A New Sufficient Condition for Stability Analysis of Nonlinear Systems Based on Differential Transform Method (DTM)

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Abstract: There are several methods to determine the stability of nonlinear systems that are fully described in control engineering resources and nonlinear control systems. The methods presented so far are not easy to analyze for the asymptotic stability of complex high-dimensional nonlinear dynamical systems. It is natural to extend the analysis of the stability of nonlinear systems and provide a method that can easily determine the asymptotic stability of a wide range of nonlinear systems, regardless of the dimensions and complexity of the system. In this paper, a new sufficient condition for asymptotic Lyapunov stability analysis of continuous nonlinear dynamical systems in the general case based on LLE is proposed. This method can easily support the asymptotic stability of a large variety of continuous high-dimensional nonlinear dynamical systems based on system parameters. Using the method outlined in this paper, in order to analyze the stability of nonlinear systems, it is easy to determine the range of system parameters where the system has stable conditions. Some numerical examples are provided to show the effectiveness of the main results.

Keywords: Differential transform method (DTM), largest Lyapunov exponent (LLE), matrix computation, matrix norm, nonlinear dynamical system, stability analysis.

1. INTRODUCTION

Surely, the most important characteristic of a control system can be its stability. The complexity of the stability analysis of dynamic systems changes with the variation of system models from linear to nonlinear systems. Therefore, of the most significant themes in the analysis of nonlinear systems, one is their stability. In recent years, much research has been done in the field of stability analysis in nonlinear dynamic systems.

A prominent method for analyzing the stability of nonlinear systems is Lyapunov's theory of stability (Massera, 1949; Rouche et al., 1997). According to the definition of stability, an equilibrium point is globally asymptotically stable (GAS) if it is stable in the sense of Lyapunov and the state converges to an equilibrium point for any initial state. One of the Lyapunov stability methods is the Lyapunov linearization method, in which a linear approximation of nonlinear system is considered around the equilibrium point and is examine its stability (Massera, 1949). This stability approach is considered only around the system equilibrium point and is not a suitable method for analyzing the stability of the nonlinear system. The major disadvantage of this method is the elimination of nonlinear dynamics of the system as well as the locality of stability.

Another method is the Lyapunov's direct method. Lyapunov functions play a vital role in control design and dynamic stability theory. It is noticeable that no general valid method is accessible for finding Lyapunov functions. The total energy stored in the physical system can be used as a Lyapunov function candidate, but the Lyapunov functions can be considered as a generalization of energy functions for nonlinear dynamical systems (Gruyitch, 1992; Rouche et al., 1997).

A robust method for the stability analysis of linear systems is the Lyapunov method, which includes Lyapunov's first method and Lyapunov's second method. The Lyapunov's second method is used to the stability analysis of nonlinear dynamic systems (Gruyitch, 1992; Rouche et al., 1997). A Lyapunov function is a scalar function defined on the phase space, which can be used to prove the stability of an equilibrium point. The Lyapunov function method is applied to study the stability of various differential equations and systems. A Lyapunov function can be defined as a scalar function in the phase space, which can be used in various applications such as stability, convergence analysis, design of model reference adaptive systems, and etc. It is difficult to find the Lyapunov function for nonlinear systems (Zhang et al., 2017). In general, there is no definite method for determining the Lyapunov function, and it should often be eliminated. Recently, many methods have been proposed to find the Lyapunov function such as Krasovskii method, modified Krasovskii method, variable gradient approach, backstepping method, modified backstepping method and dynamic programming, etc (Schultz and Gibson, 1962; Al-Bayaty, 2011; Ojha and Khandelwal, 2015; Liu and Zhang, 2017; Wang et al., 2017; Zhang et al., 2017).

The major problem with Krasovskii method is that if the number of system states is high, solving equations and determining the conditions in this way is difficult (Zhang et al., 2017). It is easier to determine the conditions in the modified Krasovskii method than that in the Krasovskii method, but it needs a large amount of computation (Al-Bayaty, 2011). The major disadvantage of this method is the elimination of nonlinear dynamics of the system as well as the locality of stability.

In the variable gradient method, the solution of the obtained equations is not simple and sometimes leads to equations, which are difficult to solve. Also, the solution obtained by this method does not differ much from the linearization method (Schultz and Gibson, 1962; Liu and Zhang, 2017). One of the advantages of the backstepping and modified backstepping methods is to prevent the removal of nonlinear dynamics of the system, but the main disadvantages of this method can be the complexity of the calculation, especially in high- dimensional dynamic systems (Ojha and Khandelwal, 2015; Wang et al., 2017).

In linear systems, the presence of all negative eigenvalues indicates stability while the presence of a single positive eigenvalue for sure indicates exponential instability. Next the result could be expanded to linearized nonlinear systems. The unique equilibrium point of the diagonalizable systems is globally asymptotically stable if a particular set of nonlinear eigenvalues is not positive at each point in the state space (Kawano and Ohtsuka, 2015). Degree of complexity of the analysis of the dynamic systems increases by changing math model from unchanging linear systems with time to changeable and nonlinear systems with time. Therefore, in recent years researchers have done the analysis of the stabilization for a nonlinear system. For example, Li et al. proposed a novel method based on mode-dependent average dwell time (MDADT) method, for the stability analysis of discrete-time switching nonlinear systems (Kawano and Ohtsuka, 2015; Li et al., 2017). In this paper as a case study, Takagi-Sugeno (TS) fuzzy model is used to approximate the switching nonlinear system (Li et al., 2017).

One of the new methods for analyzing Lyapunov is the use of the averaging functions and Steklov's averaging method (Pogromsky and Matveev, 2016). This approach is proposed for incremental stability analysis of nonlinear systems. The disadvantages of this method are the complexity and inaccuracy in the use of high-dimensional dynamical systems (Pogromsky and Matveev, 2016). Angulo et al., proposed a novel method for qualitative stability of nonlinear systems based on the negative feedback interconnection (Angulo and Slotine, 2017). The disadvantages of this method are the complexity in the use of high-dimensional dynamical systems (Angulo and Slotine, 2017). Li et al., used the Lyapunov approach for the stability analysis of nonlinear impulsive systems with delay (Li et al., 2018). In (Ren and Xiong, 2017), the Lyapunov function based on fixed dwell-time condition is exerted on the stability analysis of impulsive stochastic dynamic systems. In (Tuan and Trinh, 2018), the Lyapunov's first method is used for analyzing the stability of time delayed nonlinear fractional systems. They proposed that the expressed system is stable if the linear system around the equilibrium point is stable (Tuan and Trinh, 2018). In (Molchanov and Liu, 2002), the robust absolute stability discrete time nonlinear systems with time varying interval matrices in the linear part is considered based on the variational method and the piecewise-linear Lyapunov function (Molchanov and Liu, 2002). Jiao et al., considered the multiple Lyapunov functions for the stability analysis of switching nonlinear systems (Jiao et al., 2016). In (Liang et al., 2017), the general Lyapunov functions is considered to the stability analysis of nonlinear Multiagent systems (Liang et al., 2017). In (Aleksandrov et al., 2015), the Lyapunov method is exerted on the stability analysis of nonlinear systems via decomposition (Aleksandrov et al., 2015). Peet, used the polynomial Lyapunov function to analyze the stability of nonlinear systems (Peet, 2009). This method is used for nonlinear systems with nonlinear polynomial functions (Peet, 2009). In (Feng et al., 2015), the iterative control laws based on heuristic dynamic programming are proposed for analyzing the stability of closed loop nonlinear systems. The modern methods are utilized to do stability analysis of a class of Lipschitz nonlinear impulsive systems based on vector Lyapunov function by using Linear Matrix Inequalities (Rios et al., 2017). Xu et al., proposed a new approach for the stability analysis of elastic elliptical cylindrical shells under the uniform bending (Xu et al., 2017). In [25], the nonlinear active disturbance rejection control (ADRC) is utilized to address the stability analysis of the fast tool servo system (Li et al., 2015).

Ha et al., proposed the energy estimate method for the nonlinear stability of the incoherent solution to the Kuramoto–Sakaguchi equation [26]. Lefloch et al., studied the stability of a certain category of nonlinear systems (LeFloch and Sormani, 2015). In (Yin et al., 2011), the stochastic Lyapunov method is used to do stability analysis of stochastic nonlinear systems. Also, the state feedback controller is applied for stabilizing these systems (Yin et al., 2011). The largest Lyapunov exponent (LLE) is applied to the stability analysis of linear and nonlinear dynamical systems. The Lyapunov exponent (LE) is a quantitative criterion that determines the deviation or convergence of the two contiguous trajectories during the time evolution and has a direct relation to the stability of system state. The negative value of LLE shows that the system is stable. In (Zevin, 2015), the LLE is applied to the stability analysis of time delayed linear systems using eigenvalue of the system. In (Gavilan-Moreno and Espinosa-Paredes, 2016), the LLE is considered to analyze the instability state of boiling water reactors (Gavilan-Moreno and Espinosa-Paredes, 2016). In (Yunping et al., 2013), the LLE is proposed to analyze the stability of passive bipedal robot (Yunping et al., 2013). In (Banerjee et al., 2014), the LLE is used for the transient stability analysis of the NE 39 bus power system. Also, this method is used for the transient stability analysis of the IEEE-39 bus test system (Huang et al., 2018). As an alternative, the differential transform method (DTM) has been suggested as an analytical approach to solve differential equations (Arikoglu and Ozkol, 2005; Odibat, 2008; Ghomi Taheri et al., 2016). In (Ghomi Taheri et al., 2016), the authors presented an analytical method for calculating the LE in general terms, and, for example, examined it in two Lorenz system and Colpitts oscillator. In this paper, the chaotic Colpitts oscillator is analyzed and the system behavior has been investigated for various system parameters by the analytical method based on LE. Then, the desired circuit is constructed and the implementation results are compared with the analytical results. In all studies of the use of the LE in the stability analysis of systems, the LEs of continuous dynamical systems are estimated by numerical methods. As we have seen, a lot of research has been done on the asymptotic stability analysis of nonlinear systems based on the Lyapunov function. The disadvantages of this method include the difficulty of computing the shared Lyapunov function, the complexity of the calculation, as well as the inability to analyze the stability of the systems with high dimensions. In some cases, the Lyapunov function cannot be found for a nonlinear system, which is one of the main weaknesses of these methods. On the other hand, in a recent research on the stability analysis of a nonlinear system, new methods have been proposed for the stability analysis on a particular class of nonlinear systems, one of the main weaknesses of all the proposed methods, in the analysis of the stability of complex high-dimensional systems. In this paper, a simple method is presented to analyze the stability of nonlinear systems with unknown parameters. On the other hand, this method is capable of analyzing the stability of nonlinear systems with a high degree of complexity and high-dimensional systems.

In this note, a new and simple benchmark for analyzing the stability of continuous nonlinear systems is presented based on the estimation of the LE using DTM method. The rest of the paper is organized as follows. Section 2, the review on DTM technique is proposed. Section 3 is devoted to a novel method for calculating the LLE to do stability analysis based on DTM method. In section 4, the application of the theorem presented in section 3 is investigated in three different nonlinear systems. Discussion and conclusion about the method presented is taken into consideration in section 5.

2. REVIEW OF THE DIFFERENTIAL TRANSFORM METHOD (DTM)

A differential transformation method is an analytic method in the form of a polynomial whose computational domain is much shorter than that of the Taylor series. The *k*th derivative of the function $f(t)$ denotes its differential transform, that is defined as (Arikoglu and Ozkol, 2005; Odibat, 2008):

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(t)}{d t^k}\right]_{t=t_0}, \tag{1}$$

where $F(k)$ is the differential transformation of the function $f(t)$ and $t = t_0$ is the point at which the Taylor series expansion of the function is obtained around it. Because the function $f(t)$ is considered as a finite series, it can be expressed as:

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^k. \tag{2}$$

DTM technique actually represents signal variations. This technique is based on the expansion of the Taylor series. Some of the mathematical properties of the DTM method are shown in Table 1 (Arikoglu and Ozkol, 2005; Odibat, 2008).

Table 1. Some mathematical operations of DTM technique.

Original function	Transformed function
$f(t) = \beta x(t)$	$F(k) = \beta X(k)$
$f(t) = \frac{dx(t)}{dt}$	$F(k) = (k+1)X(k+1)$
$f(t) = x_1(t) \pm x_2(t)$	$F(k) = X_1(k) \pm X_2(k)$
$f(t) = \frac{d^m x(t)}{dt^m}$	$F(k) = (k+1)(k+2)\cdots (k+m)X(k+m)$
$f(t) = \exp(\lambda t)$	$F(k) = \frac{\lambda^k}{k!}$
$f(t) = x^m$	$F(k) = \delta(k-m)$
$f(t) = x_1(t)\, x_2(t)$	$F(k) = \sum_{k_1=0}^{k} X_1(k_1) X_2(k-k_1)$
$f(t) = x_1(t)\, x_2(t) \cdots x_{n-1}(t) x_n(t)$	$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} X_1(k_1) X_2(k_2-k_1) \cdots \cdot X_{n-1}(k_{n-1}-k_{n-2}) X_n(k-k_{n-1})$
$f(t) = \int_{t_0}^{t} x(u)du$	$F(k) = \frac{X(k-1)}{k},\ k \geq 1,\ F(0) = 0$
$f(t) = \sin(\omega t + \alpha)$	$F(k) = \frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2}+\alpha\right)$
$f(t) = \cos(\omega t + \alpha)$	$F(k) = \frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2}+\alpha\right)$

More details of expressions in Table 1 can be found in (Arikoglu and Ozkol, 2005).

3. A NEW METHOD FOR STABILITY ANALYSIS IN NONLINEAR SYSTEMS

Consider the nonlinear system $\dot{x}(t) = g(x(t))$, and assume that it can be decomposed into a linear part and a nonlinear part, i.e.,

$$\dot{x}(t) = Ax(t) + f(x(t)) \tag{3}$$

where $x(t)$ is a $m \times 1$ - matrix consisting of state vectors $\{x_1(t), x_2(t), \ldots, x_m(t)\}$, A is an $m \times m$ -matrix with coefficients in $\mathbb{R}$, and $f(x(t))$ is a nonlinear vector-valued function expression.

The next equation is obtained from (3) by applying DTM technique and Table 1.

$$(k+1)X(k+1) = AX(k) + F(k), \qquad (k = 0, 1, 2, \dots, K). \quad (4)$$

According to the DTM technique, the K-order Taylor polynomial of $x(t)$ as (Arikoglu and Ozkol, 2005; Odibat, 2008) can be obtained:

$$x(t) = X(0) + X(1)t + X(2)t^2 + \dots + X(K)t^{(K)} \quad (5)$$

LEs are the average exponential rates of divergence or convergence of nearby orbits in the state space. LEs make it possible to analyze the stability of nonlinear systems without solving them. The LE was used before the emergence of chaos theory to characterize the stability of linear and nonlinear systems. The signs of LEs characterize the stability property of the nonlinear dynamical systems. Trajectories from all directions in the state space come to the equilibrium point if all LEs are negative. So, if all LEs for system (3) are negative, then the system is asymptotically stable. In other words, the presence of a negative LLE confirms the occurrence of asymptotically stable behaviour in nonlinear dynamical systems (Wolf et al., 1985; Rosenstein et al., 1993).

If we consider a time series $x(t)$, the LLE can be defined using the following equation (Wolf *et al.*, 1985):

$$\lambda_{max} = \frac{1}{n}\sum_{t=0}^{n} Ln\left(\frac{\|\delta x(t+1)\|}{\|\delta x(t)\|}\right), \quad (6)$$

where $\delta x(t)$ is the average divergence at time t, n indicates the number of iterations or evolution times, and λ_{max} is LLE expression (Wolf et al., 1985; Rosenstein et al., 1993).

In this case, according to the DTM technique,

$$\lambda_{max} = \frac{1}{n}\sum_{k=0}^{n} Ln\left(\frac{\|X(k+1)\|}{\|X(k)\|}\right), \qquad (k = 0,1,2,\dots,n). \quad (7)$$

Note that

$$\begin{aligned}\lambda_{max} &= \frac{1}{n}\sum_{k=0}^{n} Ln\left(\frac{\|X(k+1)\|}{\|X(k)\|}\right)\\ &= \frac{1}{n}\left(Ln\left(\frac{\|X(1)\|}{\|X(0)\|}\right) + Ln\left(\frac{\|X(2)\|}{\|X(1)\|}\right) + \dots + Ln\left(\frac{\|X(n)\|}{\|X(n-1)\|}\right)\right)\\ &= \frac{1}{n}\left(Ln\left(\prod_{k=1}^{n}\left(\frac{\|X(k)\|}{\|X(k-1)\|}\right)\right)\right),\end{aligned} \quad (8)$$

so,

$$\lambda_{max} = \frac{1}{n} Ln\left(\frac{\|X(n)\|}{\|X(0)\|}\right). \quad (9)$$

where $X(n) = \begin{bmatrix} X_1(n) \\ X_2(n) \\ \vdots \\ X_m(n) \end{bmatrix}$ is a $m \times 1$ column vector, $X_i(n)$ indicates the differential transform of $x_i(t)$, and $X(0)$ is the initial point.

Equation (4) gives,

$$\begin{cases} X(1) = AX(0) + F(0) \\ X(2) = \left(\frac{1}{2!}\right)A^2X(0) + \left(\frac{1}{2!}\right)AF(0) + \left(\frac{1}{2}\right)F(1) \\ X(3) = \left(\frac{1}{3!}\right)A^3X(0) + \left(\frac{1}{3!}\right)A^2F(0) + \left(\frac{1}{3!}\right)AF(1) + \left(\frac{1}{3}\right)F(2) \\ X(4) = \left(\frac{1}{4!}\right)A^4X(0) + \left(\frac{1}{4!}\right)A^3F(0) + \left(\frac{1}{4!}\right)A^2F(1) + \left(\frac{1}{12}\right)AF(2) + \left(\frac{1}{4}\right)F(3) \end{cases} \quad (10)$$

In the general case, $X(j) = \frac{1}{j!}\left(A^j X(0) + \sum_{k=0}^{j-1} A^{(j-k-1)} k!\,F(k)\right)$ where $j = 1, 2, 3, \dots, n$.

In the special case, we have:

$$X(n) = \frac{1}{n!}\left(A^n X(0) + \sum_{k=0}^{n-1} A^{(n-k-1)} k!\,F(k)\right). \quad (11)$$

Now by replacing (11) in (9), we have:

$$\lambda_{max} = \frac{1}{n} Ln\left(\frac{\left\|\frac{1}{n!}\left(A^n X(0) + \sum_{k=0}^{n-1} A^{(n-k-1)} k!\,F(k)\right)\right\|}{\|X(0)\|}\right) \quad (12)$$

To simplify the formula, consider

$$M = \frac{\left\|\frac{1}{n!}\left(A^n X(0) + \sum_{k=0}^{n-1} A^{(n-k-1)} k!\,F(k)\right)\right\|}{\|X(0)\|} \quad (13)$$

By natural logarithm properties, if $M < 1$, then LLE is negative.

Theorem 1. Consider the nonlinear system (3) with the above assumptions. If $\|A\| < 1$ and $F(k)$ is a bounded function, then the corresponding LLE is negative.

Proof. Let M be given by (13), $\|A\| < 1$. It is sufficient to prove that $M < 1$. Now we discuss the normed space $\left(M_{n\times n}(\mathbb{R}), \|\cdot\|\right)$, where $\|\cdot\|$ is an operator norm on $n\times n$-matrices with a coefficient in $\mathbb{R}$.

According to the homogeneity and subadditivity property of matrix norm, we have:

$$M \le \left(\frac{1}{n!}\right)\frac{\|A^n X(0)\|}{\|X(0)\|} + \frac{1}{n!\|X(0)\|}\sum_{k=0}^{n-1} k!\left\|A^{(n-k-1)}F(k)\right\|. \tag{14}$$

Since $F(k)$ is bounded, there exists $\alpha > 0$ such that,

$$\|F(k)\| \le \alpha, \qquad (k = 0, 1, ..., n). \tag{15}$$

Conforming to (14), (15) and submultiplicitivity property of matrix norm, we have:

$$M \le \frac{\|A\|^n}{n!} + \frac{\alpha}{\|X(0)\|}\sum_{k=0}^{n-1}\frac{k!}{n!}\|A\|^{n-k-1}. \tag{16}$$

Because $0 \le k \le n-1$, we have:

$$\frac{k!}{n!} \le \frac{1}{n}. \tag{17}$$

From (16) and (17), we have:

$$M \le \frac{\|A\|^n}{n!} + \frac{\alpha}{n\|X(0)\|}\sum_{k=0}^{n-1}\|A\|^{n-k-1}, \tag{18}$$

and also,

$$\sum_{k=0}^{n-1}\|A\|^{n-k-1} = \sum_{i=0}^{n-1}\|A\|^{i} = \left(\frac{1-\|A\|^n}{1-\|A\|}\right). \tag{19}$$

The hypothesis $\|A\| < 1$ implies that,

$$M \le \frac{\|A\|^n}{n!} + \frac{\alpha}{n\|X(0)\|}\left(\frac{1-\|A\|^n}{1-\|A\|}\right) \le \frac{1}{n} + \frac{\alpha}{n\|X(0)\|\left(1-\|A\|\right)}. \tag{20}$$

Now, for n sufficiently large, we have:

$$\frac{1}{n}\left(1+\frac{\alpha}{\|X(0)\|\left(1-\|A\|\right)}\right) < 1. \tag{21}$$

So, by (20), $M < 1$, and this completes the proof.

4. NUMERICAL RESULTS

The following three physical and practical systems show the usefulness of the proposed Theorem. In the first example, a nonlinear system is used that can be implemented with the Colpitts oscillator. The second example is a nonlinear physical system based on the Lorenz system. In the third example, we want to show the capability of the proposed theorem for stability analysis of complex high-dimensional nonlinear dynamical systems. For this purpose, a nonlinear system is used that can be implemented by five-dimensional (*5D*) memristor-based Chua's oscillation (Vasiljevic et al., 2019; Odibat et al., 2010; Sun et al., 2018).

Example 1: Consider the following nonlinear system equation:

$$\begin{cases} \dot{y}_1(t) = -c\,y_1(t) - 0.2y_3(t) - 0.89a \cdot \left(\exp\left(\dfrac{-y_2(t)-y_1(t)}{\theta}\right) - 1\right) \\ \dot{y}_2(t) = b\,a\left(\exp\left(\dfrac{-y_2(t)-y_1(t)}{\theta}\right) - 1\right) - b\,y_3(t) \\ \dot{y}_3(t) = a\,y_1(t) \end{cases} \tag{22}$$

where $y_1(t)$, $y_2(t)$, and $y_3(t)$ are state variables, and constant parameters are $\theta = 2$, $a = 0.5$, $b = 8.34\times10^{-2}$, and $c = 51.64\times10^{-3}$.

By using DTM technique, (22) is converted as follows:

$$\begin{cases} (k+1)Y_1(k+1) = -c\,Y_1(k) - 0.2\,Y_3(k) - 0.89a \cdot \left(\dfrac{\left(\dfrac{-Y_2(k)-Y_1(k)}{2}\right)^k}{k!} - 1\right) \\ (k+1)Y_2(k+1) = b\,a\left(\dfrac{\left(\dfrac{-Y_2(k)-Y_1(k)}{2}\right)^k}{k!} - 1\right) - b\,Y_3(k) \\ (k+1)Y_3(k+1) = a\,Y_1(k) \end{cases} \tag{23}$$

where $k = 0, 1, \ldots, n$.

Now, from (4) and (23), we get:

$$F(k) = \begin{bmatrix} \dfrac{\left(\dfrac{-Y_2(k)-Y_1(k)}{2}\right)^k}{k!} \\ \dfrac{\left(\dfrac{-Y_2(k)-Y_1(k)}{2}\right)^k}{k!} \\ 0 \end{bmatrix}. \tag{24}$$

So,

$$\lim_{k\to\infty}\left(F(k)\right) = 0 \cdot \tag{25}$$

If we set the initial conditions by $|Y_1(0)| \le 1, |Y_2(0)| \le 1$, and $|Y_3(0)| \le 1$, then by the recurrence relation (23), we have

$|Y_i(k)| \le 1$ for all $k = 0,1,\dots,n$ and $i = 1,2,3$.

We use of mathematical induction to prove it. When $m=1$, it is easy to verify that $|Y_i(1)| \le 1$ for $i = 1,2,3$. Assume the inequality holds when $m=k$ for some integer $k \ge 1$; that is, assume

$|Y_i(k)| \le 1$ for all $k = 0,1,\dots,n$ and $i = 1,2,3$,

for some integer $k \ge 1$. We want to show that it also holds when $m=(k+1)$; that is, we want to show that $|Y_i(k+1)| \le 1$ for all $k = 0,1,\dots,n$ and $i = 1,2,3$, by the recurrence relation (23) we have;

$$\begin{aligned}
|Y_1(k+1)| &\le \frac{1}{(k+1)}\left(c|Y_1(k)|+0.2|Y_3(k)|+0.89a\left(\frac{\left(2/2\right)^k}{k!}+1\right)\right)\\
&\le \frac{1}{(k+1)}\left(c+0.2+0.89a\left(\frac{1}{k!}+1\right)\right)\\
&\le \frac{1}{(k+1)}\left(0.052+0.2+0.45\times 2\right)\\
&\le \frac{2}{(k+1)} \le 1
\end{aligned}$$

$$\begin{aligned}
|Y_2(k+1)| &\le \frac{1}{(k+1)}\left(ba\left(\frac{1}{k!}+1\right)+0.85\times 1\right)\\
&\le \frac{1}{(k+1)}\left(0.45\times 2+0.85\right)\\
&\le \frac{1}{(k+1)}(1+1) = \frac{2}{k+1} \le 1
\end{aligned}$$

$$|Y_3(k+1)| \le \frac{a}{(k+1)}|Y_1(k)| \le \frac{a}{(k+1)} \le 1$$

This means that $|Y_i(k)| \le 1$, where $i = 1, 2, 3$.

Form (24),

$$F(k) = \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} \text{ where } b = \left(\frac{\left(\frac{-Y_2(k)-Y_1(k)}{2}\right)^k}{k!}\right). \tag{26}$$

We have that $F(k)$ is a 3×1-matrix and by representation theory, it is a linear operator from $\mathbb{R}$ into $\mathbb{R}^3$. So by definition of the matrix norm, we have

$$\begin{aligned}
\|F(k)\| &= \sup_{\substack{\|\omega\|\le 1\\ \omega\in\mathbb{R}}} \|F(k)\omega\| = \sup_{\|\omega\|\le 1}\left\|\begin{bmatrix} b \\ b \\ 0 \end{bmatrix}\omega\right\| = \sup_{\|\omega\|\le 1}\left\|\begin{bmatrix} b\omega \\ b\omega \\ 0 \end{bmatrix}\right\|\\
&= \sup_{\|\omega\|\le 1}\sqrt{(b\omega)^2+(b\omega)^2} = \sqrt{2}\sup_{\|\omega\|\le 1}|b||\omega|\\
&= \sqrt{2}\,|b| = \sqrt{2}\left|\left(\frac{\left(\frac{-Y_2(k)-Y_1(k)}{2}\right)^k}{k!}\right)\right| \le \sqrt{2}\,\frac{\left(\frac{1+1}{2}\right)^k}{k!}\\
&= \frac{\sqrt{2}}{k!}.
\end{aligned} \tag{27}$$

With use of the Stirling's formula $\left(k! \sim \sqrt{2\pi k}\, e^{-k} k^k\right)$ for large number k,

$$\lim_{k\to\infty}\frac{\sqrt{2}}{k!} = \lim_{k\to\infty}\frac{\sqrt{2}}{\sqrt{2\pi k}\,e^{-k}k^k} = \lim_{k\to\infty}\frac{\sqrt{2}\,e^k}{\sqrt{2\pi k}\,k^k} = 0$$

So,

$$\lim_{k\to\infty}\|F(k)\| = 0 \;\equiv\; \forall\varepsilon>0\ \exists N,\ k>N\ \ \|F(k)\|\le\varepsilon.$$

Now, we show that $F(k)$ is a bounded function. For $\varepsilon = 1$, there exists a positive integer number N, such that $\|F(k)\| \le 1$, for all $k \ge N$. If we consider M,

$$M = max\{\|F(0)\|,\|F(1)\|,\dots,\|F(N)\|,1\}$$

then $\|F(k)\| \le M$, where $k = 0,1,\dots,n$ and it means that $F(k)$ is bounded.

Hence, by (3) and (22), matrix A defined is as below,

$$A = \begin{bmatrix} -c & 0 & -0.2 \\ 0 & 0 & -b \\ a & 0 & 0 \end{bmatrix}. \tag{28}$$

So, it is easy to verify that $\|A\| < 1$ and by Theorem 1, the system (26) is stable. This can be proved by calculating the LEs representation and simulating a transient time series system with numerical methods.

Fig. 1, shows that LEs are a function of time with initial conditions $|Y_i(0)| < 1$ for $i = 1, 2, 3$. The presence of negative LEs in a dynamical system indicates the stability of the system. Fig. 2, illustrates the transient time series in state variables.

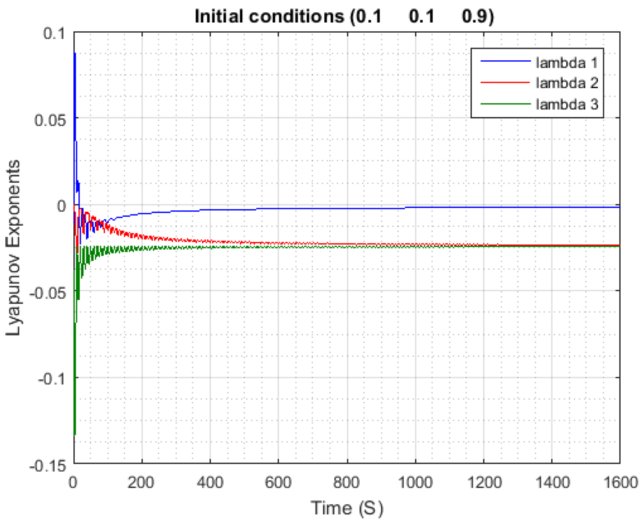

Fig. 1. The LEs calculation for system (22).

Example 2: Suppose that $x(t), y(t)$, and $z(t)$ are state variables that satisfy the following nonlinear system equations:

$$\begin{cases} \dot{x}(t)=0.1\big(y(t)-x(t)\big) \\ \dot{y}(t)=x(t)\big(0.2-z(t)-y(t)\big) \\ \dot{z}(t)=x(t)\,y(t)-0.7\,z(t) \end{cases} \tag{29}$$

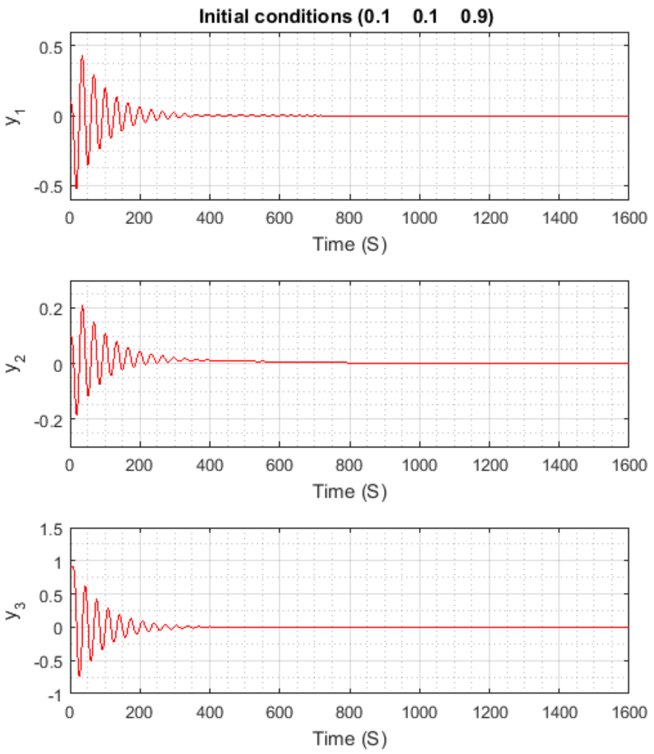

Fig. 2. Time series plot of the generated data in the present study.

By the DTM techniques on this system equations, we have:

$$\begin{cases} (k+1)X(k+1) = 0.1\,Y(k) - 0.1\,X(k) \\ (k+1)Y(k+1) = 0.2\,X(k) - \displaystyle\sum_{k_1=0}^{k} X(k_1)Y(k-k_1) \\ \qquad\qquad - \displaystyle\sum_{k_1=0}^{k} X(k_1)Z(k-k_1) \\ (k+1)Z(k+1) = \displaystyle\sum_{k_1=0}^{k} X(k_1)Y(k-k_1) - 0.7\,Z(k) \end{cases} \tag{30}$$

where $k = 0,1,\dots,n$. In the following, we give matrix representation of $F(k)$,

$$F(k) = \begin{bmatrix} 0 \\ -\displaystyle\sum_{k_1=0}^{k} X(k_1)Y(k-k_1) - \sum_{k_1=0}^{k} X(k_1)Z(k-k_1) \\ -\displaystyle\sum_{k_1=0}^{k} X(k_1)Y(k-k_1) \end{bmatrix}. \tag{31}$$

by the same calculation like the last example and with the initial condition $|X(0)|<1$, $|Y(0)|<1$, and $|Z(0)|<1$, we see that $F(k)$ is a bounded function. Now, for

$$A = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0 & -0.7 \end{bmatrix}. \tag{32}$$

it is easy to see that $\|A\|<1$ and given by the Theorem 1, the system is stable.

In order to confirm this claim, the LEs of this system is shown in Fig. 3. From the figure, it is seen that LEs are negative. As a result, system (29) is stable, which is consistent with the result from Theorem 1. The time series of the three state variables used in the present study is shown in Fig. 4.

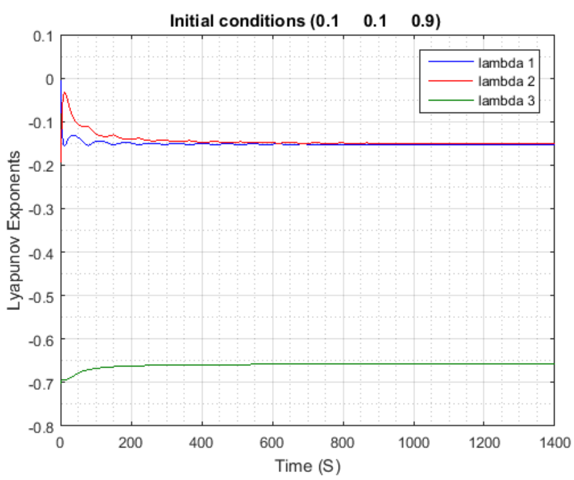

Fig. 3. Numerically calculated LEs for system (29).

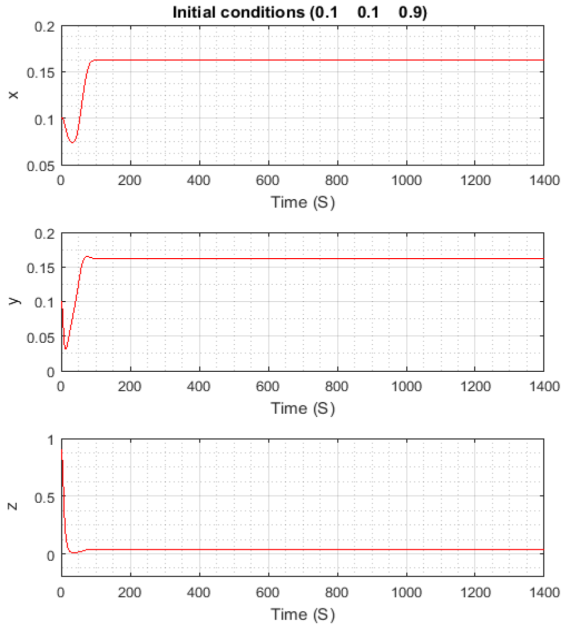

Fig. 4. Transient time series of the three state variables.

Example 3: Let the following equations be a high dimensional continues nonlinear system,

$$\begin{cases} \dot{x}_1(t) = -\alpha x_1(t) + r x_2(t) + \left(\beta x_1(t) x_5^2(t)\right) \\ \dot{x}_2(t) = \omega x_2(t) - \theta x_3(t) - \varphi x_4(t) \\ \dot{x}_3(t) = -\sigma x_1(t) + g x_2(t) - \rho x_3(t) \\ \dot{x}_4(t) = \sigma x_2(t) + s x_5(t) \\ \dot{x}_5(t) = \gamma x_1(t) \end{cases} \tag{33}$$

where $x_1(t), x_2(t), \ldots, x_5(t)$ are state variables, and constant parameters are $\alpha = 677\times10^{-4}$, $\beta = 9\times10^{-3}$, $\rho = 34\times10^{-3}$, $\omega = 25\times10^{-5}$, $\theta = 2\times10^{-2}$, $\varphi = 5.3\times10^{-3}$, $\gamma = 8.7\times10^{-4}$, $\sigma = 1\times10^{-2}$, $r = 1\times10^{-3}$, $s = 47\times10^{-3}$, and $g = 22\times10^{-2}$.

Then, with the DTM technique exploited, it is seen that

$$\begin{cases} (k+1)X_1(k+1) = -\alpha X_1(k) + r X_2(k) \\ \qquad + \beta \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} X_1(k_1) X_5(k_2-k_1) X_5(k-k_2) \\ (k+1)X_2(k+1) = \omega X_2(k) - \theta X_3(k) - \varphi X_4(k) \\ (k+1)X_3(k+1) = -\sigma X_1(k) + g X_2(k) - \rho X_3(k) \\ (k+1)X_4(k+1) = \sigma X_2(k) + s X_5(k) \\ (k+1)X_5(k+1) = \gamma X_1(k) \end{cases}, \tag{34}$$

for $k = 0, 1, \ldots, n$. Therefore, the matrix representation of $F(k)$ is as follows,

$$F(k) = \begin{bmatrix} \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} X_1(k_1) X_5(k_2-k_1) X_5(k-k_2) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{35}$$

Let us assume that the initial conditions $|X_i(0)| \le 1$ hold for $i = 1, \ldots, 5$. By the same calculations as in example 1, we see that $F(k)$ is a bounded function.

Since matrix A can be expressed as:

$$A = \begin{bmatrix} -\alpha & r & 0 & 0 & 0 \\ 0 & \omega & -\theta & -\varphi & 0 \\ -\sigma & g & -\rho & 0 & 0 \\ 0 & \sigma & 0 & 0 & s \\ \gamma & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

we see that $\|A\| < 1$ and so Theorem 1 implies the stability of this system.

Fig. 5, illustrates the existence of negative LEs confirming the stability of the system under study in this example. As shown in Fig. 6, the time series of state variables in this example is another proof of this claim.

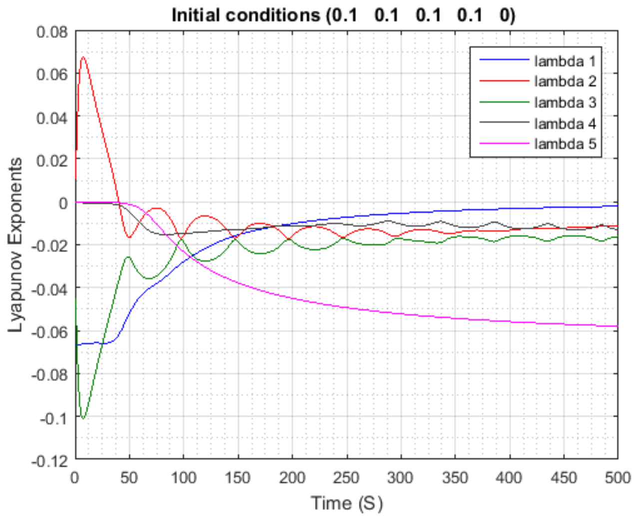

Fig. 5. The LEs as a function of time.

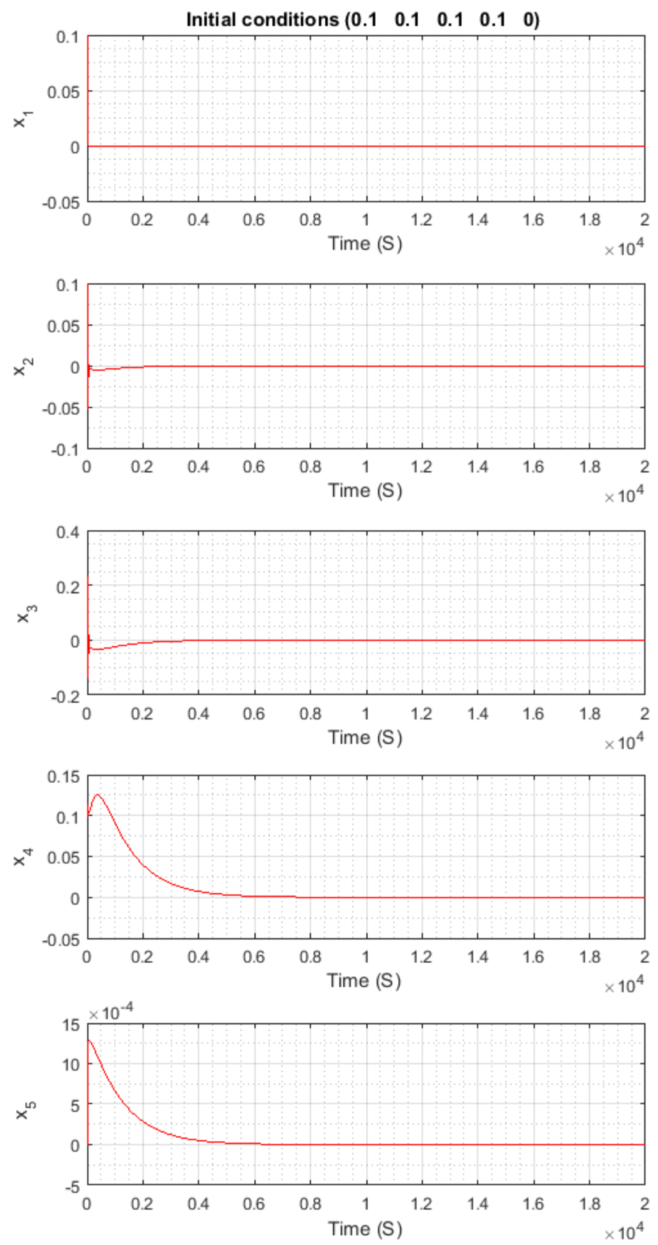

Fig. 6. Simulation of system (33) for transient time series of state variables.

6. CONCLUSIONS

This paper presents a new analytic method for asymptotically stability analysis of a wide range of nonlinear systems. In this paper, a new sufficient condition for the stability analysis of continuous nonlinear dynamical systems in the general case based on LLE has been proposed. The DTM is applicable for analytical nonlinearities, having convergent Taylor series. In this method, the DTM technique is used analytically to estimate the sign of the LLE in nonlinear systems with unknown parameters. According to Theorem 1, the nonlinearity stability sector of function $f\left(x(t)\right)$ in system (1) is not the aim of this paper, but the differential transform of nonlinear vector-valued function expression $f\left(x(t)\right)$ ($F\left(k\right)$) must be bounded. This method can be used to analyze the asymptotic stability of complex high-dimensional nonlinear dynamical systems. Some examples are applied to show the usefulness of the proposed method.

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