CEAI, Vol.23, No.1, pp. 3-12, 2021 Printed in Romania

Finite-Time Bounded Observer-Based Control for Quasi-One-Sided Lipschitz Nonlinear Systems With Time-Varying Delay

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Abstract: This paper considers the problem of finite-time bounded observer-based control for a class of quasi-one-sided Lipschitz nonlinear systems with time-varying delay, time-varying parametric uncertainties and norm-bounded disturbances. The design methodology, for the less conservative quasi-one-sided Lipschitz nonlinear systems, involves astute utilization of several matrix decompositions and Jensen's inequality. By using the delay-dependent Lyapunov-Krasovskii functional and using the matrix inequality method, the sufficient conditions are established to guarantee that the resulted closed-loop system is finite-time bounded with a prescribed H_∞ performance. Based on these results, we have developed the robust observer-based controller synthesis strategy under parametric uncertainties. The proposed methodology ensures that the resulted closed-loop system is finite-time bounded. Finally, simulate examples are given to illustrate the effectiveness of the proposed method.

Keywords: Finite-time bounded; observer-based control, quasi-one-sided Lipschitz nonlinearity, parametric uncertainty, time-varying delay.

1. INTRODUCTION

The concept of Finite-Time Stability (FTS) was introduced in 1960s. Up until now, much work has been done in this field. Given a bound on the initial condition, a system is said to be finite-time stable if the state does not exceed a certain threshold during a specified time interval. While external disturbances are considered, FTS is extended to Finite-Time Boundedness (FTB). In recent years, the problems of finite-time stability, boundedness and stabilization have interested more and more researchers because the finite-time stable systems usually demonstrate some nice features such as faster convergence rates, higher accuracies and better disturbance rejection properties and so on (Wang et al., 2015; Dong et al., 2017a; Chen and Yang, 2014; Zhang et al.,2012; Zhang et al., 2015).

Uncertainties and perturbations are frequently encountered in practical control systems. Due to factors such as environmental noises, data errors, ageing of systems, uncertain or slowly varying parameters, it is often very difficult to obtain exact mathematical models. The presence of uncertainties may cause instability and bad performances on a controlled system. Therefore, researchers have made a number of efforts to solve the problem of robust stability and stabilization of systems with parameter uncertainties. Some results generated by these efforts were presented in (Dong et al., 2017a; Zhang et al., 2015; Amato et al., 2011; Wang et al., 2016; Wang et al., 2015). (Dong et al., 2017a) studied finite-time boundedness and H_∞ control for switched neutral systems with mixed time-varying delays. (Zhang et al., 2015) considered finite-time stability and stabilization for uncertain continuous-time system with time-varying delay. (Amato et al., 2011) investigated the robust finite-time stabilization for uncertain linear systems.

In the recent years, one-sided Lipschitz nonlinear systems have gained importance as a class of nonlinear systems because: First, they are better at representing a more general form compared to the Lipschitz systems; Second, one-sided Lipschitz constant is superior than the Lipschitz constant (Cai et al., 2014; Boutayeb et al., 2012). The quasi-one-sided Lipschitz condition is shown to be an extension of one-sided Lipschitz condition and the Lipschitz condition (Hu, 2008; Fu et al., 2012), but it is less conservative. The finite-time H_∞ control problem of a class of Lipschitz nonlinear systems with parameter uncertainties was studied in (Song et al., 2015). A state feedback controller was designed to guarantee that the resulted closed-loop system is finite-time bounded. (Zhu and Hu, 2009) studied the stability for uncertain nonlinear time-delay systems with quasi-one-sided Lipschitz condition. The problem of stabilization of quasi-one-sided Lipschitz nonlinear systems was considered in (Fu et al., 2012). However, to the best of our knowledge, until now, no results have been yielded concerning the study of finite-time control of nonlinear systems with quasi-one-sided Lipschitz conditions by observer-based controller, which has motivated us to do the research presented in the paper.

Our research focuses on finite-time bounded observer-based control for quasi-one-sided Lipschitz nonlinear systems with time-varying delay, time-varying parametric uncertainties and norm-bounded disturbances. By constructing a delay-dependent Lyapunov-Krasovskii functional and using Jensen's inequality, we derive sufficient conditions to guarantee that the resulted closed-loop system is finite-time bounded and satisfies a given H_∞ constraint condition. Based on this, we further propose a robust observer-based controller synthesis strategy under parametric uncertainties. Finally, we present numerical simulation examples demonstrating the effectiveness of the proposed observer-based control scheme.

The rest of the paper is organized as follows. Section 2 gives the description of the system and several necessary lemmas and assumptions. Section 3 presents sufficient conditions of finite-time bounded and controller design method for quasi-one-sided Lipschitz nonlinear systems. Section 4 shows the application of the proposed methods to quasi-one-sided Lipschitz nonlinear systems. Finally, we recapitulate the paper in Section 5.

Notations: The notations used in this paper are standard. The superscript T denotes matrix transpose, R^n denotes the n-dimensional real Euclidean space. $R^{m\times n}$ represents the set of all $m\times n$ real matrices. The notation $P>0$ ($P<0$) means that the matrix is positive definite (negative definite). $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the minimum and maximum eigenvalues of a matrix. For $\langle x,y\rangle$ denotes $x^T y$.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following uncertain nonlinear system with time-varying delay:

$$\begin{cases} \dot{x}(t)=\left(A_1+\Delta A_1(t)\right)x(t)+\left(A_2+\Delta A_2(t)\right)x_\tau \\ \qquad +Bu(t)+f(t,x)+g(t,x_\tau)+Dw(t), \\ y(t)=Cx(t), \\ \eta(t)=H_1x(t)+H_2x_\tau+H_3w(t), \\ x(t)=\phi(t),\quad t\in[0,\bar{\tau}], \end{cases} \tag{1}$$

where $x(t)\in R^n$ is the state, $x_\tau=x(t-\tau(t)),w(t)\in L_2^q[0,+\infty]$ is the external disturbances, $u(t)\in R^p$ is the controlled input, $y(t)\in R^l$ is the measured output, $z(t)\in R^q$ is the controlled output. $\phi(t)$ is a continuous initial state. $f(t,x)$ and $g(t,x_\tau)$ are the nonlinear functions with $f(t,0)=0$ and $g(t,0)=0$. A_1,A_2,B,C,D are known real constant matrices. $\Delta A_1(t)$ and $\Delta A_2(t)$ are the unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\Delta A_1=M_1F(t)N_1,\quad \Delta A_2=M_2F(t)N_2, \tag{2}$$

where $M_i,N_i\,(i=1,2)$ are known real constants matrices and $F(t)$ is an unknown real-valued matrix function satisfying

$$F^T(t)F(t)\le I. \tag{3}$$

$\tau(t)$ is the time-varying delay satisfying

$$0\le\tau(t)\le\bar{\tau},\quad \dot{\tau}(t)\le\mu. \tag{4}$$

To begin with the main result, the following preliminary assumptions and definitions are given.

Assumption 1. The external disturbances input $w(t)$ is time -varying and satisfies

$$\int_0^{T_f} w^T(t)w(t)dt\le\delta \tag{5}$$

where δ and T_f are positive constants.

Assumption 2. The nonlinear function $f(t,x)$ is a quasi-one-sided Lipschitz function, i.e., there exist a positive definite matrix P_1 and a real symmetric matrix S_1 which relined on P_1 such that

$$\left\langle P_1\left[f(t,x_1)-f(t,x_2)\right],x_1-x_2\right\rangle\le(x_1-x_2)^T S_1(x_1-x_2), \tag{6}$$

for all $x_1,x_2\in R^n$.

Assumption 3. The nonlinear function $g(t,x_\tau)$ is a quasi-one -sided Lipschitz function, i.e., there exist a positive definite matrix P_1 a real symmetric matrix S_2 which relined on P_1 such that

$$\left\langle P_1\left[g(t,x_\tau)-g(t,\hat{x}_\tau)\right],x_\tau-\hat{x}_\tau\right\rangle\le(x_\tau-\hat{x}_\tau)^T S_2(x_\tau-\hat{x}_\tau), \tag{7}$$

for all $x_\tau,\hat{x}_\tau\in R^n$.

Now, we consider the following observer dynamics

$$\begin{cases} \dot{\hat{x}}(t)=A_1\hat{x}(t)+A_2\hat{x}_\tau+Bu(t)+f(t,\hat{x}) \\ \qquad +g(t,\hat{x}_\tau)+L\left(y(t)-\hat{y}(t)\right), \\ \hat{y}(t)=C\hat{x}(t), \end{cases} \tag{8}$$

where $\hat{x}_\tau=\hat{x}(t-\tau(t))$, $\hat{x}(t)$ and $\hat{y}(t)$ are the estimated state and output, respectively. L is the state estimator gain matrix to be designed.

Define the error $e(t)=x(t)-\hat{x}(t)$, then the error-state system is governed by

$$\begin{aligned} \dot{e}(t)=&(A_1-LC)e(t)+A_2e(t-\tau(t))+\Phi_1(t,x,\hat{x}) \\ &+\Phi_2(t,x_\tau,\hat{x}_\tau)+Dw(t)+\Delta A_1(t)x(t)+\Delta A_2(t)x_\tau, \end{aligned} \tag{9}$$

Where

$\Phi_1(t,x,\hat{x})=f(t,x)-f(t,\hat{x}),\Phi_2(t,x_\tau,\hat{x}_\tau)=g(t,x_\tau)-g(t,\hat{x}_\tau).$

We construct an observer-based controller for system (1)

$$u(t)=-K\hat{x}(t), \tag{10}$$

where $K\in R^{m\times n}$ is the controller gain. By (1), (9) and (10), we obtain the following augmented closed-loop system:

$$\dot{z}(t)=\bar{A}_1 z(t)+\bar{A}_2 z(t-\tau(t))+F_1(t,x,\hat{x}) + F_2(t,x_\tau,\hat{x}_\tau)+\bar{D}w(t), \tag{11}$$
$$\eta(t)=\bar{H}_1 z(t)+\bar{H}_2 z(t-\tau(t))+H_3 w(t),$$

where

$$z(t)=[x^T(t)\; e^T(t)]^T, \bar{H}_1=(H_1,0), \bar{H}_2=(H_2,0),$$
$$\bar{A}_1=\begin{bmatrix} A_1-BK+\Delta A_1(t) & BK \\ \Delta A_1(t) & A_1-LC\end{bmatrix}, \bar{A}_2=\begin{bmatrix} A_2+\Delta A_2(t) & 0 \\ \Delta A_2(t) & A_2\end{bmatrix},$$
$$F_1(t,x,\hat{x})=\begin{bmatrix} f(t,x) \\ \Phi_1(t,x,\hat{x})\end{bmatrix}, \quad \bar{D}=\begin{bmatrix} D \\ D\end{bmatrix},$$
$$F_2(t,x_\tau,\hat{x}_\tau)=\begin{bmatrix} g(t,x_\tau) \\ \Phi_2(t,x,\hat{x})\end{bmatrix}. \tag{12}$$

Definition 1. (Dong et al., 2017a) (Finite-Time Bounded, FTB) For given positive constants c_1,δ,T_f and a symmetric matric $R>0,$ the resulting closed-loop system (11) is said to be robustly FTB with respect to (c_1,c_2,T_f,R,δ), if there exists a constant $c_2(>c_1)$ such that

$$\sup_{-\bar{\tau}\le t\le 0}\phi^T(t)R\phi(t)\le c_1 \Rightarrow z^T(t)Rz(t)<c_2, \forall t\in[0,T_f].$$

Definition 2. (Fu et al., 2013) If there exists an observed-based controller in form (10) such that the resulting closed-loop system (11) is FTB with respect to (c_1,c_2,T_f,R,δ) and under the assumed zero initial condition, the system output satisfies the following cost function inequality for $T_f>0$ and for all admissible $w(t)$ which satisfy Assumption 1:

$$\int_0^{T_f}\eta^T(t)\eta(t)dt\le\gamma^2\int_0^{T_f}w^T(t)w(t)dt, \tag{13}$$

then the controller (10) is called as the robust finite-time H_∞ controller of the quasi-one-sided Lipschitz nonlinear system (1).

The aim of the present study is to explore finite-time observer -based robust control strategies for quasi-one-sided Lipschitz nonlinear system (1) such that the closed-loop dynamic system(11) is robustly FTB.

The following lemmas will be used to develop our main results.

Lemma 1. (Cai et al., 2014) For any constant $v>0$ and known real matrices X, Γ and U of appropriate dimensions, the inequality

$$X\Gamma(t)U+\left(X\Gamma(t)U\right)^T\le v^{-1}XX^T+vU^TU,$$

holds, where $\Gamma(t)$ is a time-varying uncertain matrix fulfilling

$$\Gamma^T(t)\Gamma(t)\le I.$$

Lemma 2. (Lee et al., 2017) For given positive definite matrix $z\in R^{n\times n}$, two scalars a, b with $a>b$ and a vector-valued function $\omega:[a,b]\to R^n$ such that the following integrals are well defined, then

$$-(b-a)\int_{-b}^{-a}\omega^T(s)Z\omega(s)ds\le-\left(\int_{-b}^{-a}\omega^T(s)ds\right)Z\left(\int_{-b}^{-a}\omega(s)ds\right),$$
$$-\frac{(b^2-a^2)}{2}\int_{-b}^{-a}\int_{t+\theta}^{t}\omega^T(s)Z\omega(s)dsd\theta \le-\left(\int_{-b}^{-a}\int_{t+\theta}^{t}\omega^T(s)dsd\theta\right)Z\left(\int_{-b}^{-a}\int_{t+\theta}^{t}\omega(s)dsd\theta\right).$$

Lemma 3. (Cai et al., 2015) Given a matrix $C\in R^{q\times n}$ with $rank(C)=q$. Assume that $X\in R^{n\times n}$ is a symmetric matrix, then there exist a matrix $\hat{X}\in R^{q\times q}$ such that $CX=\hat{X}C$, if and only if

$$X=V\begin{bmatrix}\hat{X}_{11} & 0 \\ 0 & \hat{X}_{22}\end{bmatrix}V^T,$$

where $\hat{X}_{11}\in R^{q\times q}$ and $\hat{X}_{22}\in R^{(n-q)\times(n-q)}$.

3. MAIN RESULTS

The following theorem introduces the sufficient condition to guarantee the finite-time bounded of the closed-loop system (11).

Theorem 1. Suppose that Assumption 1-3 are satisfied. For given positive constants $c_1,\delta,T_f,\alpha,$ and a symmetric matrix $R>0$, the closed-loop system (11) is FTB with respect to (c_1,c_2,T_f,R,δ), if there exist positive constant c_2 and symmetric positive definite matrices $P=diag(P_1,P_1),Q_1,Q_2$, and S such that

$$\Sigma=\begin{bmatrix}\Sigma_{11} & P\bar{A}_2 & 0 & 0 & P\bar{D} \\ * & -(1-\mu)\bar{\tau}Q_1 & 0 & 0 & 0 \\ * & * & -\bar{\tau}Q_2 & 0 & 0 \\ * & * & * & -2\alpha Q_2 & 0 \\ * & * & * & * & -\alpha I\end{bmatrix}<0, \tag{14}$$

$$2\lambda_1 c_1+2\lambda_2 c_1\bar{\tau}^2+\lambda_3 c_1\bar{\tau}^4+2\delta(1-e^{-\alpha T_f})<2\lambda_0 c_2 e^{-\alpha T_f}, \tag{15}$$

where

$$\Sigma_{11}=P\bar{A}_1+\bar{A}_1^TP+2S+\bar{\tau}Q_1+\bar{\tau}^3Q_2-\alpha P,$$
$$\tilde{P}=R^{-\frac{1}{2}}PR^{-\frac{1}{2}},\tilde{Q}_1=R^{-\frac{1}{2}}Q_1R^{-\frac{1}{2}},\tilde{Q}_2=R^{-\frac{1}{2}}Q_2R^{-\frac{1}{2}},$$
$$\lambda_0=\lambda_{\min}(\tilde{P}),\lambda_1=\lambda_{\max}(\tilde{P}),\lambda_2=\lambda_{\max}(\tilde{Q}_1),\lambda_3=\lambda_{\max}(\tilde{Q}_2).$$

Proof. For the closed-loop system (11), consider the Lyapunov-Krasovskii functional

$$\begin{aligned} V(t) &= z^T(t)Pz(t) + \bar{\tau}\int_{t-\tau(t)}^{t} z^T(s)Q_1 z(s)ds \\ &+ \bar{\tau}^2 \int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z^T(s)Q_2 z(s)dsd\theta. \end{aligned} \tag{16}$$

From Assumption 2 and 3, one has

$$\begin{aligned} &z^T(t)PF_1(t,x,\hat{x}) \le z^T(t)\bar{S}_1 z(t), \\ &z^T(t)PF_2(t,x_\tau,\hat{x}_\tau) \le z^T(t)\bar{S}_2 z(t), \end{aligned} \tag{17}$$

where $\bar{S}_1 = diag(S_1, S_1)$, $\bar{S}_2 = diag(S_2, S_2)$.

The derivative of $V(t)$ along the trajectories of system (11) is

$$\begin{aligned} \dot{V}(t) &\le z^T(t)(P\bar{A}_1 + \bar{A}_1^T P + \bar{\tau}Q_1 + \bar{\tau}^3 Q_2 + 2S)z(t) \\ &+ 2z^T(t)P\bar{A}_2 z(t-\tau(t)) + 2z^T(t)P\bar{D}w(t) - \bar{\tau}(1-\mu) \\ &\cdot z^T(t-\tau(t))Q_1 z(t-\tau(t)) - \bar{\tau}^2\int_{t-\bar{\tau}}^{t} z^T(s)Q_2 z(s)ds, \end{aligned} \tag{18}$$

where $S = \bar{S}_1 + \bar{S}_2$.

Define the following function:

$$J_1 = \dot{V}(t) - \alpha V(t) - \alpha \hat{w}^T(t)w(t). \tag{19}$$

It is easy to get from (16):

$$-\alpha V(t) \le -\alpha z^T(t)Pz(t) - \alpha\bar{\tau}^2\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z^T(s)Q_2 z(s)dsd\theta. \tag{20}$$

According to Lemma 2, it's easy to get:

$$-\bar{\tau}^2\int_{t-\bar{\tau}}^{t} z^T(s)Q_2 z(s)ds \le -\bar{\tau}\left(\int_{t-\bar{\tau}}^{t} z^T(s)ds\right)Q_2\left(\int_{t-\bar{\tau}}^{t} z(s)ds\right), \tag{21}$$

$$\begin{aligned} &-\alpha\bar{\tau}^2\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z^T(s)Q_2 z(s)dsd\theta \\ &\le -2\alpha\left(\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z^T(s)dsd\theta\right)Q_2\left(\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z(s)dsd\theta\right). \end{aligned} \tag{22}$$

From (18)-(22), one can get

$$\begin{aligned} J_1 &\le z^T(t)(P\bar{A}_1 + \bar{A}_1^T P + \bar{\tau}Q_1 + \bar{\tau}^3 Q_2 + 2S)z(t) \\ &+ 2z^T(t)P\bar{A}_2 z(t-\tau(t)) + 2z^T(t)P\bar{D}w(t) \\ &- \bar{\tau}(1-\mu)z^T(t-\tau(t))Q_1 z(t-\tau(t)) \\ &- \bar{\tau}^2\int_{t-\bar{\tau}}^{t} z^T(s)Q_2 z(s)ds - \alpha V(t) - \alpha w^T(t)w(t) \\ &\le z^T(t)(P\bar{A}_1 + \bar{A}_1^T P + \bar{\tau}Q_1 + \bar{\tau}^3 Q_2 + 2S - \alpha P)z(t) \\ &+ 2z^T(t)P\bar{A}_2 z(t-\tau(t)) + 2z^T(t)P\bar{D}w(t) \\ &- \bar{\tau}(1-\mu)z^T(t-\tau(t))Q_1 z(t-\tau(t)) \\ &- \bar{\tau}^2\int_{t-\bar{\tau}}^{t} z^T(s)Q_2 z(s)ds - \alpha w^T(t)w(t) \\ &- 2\alpha\left(\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z^T(s)dsd\theta\right)Q_2\left(\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z(s)dsd\theta\right) \\ &\le \zeta^T(t)\Sigma\zeta(t), \end{aligned} \tag{23}$$

where

$$\zeta(t) = [z^T(t), z^T(t-\tau(t)), \int_{t-\bar{\tau}}^{t} z^T(s)ds, \int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} z^T(s)dsd\theta, w^T(t)]^T$$

and Σ is given by (14). The condition inequality (14) implies

$$\dot{V}(t) \le \alpha V(t) + \alpha w^T(t)w(t).$$

Multiplying the above inequality by $e^{-\alpha t}$, one get

$$\frac{d}{dt}\left(e^{-\alpha t}V(t)\right) < \alpha e^{-\alpha t}w^T(t)w(t).$$

By integrating the aforementioned inequality between 0 and t, we derive

$$V(t) < e^{\alpha t}V(0) + \alpha\int_0^t e^{-\alpha s}w^T(s)w(s)ds. \tag{24}$$

From (16), one has

$$\begin{aligned} e^{\alpha t}V(0)) &\le e^{\alpha T_f} z^T(0)R^{\frac{1}{2}}\tilde{P}R^{\frac{1}{2}}z(0) \\ &+ e^{\alpha T_f}\bar{\tau}\int_{-\tau(0)}^{0} z^T(s)R^{\frac{1}{2}}\tilde{Q}_1 R^{\frac{1}{2}}z(s)ds \\ &+ e^{\alpha T_f}\bar{\tau}^2\int_{-\bar{\tau}}^{0}\int_{\theta}^{0} z^T(s)R^{\frac{1}{2}}\tilde{Q}_2 R^{\frac{1}{2}}z(s)dsd\theta \\ &\le \lambda_{\max}(\tilde{P})e^{\alpha T_f}z^T(0)Rz(0) + \lambda_{\max}(\tilde{Q}_1)e^{\alpha T_f}\bar{\tau}\int_{-\bar{\tau}}^{0} z^T(s)Rz(s)ds \\ &+ \lambda_{\max}(\tilde{Q}_2)e^{\alpha T_f}\bar{\tau}^2\int_{-\bar{\tau}}^{0}\int_{\theta}^{0} z^T(s)Rz(s)dsd\theta \\ &\le \lambda_1 c_1 e^{\alpha T_f} + \lambda_2 c_1\bar{\tau}^2 e^{\alpha T_f} + \frac{1}{2}\lambda_3 c_1\bar{\tau}^4 e^{\alpha T_f}. \end{aligned} \tag{25}$$

On the one hand, the following inequality holds:

$$\begin{aligned} \alpha e^{\alpha t}\int_0^t e^{-\alpha s}w^T(s)w(s)ds &\le -\delta e^{\alpha T_f}(e^{-\alpha T_f} - 1) \\ &= \delta e^{\alpha T_f} - \delta \end{aligned} \tag{26}$$

From (24)-(26), one can get

$$V(t) < \lambda_1 c_1 e^{\alpha T_f} + \lambda_2 c_1\bar{\tau}^2 e^{\alpha T_f} + \frac{1}{2}\lambda_3 c_1\bar{\tau}^4 e^{\alpha T_f} + \delta e^{\alpha T_f} - \delta. \tag{27}$$

From (16), one can get

$$\begin{aligned} V(t) &\ge z^T(t)Pz(t) \\ &\ge \lambda_{\min}(\tilde{P})z^T(t)Rz(t) \\ &= \lambda_0 z^T(t)Rz(t). \end{aligned} \tag{28}$$

From (27) and (28), one can obtain

$$z^T(t)Rz(t) \le \frac{2\lambda_1 c_1 + 2\lambda_2 c_1\bar{\tau}^2 + \lambda_3 c_1\bar{\tau}^4 + 2\delta - 2\delta e^{-\alpha T_f}}{2\lambda_0 e^{-\alpha T_f}}. \tag{29}$$

Condition (15) implies that

$$z^T(t)Rz(t) < c_2,\ \forall t \in \left[0, T_f\right],$$

which implied that the system (11) is FTB with respect to $(c_1, c_2, T_f, R, \delta)$. The proof is completed.

Furthermore, based on the proof of Theorem 1, the following corollary can be obtained easily.

Corollary 1. Suppose that Assumption 1-3 are satisfied. For given positive constants c_1, δ, T_f, α, and a symmetric matrix $R > 0$, the closed-loop system (11) with $\Delta A_1(t) = \Delta A_2(t) = 0$ is FTB with respect to $(c_1, c_2, T_f, R, \delta)$, if there exist positive constant c_2 and symmetric positive definite matrices $P = diag(P_1, P_1), Q_1, Q_2$ and S such that

$$\bar{\Sigma} = \begin{bmatrix} \bar{\Sigma}_{11} & P\tilde{A}_2 & 0 & 0 & P\bar{D} \\ * & -(1-\mu)\bar{\tau}Q_1 & 0 & 0 & 0 \\ * & * & -\bar{\tau}Q_2 & 0 & 0 \\ * & * & * & -2\alpha Q_2 & 0 \\ * & * & * & * & -\alpha I \end{bmatrix} < 0, \quad (30)$$

$$2\lambda_1 c_1 + 2\lambda_2 c_1 \bar{\tau}^2 + \lambda_3 c_1 \bar{\tau}^4 + 2\delta(1 - e^{-\alpha T_f}) < 2\lambda_0 c_2 e^{-\alpha T_f},$$

where

$$\Sigma_{11} = P\bar{A}_1 + \bar{A}_1^T P + 2S + \bar{\tau}Q_1 + \bar{\tau}^3 Q_2 - \alpha P,$$

$$\bar{A}_1 = \begin{bmatrix} A_1 - BK & BK \\ 0 & A_1 - LC \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix},$$

$$\tilde{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}}, \tilde{Q}_1 = R^{-\frac{1}{2}} Q_1 R^{-\frac{1}{2}}, \tilde{Q}_2 = R^{-\frac{1}{2}} Q_2 R^{-\frac{1}{2}},$$

$$\lambda_0 = \lambda_{\min}(\tilde{P}), \lambda_1 = \lambda_{\max}(\tilde{P}), \lambda_2 = \lambda_{\max}(\tilde{Q}_1), \lambda_3 = \lambda_{\max}(\tilde{Q}_2).$$

Theorem 2. Suppose that Assumption 1-3 are satisfied. For given positive constants c_1, δ, T_f, α, and a symmetric matrix $R > 0$, the closed-loop system (11) is FTB with respect to $(c_1, c_2, T_f, R, \delta)$ and satisfies the cost function (13), if there exist positive constants c_2, κ and symmetric positive definite matrices $P = diag(P_1, P_1), Q_1, Q_2$, and S such that the following inequality and condition (15) hold:

$$\begin{bmatrix} \Sigma_{11} & P\bar{A}_2 & 0 & 0 & P\bar{D} & \bar{H}_1^T \\ * & -(1-\mu)\bar{\tau}Q_1 & 0 & 0 & 0 & \bar{H}_2^T \\ * & * & -\bar{\tau}Q_2 & 0 & 0 & 0 \\ * & * & * & -2\alpha Q_2 & 0 & 0 \\ * & * & * & * & -\alpha I & H_3^T \\ * & * & * & * & * & -\kappa I \end{bmatrix} < 0, \quad (31)$$

where

$$\Sigma_{11} = P\bar{A}_1 + \bar{A}_1^T P + 2S + \bar{\tau}Q_1 + \bar{\tau}^3 Q_2 - \alpha P.$$

Proof. Select the same Lyapunov-Krasovskii functional candidate as Theorem 1 and define the following function

$$J = \dot{V}(t) - \alpha V(t) - \alpha \hat{w}(t) w(t) + \kappa^{-1} \eta^T(t) \eta(t).$$

We have

$$J \le \zeta^T(t) \Sigma \zeta(t) + \kappa^{-1} \eta^T(t) \eta(t).$$

Applying Schur complement, from (31) we have

$$J < 0.$$

So, it follows that

$$\dot{V}(t) - \alpha V(t) \le \alpha w^T(t) w(t) - \kappa^{-1} \eta^T(t) \eta(t).$$

Multiplying the above inequality by $e^{-\alpha t}$ results in

$$\frac{d}{dt}\left(e^{-\alpha t} V(t)\right) < e^{-\alpha t}\left(\alpha w^T(t) w(t) - \kappa^{-1} \eta^T(t) \eta(t)\right).$$

By integrating the aforementioned inequality between 0 and T_f, under the assumed zero initial condition, we get

$$\begin{aligned} V(T_f) &\le e^{\alpha T_f} \int_0^{T_f} e^{-\alpha s} (\alpha w^T(s) w(s) - \kappa^{-1} \eta^T(t) \eta(t)) ds \\ &\le e^{\alpha T_f} \int_0^{T_f} (\alpha w^T(s) w(s) - \kappa^{-1} \eta^T(t) \eta(t)) ds. \end{aligned}$$

So, we have

$$\int_0^{T_f} \eta^T(t) \eta(t) dt \le \alpha\kappa \int_0^{T_f} w^T(t) w(t) dt.$$

Leting $\gamma = \sqrt{\alpha\kappa}$, we get that the closed-loop system (11) is FTB with respect to $(c_1, c_2, T_f, R, \delta)$ and satisfies the cost function (13). This completes the proof.

Remark 1. The conditions (14) and (15) are nonlinear matrix inequalities that difficult to be solved directly. To overcome this limitation, the following theorem give some solvable LMI-conditions.

Theorem 3. Suppose that Assumption 1-3 are satisfied. For given positive constants $c_1, \delta, T_f, \alpha, v_1$, and a symmetric matrix $R > 0$, the closed-loop system (11) is FTB with respect to $(c_1, c_2, T_f, R, \delta)$, if there exist positive constants $c_2, v_2, \beta_1, \beta_2, \beta_3$, symmetric positive-definite matrices $\bar{Q}_1, \bar{Q}_2, \bar{S}$ and $X = diag(X_1, X_1))$, real matrices Y, G, such that the following LMIs hold:

$$\tilde{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_1 & \tilde{\Sigma}_2 \\ * & \tilde{\Sigma}_3 \end{bmatrix} < 0, \quad (32)$$

$$\beta_1 R^{-1} < X < R^{-1}, \beta_2 R^{-1} < \bar{Q}_1 < R^{-1}, \beta_3 R^{-1} < \bar{Q}_2 < R^{-1}, \quad (33)$$

$$\begin{bmatrix} \delta(1 - e^{-\alpha T_f}) - c_2 \bar{\tau}^2 e^{-\alpha T_f} & c_1^{\frac{1}{2}} & \bar{\tau} c_1^{\frac{1}{2}} & \bar{\tau}^2 c_1^{\frac{1}{2}} \\ * & -\beta_1 & 0 & 0 \\ * & * & -\beta_2 & 0 \\ * & * & * & -2\beta_3 \end{bmatrix} < 0, \quad (34)$$

where

$$\tilde{\Sigma}_1=\begin{bmatrix}\tilde{\Sigma}_{11} & \tilde{A}_2\bar{Q}_1 & 0 & 0 & \bar{D}\\ * & -(1-\mu)\bar{\tau}\bar{Q}_1 & 0 & 0 & 0\\ * & * & -\bar{\tau}\bar{Q}_2 & 0 & 0\\ * & * & * & -2\alpha\bar{Q}_2 & 0\\ * & * & * & * & -\alpha I\end{bmatrix},$$

$$\tilde{\Sigma}_2=\begin{bmatrix}\bar{M}_1 & v_1X\bar{N}_1 & \bar{M}_2 & 0 & X & X\\ 0 & 0 & 0 & \tilde{\Sigma}_{29} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\tilde{\Sigma}_3=diag(-v_1I,-v_1I,-v_2I,-v_2I,-\bar{\tau}^{-1}\bar{Q}_1,-\bar{\tau}^{-3}\bar{Q}_2).$$

$$\bar{M}_1=\begin{bmatrix}M_1\\ M_1\end{bmatrix},\bar{M}_2=\begin{bmatrix}M_2\\ M_2\end{bmatrix},\bar{N}_1=\begin{bmatrix}N_1 & 0\end{bmatrix},\tilde{A}_2=\begin{bmatrix}A_2 & 0\\ 0 & A_2\end{bmatrix},$$
$$\tilde{\Sigma}_{11}=\tilde{A}_1X+X\tilde{A}_1^T+2\bar{S}-\alpha X,\bar{N}_2=\begin{bmatrix}N_2 & 0\end{bmatrix},$$
$$\tilde{\Sigma}_{29}=v_2\bar{Q}_1\bar{N}_2^T,\ \tilde{A}_1X=\begin{bmatrix}A_1X_1-BY & BY\\ 0 & A_1X_1-GC\end{bmatrix},$$
$$CX_1=\hat{X}_1C,\quad X_1=V\begin{bmatrix}\hat{X}_{11} & 0\\ 0 & \hat{X}_{22}\end{bmatrix}V^T.$$

Furthermore, if the conditions (32)-(34) have feasible solutions, the desired observer gain L and controller gain K can
be given by $K=YX_1^{-1},L=G\hat{X}_1^{-1}$.

Proof. Considering the uncertainties in equality (14), (14) can be rewritten as

$$\Sigma=\bar{\Sigma}+\Delta\bar{\Sigma}<0, \tag{35}$$

where $\bar{\Sigma}$ is given by (30) and

$$\Delta\bar{\Sigma}=\begin{bmatrix}P\Delta\tilde{A}_1+\Delta\tilde{A}_1^TP & P\Delta\tilde{A}_2 & 0 & 0 & 0\\ * & 0 & 0 & 0 & 0\\ * & * & 0 & 0 & 0\\ * & * & * & 0 & 0\\ * & * & * & * & 0\end{bmatrix},$$

$$\Delta\tilde{A}_1=\begin{bmatrix}\Delta A_1 & 0\\ \Delta A_1 & 0\end{bmatrix},\quad \Delta\tilde{A}_2=\begin{bmatrix}\Delta A_2 & 0\\ \Delta A_2 & 0\end{bmatrix}.$$

Noticing the uncertainties which described as the form in (2), one has

$$\Delta\bar{\Sigma}=\Theta_1F(t)\Gamma_1+\left(\Theta_1F(t)\Gamma_1\right)^T+\Theta_2F(t)\Gamma_2+\left(\Theta_2F(t)\Gamma_2\right)^T.$$

where

$$\Theta_1=\begin{bmatrix}(P\bar{M}_1)^T & 0 & 0 & 0 & 0\end{bmatrix}^T,\Gamma_1=\begin{bmatrix}\bar{N}_1 & 0 & 0 & 0 & 0\end{bmatrix},$$
$$\Theta_2=\begin{bmatrix}(P\bar{M}_2)^T & 0 & 0 & 0 & 0\end{bmatrix}^T,\Gamma_2=\begin{bmatrix}0 & \bar{N}_2 & 0 & 0 & 0\end{bmatrix}.$$

Using Lemma 1, we can obtain

$$\Delta\bar{\Sigma}\le v_1^{-1}\Theta_1\Theta_1^T+v_1\Gamma_1^T\Gamma_1+v_2^{-1}\Theta_2\Theta_2^T+v_2\Gamma_2^T\Gamma_2. \tag{36}$$

Then, the inequality (35) can be guaranteed by

$$\bar{\Sigma}+v_1^{-1}\Theta_1\Theta_1^T+v_1\Gamma_1^T\Gamma_1+v_2^{-1}\Theta_2\Theta_2^T+v_2\Gamma_2^T\Gamma_2<0. \tag{37}$$

Applying Schur's complement, (37) holds if and only if the following inequality holds:

$$\begin{bmatrix}\bar{\Sigma}_1 & \bar{\Sigma}_2\\ * & \bar{\Sigma}_3\end{bmatrix}<0, \tag{38}$$

where

$$\bar{\Sigma}_1=\begin{bmatrix}\bar{\Sigma}_{11} & P\tilde{A}_2 & 0 & 0\\ * & -(1-\mu)\bar{\tau}Q_1 & 0 & 0\\ * & * & -\bar{\tau}Q_2 & 0\\ * & * & * & -2\alpha Q_2\end{bmatrix},$$

$$\bar{\Sigma}_2=\begin{bmatrix}P\bar{D} & P\bar{M}_1 & v_1\bar{N}_1^T & P\bar{M}_2 & 0\\ 0 & 0 & 0 & 0 & v_2\bar{N}_2^T\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\bar{\Sigma}_3=diag(-\alpha I,-v_1I,-v_1I,-v_2I,-v_2I).$$

From Lemma 2, there exists an appropriate dimensions matrix $\hat{X}_1$ such that $CX_1=\hat{X}_1C$. Let $G=L\hat{X}_1$.

Pre- and post-multiplying above result inequality (38) by block-diagonal matrix $diag(P^{-1},I,I,I,I,I,I,I)$, and letting

$$\begin{bmatrix}X_1 & 0\\ 0 & X_1\end{bmatrix}X=P^{-1},\quad Y=KX_1,$$

one get

$$\begin{bmatrix}\hat{\Sigma}_1 & \hat{\Sigma}_2\\ * & \hat{\Sigma}_3\end{bmatrix}<0, \tag{39}$$

where

$$\hat{\Sigma}_1=\begin{bmatrix}\hat{\Sigma}_{11} & \tilde{A}_2 & 0 & 0 & \bar{D}\\ * & -(1-\mu)\bar{\tau}Q_1 & 0 & 0 & 0\\ * & * & -\bar{\tau}Q_2 & 0 & 0\\ * & * & * & -2\alpha Q_2 & 0\\ * & * & * & * & -\alpha I\end{bmatrix},$$

$$\hat{\Sigma}_2=\begin{bmatrix}\bar{M}_1 & v_1X\bar{N}_1 & \bar{M}_2 & 0\\ 0 & 0 & 0 & v_2\bar{N}_2^T\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$\hat{\Sigma}_3=diag(-v_1I,-v_1I,-v_2I,-v_2I),$$
$$\hat{\Sigma}_{11}=\tilde{A}_1X+X\tilde{A}_1^T+2XSX+\bar{\tau}XQ_1X+\bar{\tau}^3XQ_2X-\alpha X,$$

Applying the schur's complement, we have that (38) holds if following inequality holds:

$$\tilde{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_1 & \tilde{\Sigma}_2 \\ * & \tilde{\Sigma}_3 \end{bmatrix} < 0, \quad (40)$$

where

$$\tilde{\Sigma}_1 = \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{A}_2 & 0 & 0 & \bar{D} \\ * & -(1-\mu)\bar{\tau}Q_1 & 0 & 0 & 0 \\ * & * & -\bar{\tau}Q_2 & 0 & 0 \\ * & * & * & -2\alpha Q_2 & 0 \\ * & * & * & * & -\alpha I \end{bmatrix},$$

$$\tilde{\Sigma}_2 = \begin{bmatrix} \bar{M}_1 & v_1 X\bar{N}_1 & \bar{M}_2 & 0 & X & X \\ 0 & 0 & 0 & v_2\bar{N}_2^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\Sigma}_3 = diag(-v_1 I, -v_1 I, -v_2 I, -v_2 I, -\bar{\tau}^{-1}\bar{Q}_1, -\bar{\tau}^{-3}\bar{Q}_2),$$
$$\hat{\Sigma}_{11} = \tilde{A}_1 X + X\tilde{A}_1^T + 2\bar{S} - \alpha X,$$
$$\bar{Q}_1 = Q_1^{-1}, \bar{Q}_2 = Q_2^{-1}, \bar{S} = XSX.$$

Pre- and post-multiplying above result inequality (40) by block-diagonal matrix $diag(I, \bar{Q}_1, \bar{Q}_2, \bar{Q}_2, I, I, I, I, I, I)$, the inequality (32) can be obtained by means of simple manipul-tion.

Let

$$\tilde{X} = R^{\frac{1}{2}} X R^{\frac{1}{2}}, \hat{Q}_1 = R^{\frac{1}{2}}\bar{Q}_1 R^{\frac{1}{2}}, \hat{Q}_2 = R^{\frac{1}{2}}\bar{Q}_2 R^{\frac{1}{2}},$$

and set

$$\beta_1 \le \lambda_{\min}(\tilde{X}), \quad \lambda_{\max}(\tilde{X}) < 1, \ \beta_2 \le \lambda_{\min}(\hat{Q}_1),$$
$$\lambda_{\max}(\hat{Q}_1) < 1, \ \beta_3 \le \lambda_{\min}(\hat{Q}_2), \ \lambda_{\max}(\hat{Q}_2) < 1,$$

then (15) can be held by the following inequality:

$$c_1\beta_1^{-1} + c_1\bar{\tau}^2\beta_2^{-1} + (1/2)c_1\bar{\tau}^4\beta_3^{-1} + \delta(1 - e^{-\alpha T_f}) < c_2 e^{-\alpha T_f}. \quad (41)$$

Using Schur's complement and eigenvalue transformation, we can get LMIs (33) and (34). This completes the proof.

According to Theorem 2, the following theorem can be obtained.

Theorem 4. Suppose that Assumption 1-3 are satisfied. For given positive constants c_1, δ, T_f, α, and a symmetric matrix $R > 0$, the closed-loop system (11) is FTB with respect to $(c_1, c_2, T_f, R, \delta)$ and satisfies the cost function (13), if there exist positive constants $c_2, v_1, v_2, \beta_1, \beta_2, \beta_3, \kappa$, symmetric positive-definite matrices $\bar{Q}_1, \bar{Q}_2, \bar{S}$ and $X = diag(X_1, X_1)$), real matrices *Y*, *G*, such that the following matrix inequality and conditions (33) and (34) hold:

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix} < 0, \quad (42)$$

where

$$\Pi_1 = \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{A}_2\bar{Q}_1 & 0 & 0 & \bar{D} & X\bar{H}_1^T \\ * & -(1-\mu)\bar{\tau}\bar{Q}_1 & 0 & 0 & 0 & \bar{Q}_1\bar{H}_2^T \\ * & * & -\bar{\tau}\bar{Q}_2 & 0 & 0 & 0 \\ * & * & * & -2\alpha\bar{Q}_2 & 0 & 0 \\ * & * & * & * & -\alpha I & H_3^T \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} \bar{M}_1 & v_1 X\bar{N}_1 & \bar{M}_2 & 0 & X & X \\ 0 & 0 & 0 & \tilde{\Sigma}_{29} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_3 = diag(-\kappa I, -v_1 I, -v_1 I, -v_2 I, -v_2 I, -\bar{\tau}^{-1}\bar{Q}_1, -\bar{\tau}^{-3}\bar{Q}_2).$$

$$\bar{M}_1 = \begin{bmatrix} M_1 \\ M_1 \end{bmatrix}, \bar{M}_2 = \begin{bmatrix} M_2 \\ M_2 \end{bmatrix}, \bar{N}_1 = \begin{bmatrix} N_1 & 0 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix},$$

$$\tilde{\Sigma}_{11} = \tilde{A}_1 X + X\tilde{A}_1^T + 2\bar{S} - \alpha X, \bar{N}_2 = \begin{bmatrix} N_2 & 0 \end{bmatrix},$$

$$\tilde{\Sigma}_{29} = v_2\bar{Q}_1\bar{N}_2^T, \ \tilde{A}_1 X = \begin{bmatrix} A_1 X_1 - BY & BY \\ 0 & A_1 X_1 - GC \end{bmatrix},$$

$$CX_1 = \hat{X}_1 C, \quad X_1 = V\begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & \hat{X}_{22} \end{bmatrix} V^T.$$

Furthermore, if the conditions (33), (34) and (42) have feasible solutions, the desired observer gain *L* and controller gain *K* can be given by $K = YX_1^{-1}, L = G\hat{X}_1^{-1}$.

Proof. The proof is similar to the proof of Theorem 3. Thus, it is omitted.

According to Theorem 2, the following corollary can be obtained.

Corollary 2. Suppose that Assumption 1-3 are satisfied.

For given positive constants c_1, δ, T_f, α, and a symmetric matrix $R > 0$, the closed-loop system (11) with $\Delta A_1(t) = \Delta A_2(t) = 0$ is FTB with respect to $(c_1, c_2, T_f, R, \delta)$ and satisfies the cost function (13), if there exist positive constants $c_2, \beta_1, \beta_2, \beta_3, \kappa$, symmetric positive-definite matrices $\bar{Q}_1, \bar{Q}_2, \bar{S}$ and $X = diag(X_1, X_1)$, real matrices *Y*, *G*, such that the following matrix inequality and conditions (33) and (34) hold:

$$\begin{bmatrix} \tilde{\Sigma}_1 & \tilde{\Sigma}_2 \\ * & \tilde{\Sigma}_3 \end{bmatrix} < 0, \quad (43)$$

where

$$\tilde{\Sigma}_1 = \begin{bmatrix} \tilde{\Sigma}_{11} & \tilde{A}_2\bar{Q}_1 & 0 & 0 & \bar{D} \\ * & -(1-\mu)\bar{\tau}\bar{Q}_1 & 0 & 0 & 0 \\ * & * & -\bar{\tau}\bar{Q}_2 & 0 & 0 \\ * & * & * & -2\alpha\bar{Q}_2 & 0 \\ * & * & * & * & -\alpha I \end{bmatrix},$$

$$\tilde{\Sigma}_2 = \begin{bmatrix} X\bar{H}_1^T & X & X \\ \bar{Q}_1\bar{H}_2^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ H_3^T & 0 & 0 \end{bmatrix}, \tilde{\Sigma}_3 = diag(-\kappa I, -\bar{\tau}^{-1}\bar{Q}_1, -\bar{\tau}^{-3}\bar{Q}_2).$$

$$\tilde{A}_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix}, \tilde{\Sigma}_{11} = \tilde{A}_1 X + X\tilde{A}_1^T + 2\bar{S} - \alpha X, CX_1 = \hat{X}_1 C,$$

$$\tilde{A}_1 X = \begin{bmatrix} A_1X_1 - BY & BY \\ 0 & A_1X_1 - GC \end{bmatrix}, X_1 = V\begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & \hat{X}_{22} \end{bmatrix}V^T.$$

Furthermore, if the conditions (33), 34) and (43) have feasible solutions, the desired observer gain L and controller gain K can be given by $K = YX_1^{-1}, L = G\hat{X}_1^{-1}$.

Remark 2. Linear and nonlinear state-feedback control approaches for different forms of one-sided (or quasi-one-sided) Lipschitz nonlinear system have been investigated in recent works (Song and He, 2015; Cai et al., 2014; Cai et al. 2015; Fu et al., 2013). These techniques cannot be applied to the stabilization of nonlinear systems and finite-time bounded of non-linear systems if the full state vector is not known. The present work, contrastingly, employs observers for state-estimation of nonlinear systems, the proposed control strategy utilizing the estimated states rather than the true values. We derive a finite-time bounded condition, which guarantees simultaneous finite-time bounded of the system's state vector and the estimation error, for the overall closed-loop system.

Remark 3. (Dong et al., 2015) considered the problem of state observer design for a class of nonlinear dynamical systems with interval time-varying delay. (Dong et al., 2017b) investigated the nonlinear observer design for one-sided Lipschitz systems with time-varying delay and uncertainties. But, the problem of finite-time bounded for nonlinear system was not considered in (Dong et al., 2015, 2017b). In this paper, we study the problem of finite-time bounded observer-based control for a class of quasi-one-sided Lipschitz nonlinear systems with time-varying delay.

4. SIMULATION EXAMPLES

In this section, we give simulation examples to illustrate the effectiveness of the proposed methods.

Example 1. Consider system (1) with the following parameters:

$$A_1 = \begin{bmatrix} -322 & -134 \\ -150 & -300 \end{bmatrix}, A_2 = \begin{bmatrix} -43.25 & 0 \\ 0 & 54.25 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, f(t,x) = \begin{bmatrix} 0 \\ 0.6\sin(x_2) \end{bmatrix},$$

$$g(t, x_\tau) = 0, M_1 = \begin{bmatrix} 0.07 \\ 0.07 \end{bmatrix}, N_1 = \begin{bmatrix} 0.1 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.03 \\ 0.03 \end{bmatrix}, N_2 = \begin{bmatrix} 0.01 & 0.15 \end{bmatrix},$$

Observing that

$$\langle P_1 f(t,x), x \rangle \le 0.6x^T P_1 x,$$

we have $S_1 = 0.6P_1$, then $\bar{S} = XSX = 0.6X$.

Taking $c_1 = 0.01, \alpha = 1.8, \beta_1 = 0.1, \beta_2 = 0.2, \beta_3 = 0.1, T_f = 3s,$

$$\mu = 0.7, v_1 = 0.002, v_2 = 0.001, \delta = 0.1, R = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}.$$

Using the Matlab LMI control toolbox, we solve (32)-(34) and obtain a set of feasible solutions as follows:

$$X = \begin{bmatrix} 445.9752 & -25.6030 & 0 & 0 \\ -25.6030 & 462.8791 & 0 & 0 \\ 0 & 0 & 445.9752 & -25.6030 \\ 0 & 0 & -25.6030 & 462.8791 \end{bmatrix},$$

$$Y = \begin{bmatrix} -561490 & -467570 \end{bmatrix},$$

$$\bar{Q}_1 = \begin{bmatrix} 1.6493 & -0.0755 & 0.2227 & -0.0069 \\ -0.0755 & 1.4901 & -0.0069 & 0.2566 \\ 0.2227 & -0.0069 & 1.6458 & -0.1158 \\ -0.0069 & 0.2566 & -0.1158 & 1.6517 \end{bmatrix},$$

$$\bar{Q}_2 = \begin{bmatrix} 0.1478 & 0.0052 & 0.0128 & -0.0016 \\ 0.0052 & 0.1434 & -0.0016 & 0.0080 \\ 0.0128 & -0.0016 & 0.1458 & -0.0001 \\ -0.0016 & 0.0080 & -0.0001 & 0.1499 \end{bmatrix},$$

$$G = \begin{bmatrix} -431420 \\ -119580 \end{bmatrix}, \quad \hat{X}_1 = 445.9752.$$

The observer gain and state feedback controller gain can be given by

$$K = \begin{bmatrix} -1321.2 & -1083.2 \end{bmatrix}, L = \begin{bmatrix} -967.3567 \\ -268.1263 \end{bmatrix}, c_2 = 47.4192.$$

We assume the disturbance input $w(t) = \begin{bmatrix} 0.8e^{-5t} & 0 \end{bmatrix}^T$. Fig.1 shows the closed-loop system state. Fig.2 shows the time history of $x^T(t)Rx(t)$. From these figures, it is easy to see that the closed-loop system is finite-time bounded via the obtained observer-based controllers.

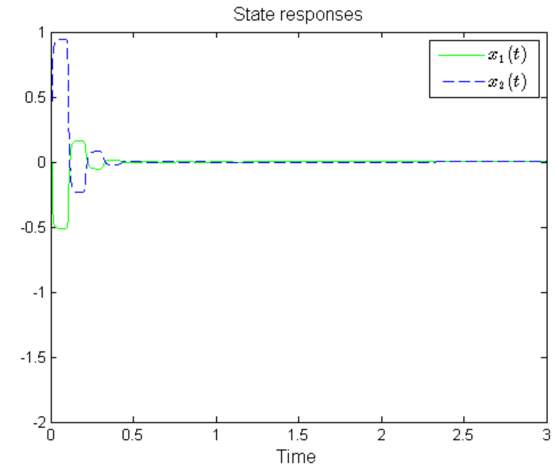

Fig. 1. The trajectories of $x_1(t)$ and $x_2(t)$ in Example 1.

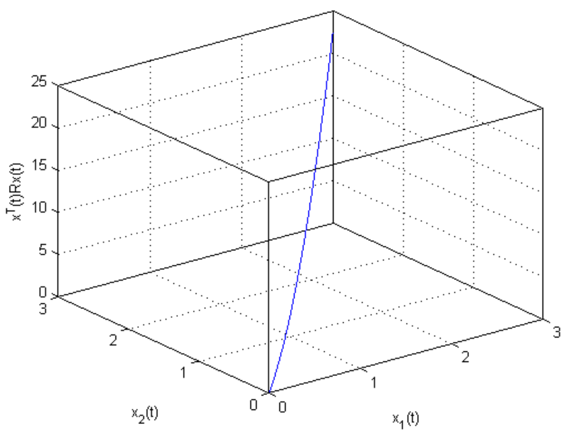

Fig. 2. Time histories of $x^T(t)Rx(t)$ during the finite-time interval $[0,T_f]$ in Example 1.

Example 2. Consider system (1) with $\Delta A_1(t)=0, \Delta A_2(t)=0$ and the following parameters:

$$A_1=\begin{bmatrix}-120 & -14\\ -15 & -70\end{bmatrix}, A_2=\begin{bmatrix}-34.25 & 0\\ 0 & -24.25\end{bmatrix}, B=\begin{bmatrix}0\\ 0.1\end{bmatrix},$$
$$H_1=[0 \quad 0.1], H_2=[0 \quad 0.1], H_3=[0 \quad 0], g(t,x_\tau)=0,$$
$$C=[0.1 \quad 0], D=\begin{bmatrix}0.01 & 0\\ 0 & 0.01\end{bmatrix}, f(t,x)=\begin{bmatrix}0\\ 0.6\sin(x_2)\end{bmatrix}.$$

We have $S_1=0.6P_1$, then $\bar{S}=XSX=0.6X$.

Taking $c_1=0.01, \alpha=1.6, \beta_1=0.1, \beta_2=0.2,$

$$\beta_3=0.1, \mu=0.7, \bar{\tau}=0.4, \delta=0.1, R=\begin{bmatrix}1 & 0.2\\ 0.2 & 1\end{bmatrix}, T_f=2s.$$

Using the Matlab LMI control toolbox, we solve (40), (33) and (34) and obtain a set of feasible solutions as follows:

$$X=\begin{bmatrix}5.5785 & -0.5706 & 0 & 0\\ -0.5706 & 11.3220 & 0 & 0\\ 0 & 0 & 5.5785 & -0.5706\\ 0 & 0 & -0.5706 & 11.3220\end{bmatrix},$$
$$Y=[-726.5204 \quad -710.0484],$$

$$\bar{Q}_1=\begin{bmatrix}0.0876 & 0.0098 & 0.0233 & 0.0019\\ 0.0098 & 0.1881 & 0.0019 & 0.0500\\ 0.0233 & 0.0019 & 0.1267 & -0.0013\\ 0.0019 & 0.0500 & -0.0013 & 0.2228\end{bmatrix},$$
$$\bar{Q}_2=\begin{bmatrix}75.7052 & 0.0365 & 25.2437 & 0.0285\\ 0.0365 & 75.7227 & 0.0285 & 25.2462\\ 25.2437 & 0.0285 & 62.9374 & 0.0350\\ 0.0285 & 25.2462 & 0.0350 & 62.9663\end{bmatrix},$$
$$G=\begin{bmatrix}1942.7\\ -1471.4\end{bmatrix}, \hat{X}_1=5.5785, \kappa=295.4756.$$

According to Corollary 2, the system (11) is FTB with respect to (c_1,c_2,T_f,R,δ) and satisfies the cost function (13) with γ = 21.7431. The observer gain and state feedback controller gain can be given by

$$K=[-137.3588 \quad -69.6367],$$
$$L=10^8\begin{bmatrix}348.2505\\ -263.7622\end{bmatrix}, c_2=43.5474.$$

We assume the disturbance input $w(t)=\begin{bmatrix}0.8e^{-5t} & 0\end{bmatrix}^T$. Fig.3 shows the closed-loop system state. Fig.4 shows the time history of $x^T(t)Rx(t)$. From these figures, it is easy to see that the closed-loop system is finite-time bounded via the obtained observer-based controllers.

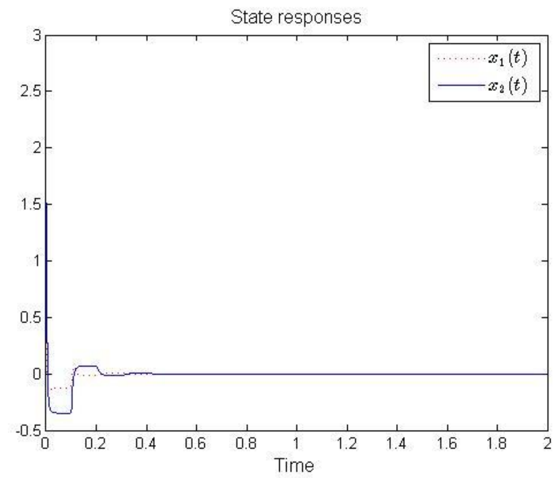

Fig. 3. State trajectories of the closed-loop system in Example 2.

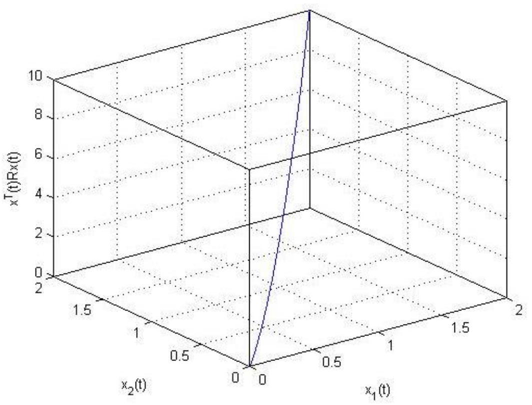

Fig. 4. Time histories of $x^T(t)Rx(t)$ during the finite-time interval $\left[0, T_f\right]$ in Example 2.

5. CONCLUSION

In this paper, we investigated the finite-time bounded observer-based control problem for a class of continuous-time nonlinear systems with time-varying delay, time-varying norm-bounded parameter uncertainties and admissible external disturbances. The nonlinearities are assumed to satisfy the quasi-one-sided Lipschitz nonlinear constraint conditions. Some sufficient conditions for the finite-time bounded of the resulted closed-loop system are developed. On this basis, robust observer-based controller synthesis strategy under parametric uncertainties is proposed. Illustrative examples are given to illustrate the effectiveness and applicability of the proposed design method.

ACKNOWLEDGMENTS

This work was supported by the Natural Science Foundation of Tianjin under Grant No. 18JCYBJC88000 and the National Nature Science Foundation of China under Grant No. 61873186.

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