is the last time step of previous draining cycle. The tank operates in filling status from time instant $j+1$ until to time instant k. Then from time $k+1$ the tank switches to draining status again.

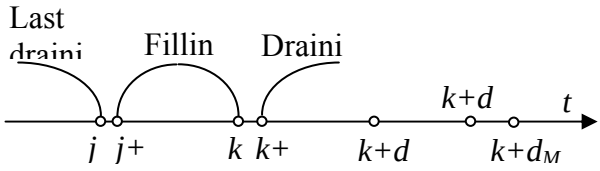

Fig. 6. The time sequence of draining and filling cycle switching

The chlorine residual model with tank effects must to be established from time $k+d_m$, where d_m and d_M are the minimum and maximum transport delays from the tank to the monitored node, respectively. Using the input-output model obtained in the previous section, the following equations can be derived [2], [5]:

Last draining:

$$y(j)=\sum_{i\in D_0} dr_i(j)c_T(j-i)+\sum_{i\in I_0} a_i(j)u(j-i) \quad (45)$$

$$c_T=\alpha_d c_T(t-1),\ t=...,j \quad (46)$$

Filling: $c_T(t)=\alpha_f(t)c_T(t-1)+\sum_{i\in F} fl_i(t)u(t-i)$,

$$t=j+1,...,k \quad (47)$$

Draining:

$$c_T(t)=\alpha_d c_T(t-1),\quad t=k+1,.... \quad (48)$$

$$y(t)=\sum_{i\in I} a_i^0(t)u(t-i),\ t=k+1,...,k+d_m-1 \quad (49)$$

$$y(t)=\sum_{i\in D} dr_i(t)c_T(t-i)+\sum_{i\in I} a_i^0(t)u(t-i),$$

$$t=k+d_m,\ \ldots \quad (50)$$

where D_0 and I_0 are the delay number set corresponding to paths from tank to the monitored node and paths from the injection node to the monitored node, respectively, during the last draining cycle. Similarly, F is defined for delays during filling cycle, D and I are defined for the current draining cycles, and other parameters such as fl_i, dr_i, a_i, a_d and a_f are defined like in previous sections. The following numbers are defined:

$$d_m=\min(D),\ d_M=\max(D),\ d_M^0=\max(D_0) \quad (51)$$

First, using (45) and (46), we shall represent $c_T(j)$ by $y(j)$, yielding:

$$c_T(j-i)=\alpha_d^{(d_M^0-i)}c_T(j-d_M^0)=\alpha_d^{(-i)}c_T(j) \quad (52)$$

and substitute into (45) it can be obtained:

$$c_T(j)=\frac{y(j)-\sum_{i\in I_0} a_i(j)u(j-1)}{\sum_{i\in D_0} dr_i(j)\alpha_d^{(-1)}} \quad (53)$$

Substituting (53) into (47), for $c_T(t)$ during filling cycle is obtained [2], [5]:

$$c_T(t)=b'(t)y(j)+\sum_{i\in Id(t)} fl_i''(t)u(t-i),\ t=j+1,....,k$$

$$Id(t)=(I_0+t-j)\ \Upsilon_{t'=j+1,...,t}(I_F+t'-j-1) \quad (54)$$

where $b'(t)$ and $fl_i'(t)$ are new coefficients, and $Id(t)$ is the delay number set.

During the draining cycle, when time moves from $t=k+d_m+1$ to $t=k+d_M$,K , there exist tank concentrations described by both (48) and (54); substituting them into (50) one obtains [2], [5]:

$$y(t)=b''(t)y(j)+\sum_{i\in I_S(t)} fl_i''(t)u(t-i)$$

$$+\sum_{i\in I} a_i^0(t)u(t-i),\quad \text{for } t=k+d_m,...,k+d_M \quad (55)$$

$$y(t)=\sum_{i\in I} a_i^0(t)u(t-i)+\alpha_d'(t)c_T(t-d_M),\ \text{for}$$

$$t=k+d_M+1,.... \quad (56)$$

where $I_S(t)=\Upsilon_{i\in D}\{Id(t-i)+i+1\}$ is the delay number set during the switching course, and b'', fl_i'' and α_d' are new introduced parameters.

Now, using the same method as in the previous section, $y(j)$ and $c_T(t-d_M)$ are cancelled, (55) and (56) can be written into (57) and (58) introducing new parameters b, fl''' and a_i, respectively [2], [5]:

$$y(t)=b(t)y(t-1)+\sum_{i\in(I_S+1)\Upsilon I_S} fl_i'''(t)u(t-i)$$

$$+\sum_{i\in(I+1)\Upsilon I} a_i(t)u(t-i),$$

$$\text{for } t=k+d_m+1,\text{K},k+d_M \quad (57)$$

$$y(t)=b(t)y(t-1)+\sum_{i\in(I_S+1)\Upsilon I} a_i(t)u(t-i),$$

$$\text{for } t=k+d_M+1,\text{K} \quad (58)$$

where (57) denotes the model around switching course of the tank operation status and (58) described the model after switching. Comparing delay number set I_S and I defined by (44), it can be seen that the existence of the switching type tank make the model structure complicated and more parameters are needed around the switching course.

5. CONCLUSIONS

In order to obtain useful mathematical models for control, in this paper some aspects of modelling of chlorine residuals in drinking water distribution networks were presented.

The equations of chlorine residual concentration dynamics in DWDNs are obtaind based on the

mass conservation and reaction kinetics. The input-output relationship between chlorine concentrations at an injection node (input) and at a monitored node (output) was modeled as a linear discrete-time system with unknown parameters. The model was developed beginning with the case of one single pipe following with a water network with any number of pipes connected in series or in parallel and finally with a complex water network with storage facilities.

The models were formulated in discrete-time as ARMA (auto-regressive moving average) models with time-varying coefficients. The discrete-time formulation is suitable for handling the transport delay, which is inherently associated with the delivery of water and can be used both for estimation and control using prediction methods.

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