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Solving parameterized Non-symmetric Algebraic Riccati Equations: A Matrix Sign Function Approach ⋆

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Abstract: We study numerical solutions of parameterized non–symmetric continuous–time algebraic Riccati equations (PNAREs) related to issues in control system theory and differential games. Thanks to good regularity properties of the solutions of PNAREs with respect to the parameter, our main tool involving an analytically determined matrix sign function is applied to the computation of various solutions, including the strongly stabilizing, the reverse dichotomic and the dichotomic ones. The approach applies also to the parameterized non–symmetric discrete–time algebraic Riccati equations. The efficiency of the proposed method is illustrated by several numerical experiments.

Keywords: Non-symmetric Algebraic Riccati Equations, Matrix Sign Function, parameterized Solutions.

1. INTRODUCTION

Algebraic equations which are quadratic in the variable are usually called *algebraic Riccati equations (ARE)*, to pay tribute to Count Jacopo Riccati (Bittanti et al., 1991; Jungers, 2017) who initiated their study. In their most notorious form, they feature matrix coefficients with Hermitian symmetry and one is usually interested to solve them for the square Hermitian unknown matrix. In contrast to linear equations, one of the prominent AREs hallmarks is that they usually have several solutions with quite distinct properties. However, even though the AREs are essentially quadratic in the unknown, specific linear algebra tools are usually involved in their solving. Both the theoretic study and the numerical computation of the solutions to AREs have generated a rich body of literature. For example, in the symmetric case, several approaches have been provided to characterize all possible solutions (Ran and Rodman, 1992; Abou-Kandil et al., 2003; Lancaster and Rodman, 1995), while their properties proved to be keystone in several applications, like optimal control for linear systems subject to quadratic cost functions, optimal filtering, and robust stabilization and control, to name just a few (see (Ionescu et al., 1999; Dorato et al., 1995; Anderson and Moore, 1989)). In several contexts in control system theory, the issue to solve is parameter dependent. This is for instance the case of global stabilizing of linear systems with input nonlinearities (Teel, 1995) or of gain scheduling control (Apkarian and Gahinet, 1995). The solution of these problems is obtained by the solution of a parameterized ARE. The interest in this type of parameterized ARE goes back to the seventies and eighties: this question has been treated numerically with imbedding equation in (Jamshidi et al., 1970). The continuity properties of the ARE with respect to the parameter has been investigated in (Faibusovich, 1986), (Trentelman, 1987) and also in (Ran and Rodman, 1988).

Other different and quite distinct AREs – called non–symmetric algebraic Riccati equations (NARE) – feature no symmetry in the coefficients, while the unknown may be a rectangular (instead of square) matrix (see for example (Abou-Kandil et al., 2003; Bini et al., 2012)). They are usually encountered in game (Başar and Olsder, 1995; Ho, 1970; Abou-Kandil et al., 2003) or transport theory (Juang, 1995; Lu, 2005; Weng et al., 2012). For example, when considering a linear differential game with quadratic cost functions, the Nash (Freiling et al., 1999; Engwerda, 2005) or Stackelberg (Abou-Kandil et al., 2003; Jungers, 2008) strategies with open-loop information structure may be solved via coupled AREs, which, in turn, can be rewritten equivalently as NAREs. Precisely as the symmetric AREs, NAREs may have several distinct solutions, while their classification is considerably more difficult. To have one example only, the stabilizing solution to symmetric AREs, provided it exists, is unique, while a NARE may have several different stabilizing solutions featuring different properties.

Among the various solutions to NAREs, perhaps the most well–known is the *strongly stabilizing* one (see (Engwerda, 2005, Definition 7.2)) which allows to stabilize the state and anti-stabilize the co-state closed-loop matrices. The uniqueness of this particular solution triggers certain properties in the robust control theory of the underlying system. Medanic introduced in (Medanic, 1982) other interesting solutions, like the *dichotomic* and *reverse dichotomic*, which substantiate the most stable and anti–stable solutions of a given NAREs and proved to be of particular interest in control systems or game theory. Moreover, they play a crux role in solving AREs by integrating

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the related differential Riccati equation (see (Medanic, 1982, Theorem 5)).

Apart from the theoretical study, the lack of symmetry of NAREs is a source of difficulties when solving them numerically. Several methods originally developed for symmetric AREs have been modified and extended to cope with NAREs, for example the Popov function (Ionescu and Weiss, 1993; Ionescu and Oară, 1996; Kremer, 2003; Jungers and Oară, 2012; Jungers et al., 2009), Newton's iterative algorithm (Sandell, 1974; Guo and Laub, 2000), the invariant subspace method (Van Dooren, 1981; Abou-Kandil et al., 2003; Guo, 2001; Laub, 1979), or more generally, the deflating subspace method for matrix pencils, in both continuous-time (Jungers and Oară, 2012) and discrete–time settings (Jungers et al., 2009).

The main focus of this paper is on an alternative method based on the matrix sign function (see (Roberts, 1980; Denman and Beavers, 1976) and the refinements in (Gardiner and Laub, 1986)) which allows to linearize the quadratic NARE into an overdetermined system of linear equations. Note that recently the matrix sign function has been extended to the case of a particular class of parameterized ARE (Guerra et al., 2012, 2015). The matrix sign function has been already applied in the framework of NARE. One can cite mainly the contributions (Guo and Bai, 2005) and (Bini et al., 2012, Section 3.5). It should be emphasized that these two results are possible thanks to strong and restrictive assumptions: (Bini et al., 2012, Section 3.5) assumes that the characteristic matrix has a suitable eigenvalues splitting and (Guo and Bai, 2005) (or refinements provided in (Bai et al., 2006)) assumes that the characteristic matrix is a M-matrix implying the same eigenvalues splitting. In other words, the matrix sign function applies naturally in these cases because there is one and only one stabilizing solution (that is the strongly stabilizing solution) of the NARE. Our main contribution is to extend this approach by providing an analytical parameterized strongly stabilizing solution to a parameterized non–symmetric algebraic Riccati equations (PNARE) to possibly include other specific solutions of NAREs.

The paper is focused on the continuous-time case of PNARE and is organized as follows. Section 2 introduces the PNAREs and various solutions of interest, and their main properties. The main results on computing analytically the parameterized strongly stabilizing solution of PNAREs are given in Section 4 and exemplified through several relevant numerical experiments in Section 5. Some concluding remarks are contained in Section 6.

Notation: By $\mathbb{R}$ and $\mathbb{C}$ we denote the real axis and the complex plane, respectively. The open left–half plane (the stability domain in continuous–time) is denoted by $\mathbb{C}^-$, the open right–half plane by $\mathbb{C}^+$ and the open unit-disk by $\mathbb{D}$. $\mathbb{N}$ and $\mathbb{N}^*$ are respectively the set of non-negative integers and the set of positive integers. For a constant matrix A we denote by A' its transpose and if A is invertible by A^{-1} its inverse. $\Lambda(A)$ is the spectrum of a square matrix A, $\mathrm{rank}(A)$ its rank and $\mathrm{trace}(A)$ its trace. An eigenvalue of A is called *stable* provided it is inside $\mathbb{C}^-$ for continuous-time and inside $\mathbb{D}$ for discrete-time, and *antistable* otherwise. I_{n_1} and $0_{n_1\times n_2}$ denote the identity matrix of size $n_1\times n_1$ and the null matrix of size $n_1\times n_2$, respectively. The diagonal matrix with the scalars $\alpha_1,\cdots,\alpha_n$ on the diagonal is denoted $\mathrm{diag}(\alpha_1,\cdots,\alpha_n)$. For a complex α, $\mathrm{Re}(\alpha)$ denotes its real part.

2. PARAMETERIZED NARES

In this section we introduce the parameterized non-symmetric algebraic Riccati equations, define several solutions of interest, and give several basic results.

Let us define $\mathscr{U}\subseteq\mathbb{R}$ the set of admissible parameters. The algebraic quadratic equation ($n,p\in\mathbb{N}^*$), $\forall\alpha\in\mathscr{U}$

$$M_{21}(\alpha)+M_{22}(\alpha)K(\alpha)-K(\alpha)M_{11}(\alpha) \\ -K(\alpha)M_{12}(\alpha)K(\alpha)=0_{p\times n} \quad (1)$$

in the unknown $K(\alpha)\in\mathbb{R}^{p\times n}$, where $M_{11}(\alpha)\in\mathbb{R}^{n\times n}$, $M_{12}(\alpha)\in\mathbb{R}^{n\times p}$, $M_{21}(\alpha)\in\mathbb{R}^{p\times n}$, and $M_{22}(\alpha)\in\mathbb{R}^{p\times p}$ is called the *Parameterized Continuous–time Non-symmetric Algebraic Riccati Equation* (PNARE). Since p and n are not necessary equal, the unknown $K(\alpha)$ is, in general, a rectangular matrix. Equation (1) is associated with the characteristic matrix

$$M(\alpha)=\left[\begin{array}{c:c} M_{11}(\alpha) & M_{12}(\alpha) \\ \hdashline M_{21}(\alpha) & M_{22}(\alpha)\end{array}\right]\in\mathbb{R}^{(n+p)\times(n+p)}, \quad (2)$$

which plays an essential role for PNARE. The PNARE (1) could be rewritten equivalently as the following equation:

$$M(\alpha)\begin{bmatrix} I_n \\ K(\alpha)\end{bmatrix}=\begin{bmatrix} I_n \\ K(\alpha)\end{bmatrix}(M_{11}(\alpha)+M_{12}(\alpha)K(\alpha)). \quad (3)$$

To be more precise, in the latter equation, the first block is trivial and the second one is equivalent to the PNARE (1).

Finding a solution $K(\alpha)$ to PNARE (1) aims at determining an equivalence transformation matrix

$$T(\alpha):=\begin{bmatrix} I & 0 \\ K(\alpha) & I\end{bmatrix}, \quad (4)$$

leading without assumption on $K(\alpha)$ to the inverse

$$T^{-1}(\alpha)=\begin{bmatrix} I & 0 \\ -K(\alpha) & I\end{bmatrix}, \quad (5)$$

such that $T^{-1}(\alpha)M(\alpha)T(\alpha)$ is block upper triangular.

Remark 1. In particular, if $n=p$, $M_{11}(\alpha)=-M_{22}'(\alpha)$, and $M_{12}(\alpha)$ and $M_{21}(\alpha)$ are both symmetric, we recover the well–known parameterized symmetric algebraic Riccati equation for which the characteristic matrix $M(\alpha)$ is Hamiltonian (Laub and Meyer, 1974), the eigenvalues of $M(\alpha)$ have symmetry with respect to the imaginary axis (see (Kučera, 1972)), and one usually seeks Hermitian solutions $K(\alpha)=K'(\alpha)$.

The following two theorems are straightforward consequences of Radon's Lemma (Radon, 1927) (see also (Abou-Kandil et al., 2003, Theorem 6.2.2 and Theorem 6.2.4) and (Bini et al., 2012; Engwerda, 2005)).

Theorem 1. Let $K(\alpha)\in\mathbb{R}^{p\times n}$ be a solution to the PNARE over $\alpha\in\mathscr{U}$. Then there are three parameterized matrices $X(\alpha)\in\mathbb{R}^{n\times n}$, $Y(\alpha)\in\mathbb{R}^{p\times n}$ and $J(\alpha)\in\mathbb{R}^{n\times n}$, with $X(\alpha)$ invertible and $J(\alpha)$ a Jordan matrix, verifying

$$M(\alpha)\begin{bmatrix} X(\alpha) \\ Y(\alpha)\end{bmatrix}=\begin{bmatrix} X(\alpha) \\ Y(\alpha)\end{bmatrix}J(\alpha), \quad (6)$$

and

$$Y(\alpha)=K(\alpha)X(\alpha). \quad (7)$$

□

Theorem 2. Let $\mathscr{V}(\alpha)\subset\mathbb{R}^{(n+p)}$ be an invariant subspace of the characteristic matrix $M(\alpha)$ of dimension n over $\alpha\in\mathscr{U}$. Then there are three parameterized matrices $X(\alpha)\in\mathbb{R}^{n\times n}$, $Y(\alpha)\in\mathbb{R}^{p\times n}$ and a Jordan matrix $J(\alpha)\in\mathbb{R}^{n\times n}$ verifying

$$M(\alpha)\begin{bmatrix} X(\alpha) \\ Y(\alpha) \end{bmatrix} = \begin{bmatrix} X(\alpha) \\ Y(\alpha) \end{bmatrix} J(\alpha). \tag{8}$$

Moreover, if $X(\alpha)$ is invertible, then the set $\mathscr{V}(\alpha)$ is a graph invariant subspace of the matrix $M(\alpha)$ of dimension n and $K(\alpha) = Y(\alpha)X^{-1}(\alpha)$ is a solution to the PNARE. This solution is independent of the choice of the basis matrices $X(\alpha)$ and $Y(\alpha)$ and depends only on the spectrum $\Lambda(J(\alpha))$. □

Proof: For each value of $\alpha \in \mathscr{U}$, the proofs of Theorem 1 and Theorem 2 are known in the literature. See (Abou-Kandil et al., 2003) and (Bini et al., 2012). ■

It should be emphasized that Theorem 1 and 2 stand for a given value of the parameter α. It is interesting to notice that the regularity of the function $K(\alpha)$ is induced by the one of the characteristic matrix $M(\alpha)$ with respect to the parameter α. The behavior of the parameterized solutions in a neighborhood of a given value is related to Theorem 3.

Theorem 3. (Lemma 2.3 in (Karow and Kressner, 2014)). Consider a fixed value of the admissible parameter $\alpha_0 \in \mathscr{U}$. Suppose that there exists $K_0 \in \mathbb{R}^{p\times n}$ a solution of the PNARE (1) for $\alpha = \alpha_0$. If the condition

$$\Lambda(M_{11}(\alpha_0) + M_{12}(\alpha_0)K_0) \\ \cap \Lambda(M_{22}(\alpha_0) - K_0 M_{12}(\alpha_0)) = \emptyset \tag{9}$$

is verified, then there exist an open neighborhood $\mathscr{E} \subset \mathbb{C}^{(n+p)\times(n+p)}$ of 0 and an open neighborhood $\mathscr{K} \subset \mathbb{R}^{p\times n}$ of K_0 such that for each $E \in \mathscr{E}$ the PNARE (1) with $M = M(\alpha_0) + E$ has a unique solution $K_E \in \mathscr{K}$. Moreover K_E depends holomorphically on E and admits the first-order expansion

$$K_E = K_0 + \mathbb{T}_K^{-1}(E_{21}) + O(\|E\|^2),$$

with the Sylvester operator

$$\mathbb{T}_K : \Delta K \mapsto \Delta K(M_{11}(\alpha_0) + M_{12}(\alpha_0)K_0) \\ - (M_{22}(\alpha_0) - K_0 M_{12}(\alpha_0))\Delta K.$$

□

The continuity property of each solution when the separation condition (9) is verified motivates results dealing with analytical solutions of PNARE (1) instead of numerical ones.

Notice that $\Lambda(J(\alpha))$ is always included in $\Lambda(M(\alpha))$ and actually differentiates among various solutions of the PNARE. To define specifically several important solutions assume the eigenvalues of the matrix M are ordered such that their real parts form a nondecreasing sequence,

$$\mathrm{Re}(\lambda_1(M(\alpha))) \le \mathrm{Re}(\lambda_2(M(\alpha))) \le \\ \le \cdots \le \mathrm{Re}(\lambda_{n+p}(M(\alpha))). \tag{10}$$

Definition 4. Following the definitions in (Medanic, 1982; Engwerda, 2005):

- If $\mathrm{Re}(\lambda_p(M(\alpha))) < \mathrm{Re}(\lambda_{p+1}(M(\alpha)))$, then the solution $K(\alpha)$ related to $\{\lambda_{p+1}(M(\alpha)), \ldots, \lambda_{n+p}(M(\alpha))\}$ is called *dichotomic*;
- If $\mathrm{Re}(\lambda_n(M(\alpha))) < \mathrm{Re}(\lambda_{n+1}(M(\alpha)))$, then the solution $K(\alpha)$ related to $\{\lambda_1(M(\alpha)), \ldots, \lambda_n(M(\alpha))\}$ is called *reverse dichotomic*;
- A solution $K(\alpha)$ is called *stabilizing* if
$$\Lambda(M_{11}(\alpha) + M_{12}(\alpha)K(\alpha)) \subset \mathbb{C}^-,$$
and *strongly stabilizing* if, in addition,
$$\Lambda(M_{22}(\alpha) - K(\alpha)M_{12}(\alpha)) \subset \mathbb{C}^+.$$
The strongly stabilizing solution is also referred as (n,p)–splitting solution.

By the latter definitions and Theorem 2, a strongly stabilizing solution exists only if

$$\mathrm{Re}(\lambda_n(M)) < 0 < \mathrm{Re}(\lambda_{n+1}(M)),$$

and $M(\alpha)$ admits a graph invariant subspace related to the eigenvalues $\{\lambda_1(M(\alpha)), \ldots, \lambda_n(M(\alpha))\}$, see (Abou-Kandil et al., 2003). When a strongly stabilizing solution exists it is unique.

In order to illustrate the context of continuity properties of PNARE (1) we provide an academic illustration, with $n = 2$ and $p = 1$:

$$M(\alpha) = \left[\begin{array}{cc:c} 1 & \alpha & \max(\alpha,0) \\ \alpha & 1 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right].$$

The eigenvalues of $M(\alpha)$ are $\{2, 1-\alpha, 1+\alpha\}$ and $M(\alpha)$ is continuous with respect to the parameter and such Theorem 3 applies.

For $\alpha \in]-\infty,-1[\cup]-1,0[$, the solution is unique and trivial $K_a(\alpha) = [\,0\ \ 0\,]$ by selecting $\{1-\alpha, 1+\alpha\}$. It is reverse dichotomic only when $\alpha \in]-1,0[$.

For $\alpha = -1$, the eigenvalues are as follows: 2 is double and 0 is simple. The solutions are of infinite number and given by selecting $\{0,2\}$, $K_b(\alpha,\beta) = [\,-\beta\ \ +\beta\,]$, for any $\beta \in \mathbb{R}$.

For $\alpha = 0$, the eigenvalues are as follows: 2 is simple and 1 is double. The solutions are

- by selecting $\{1,1\}$, $K_a(\alpha) = [\,0\ \ 0\,]$, which is reverse dichotomic,
- by selecting $\{1,2\}$, $K_c(\alpha,\beta) = [\,1\ \ \beta\,]$, for any $\beta \in \mathbb{R}$.

For $\alpha \in]0,1[\cup]1,+\infty[$, the solutions are

- by selecting $\{1+\alpha, 1-\alpha\}$, $K_d(\alpha) = [\,0\ \ 0\,]$, which is reverse dichotomic when $\alpha \in]0,1[$,
- by selecting $\{2, 1-\alpha\}$, $K_e(\alpha) = [\,-1+1/\alpha\ \ -1+1/\alpha\,]$,
- by selecting $\{2, 1+\alpha\}$, $K_f(\alpha) = [\,1+1/\alpha\ \ -1-1/\alpha\,]$, which is dichotomic.

For $\alpha = 1$, the eigenvalues are as follows: 2 is double and 0 is simple. The solutions are

- by selecting $\{0,2\}$, $K_a(\alpha) = [\,0\ \ 0\,]$,
- by selecting $\{2,2\}$, $K_g(\alpha) = [\,2\ \ -2\,]$, which is dichotomic.

On this numerical illustration, we can see that when the separation condition (9) is not verified, it seems very difficult to formalize the number of solutions, which depends not only on the eigenvalues but on the eigenstructure of the matrix $M(\alpha)$. For $\alpha = -1$ the number of solutions is infinite but for $\alpha = 1$, there are only two solutions. For $\alpha = 1$, we face to a bifurcation of the solutions.

Invariant subspaces are closely associated with the solution of a NARE. The matrix sign function conserving the invariant subspaces will be useful to solve the NARE. The following section details this operator.

3. MATRIX SIGN FUNCTION

The *sign function* for a scalar $z \in \mathbb{C}$ is given by

$$\mathrm{sign}(z) = \begin{cases} +1 \text{ if } \mathrm{Re}(z) > 0, \\ -1 \text{ if } \mathrm{Re}(z) < 0, \\ \text{undefined otherwise.} \end{cases} \tag{11}$$

The *matrix sign function* is an extension of the sign function defined for matrices that do not feature eigenvalues on the imaginary axis (for more details see (Kenney and Laub, 1995)). Let $A \in \mathbb{R}^{(n+p)\times(n+p)}$,

$$A = V(D+J)V^{-1} \tag{12}$$

be its Jordan canonical form, where $D = \mathrm{diag}(d_1, \cdots, d_{n+p})$, J is nilpotent, J and D are commutative with respect to multiplication, and n and p are the number of eigenvalues with negative and positive real parts, respectively. Then

$$\mathrm{sign}(A) := V\mathrm{diag}(\mathrm{sign}(d_1), \cdots, \mathrm{sign}(d_{n+p}))V^{-1}. \tag{13}$$

The properties of the matrix sign function are detailed in (Roberts, 1980; Kenney and Laub, 1995). Here we will only recall the main ones that will be used in the sequel of the paper. Let $A \in \mathbb{R}^{(n+p)\times(n+p)}$:

- $\mathrm{sign}(A)$ is diagonalizable;
- $\mathrm{sign}(A)$ is a square root of the identity matrix;
- $\mathrm{sign}(A)$ and A are commutative;
- $\mathrm{rank}(\mathrm{sign}(A) - I_{n+p})$ equals the number of anti–stable eigenvalues (multiplicity counted) of A and $\mathrm{rank}(\mathrm{sign}(A) + I_{n+p})$ equals the number of stable eigenvalues of A;
- If A is Hurwitz, $\mathrm{sign}(A) = -I_{n+p}$;
- For any real scalar $\alpha \neq 0$, $\mathrm{sign}(\alpha A) = \mathrm{sign}(\alpha)\mathrm{sign}(A)$, (the *scaling property*);
- The matrices A and $\mathrm{sign}(A)$ share the same invariant subspaces, i.e.,
$$AV = VS \Rightarrow \mathrm{sign}(A)V = V\mathrm{sign}(S), \tag{14}$$
where V and S are two matrices of appropriate dimensions (for more details see (Bini et al., 2012, Lemma 1.7))

These properties are crucial to compute invariant subspaces of matrices and to solve NAREs by transforming the quadratic equation into an over–determined linear system of equations (see for example (Barraud, 1979; Byers, 1987)). The matrix sign function of a parameterized matrix $M(\alpha)$ is well defined only when $M(\alpha)$ does not admit imaginary eigenvalues. Hence, the set of parameters $\mathscr{U}$ should exclude these values.

Several methods are available to obtain matrix sign function of a given matrix besides the definition (13). An analytical way to compute the matrix sign function has been introduced in (Roberts, 1980) and is given by

$$\begin{aligned}\mathrm{sign}(M(\alpha)) &= \frac{2}{\pi}M(\alpha)\int_0^{+\infty}(y^2 I_{n+p} + M^2(\alpha))^{-1}\mathrm{d}y, \\ &= \frac{2M(\alpha)}{\pi}\int_0^{+\infty}\frac{\mathrm{adj}(y^2 I_{n+p} + M^2(\alpha))\mathrm{d}y}{\det(y^2 I_{n+p} + M^2(\alpha))}, \end{aligned} \tag{15}$$

where the adjugate $\mathrm{adj}(\cdot)$ is the transpose of the cofactor matrix of the argument. Computing $\mathrm{sign}(M(\alpha))$ leads to calculate the integral of a rational fraction in the unknown y^2. Computing the integral of such a rational fraction can be done thanks to the factorization of the denominator $\det(y^2 I_{n+p} + M^2(\alpha))$ and integral of simple elements or, alternatively, by considering a residual approach. The formula (15) is of interest in the parameterized case because it allows to keep an explicit dependency with respect to the parameter α. Notice that contrary to (Guerra et al., 2012), we make no assumption on the dependency of $M(\alpha)$ on the parameter α .

When the value of the parameter α is fixed, a large collection of advanced algorithms are available for the computation of the matrix sign function of a matrix featuring no eigenvalues on the imaginary axis. A widespread source of iterative algorithm is based on the observation that $(\mathrm{sign}(M))^2 = I_{n+p}$ and on Newton–Kantorovich procedure with additional refinements. The current paper being focused on parameterized NARE, the numerical algorithms for a specific parameter are not recalled, see (Kenney and Laub, 1995) or (Benner and Byers, 2006) for details.

4. COMPUTATION OF SOLUTIONS OF (P)NARES: A MATRIX SIGN FUNCTION APPROACH

The strongly stabilizing solution of a PNARE will be considered first in Subsection 4.1 as a keystone for all other types of solutions in continuous-time domain, investigated further into Subsection 4.2. The main guidelines to allow to treat parameter non-symmetric discrete-time algebraic Riccati equations will be provided in Subsection 4.3.

4.1 Computation of the strongly stabilizing solution to the PNARE

When considering the particular case of strongly stabilizing solution to the PNARE, thanks to the use of the matrix sign function, we can refine Theorems 1 and 2 as follows.

Theorem 5. If the PNARE (1) has a strongly stabilizing solution $K(\alpha)$ over $\alpha \in \mathscr{U}$, then the characteristic matrix $M(\alpha)$ admits a matrix sign function, denoted

$$\mathrm{sign}(M(\alpha)) = \left[\begin{array}{c:c} W_{11}(\alpha) & W_{12}(\alpha) \\ \hdashline W_{21}(\alpha) & W_{22}(\alpha)\end{array}\right], \quad \forall \alpha \in \mathscr{U}, \tag{16}$$

with $W_{11}(\alpha) \in \mathbb{R}^{n\times n}$, $W_{12}(\alpha) \in \mathbb{R}^{n\times p}$, $W_{21}(\alpha) \in \mathbb{R}^{p\times n}$ and $W_{22}(\alpha) \in \mathbb{R}^{p\times p}$, and that verifies

$$\mathrm{Trace}(\mathrm{sign}(M(\alpha))) = p - n. \tag{17}$$

In addition, $K(\alpha)$ is the unique solution of the over–determined system

$$\begin{bmatrix} W_{12}(\alpha) \\ W_{22}(\alpha) + I_p \end{bmatrix} K(\alpha) = -\begin{bmatrix} W_{11}(\alpha) + I_n \\ W_{21}(\alpha) \end{bmatrix}. \tag{18}$$

□

Proof: The proof follows the lines of Theorem 3.9 in (Bini et al., 2012) by considering the parameterized NARE. When a strongly stabilizing solution exists over $\mathscr{U}$, the spectrum of the characteristic matrix $M(\alpha)$ contains exactly n stable eigenvalues and p antistable eigenvalues (multiplicity counted). By continuity of $M(\alpha)$ with respect to the parameter α, this splitting is verified over $\mathscr{U}$. $M(\alpha)$ has no purely imaginary eigenvalues and admits a matrix sign function over $\mathscr{U}$. The PNARE reads

$$M(\alpha)\begin{bmatrix} I_n \\ K(\alpha)\end{bmatrix} = \begin{bmatrix} I_n \\ K(\alpha)\end{bmatrix}(M_{11}(\alpha) + M_{12}(\alpha)K(\alpha)), \tag{19}$$

with $(M_{11}(\alpha) + M_{12}(\alpha)K(\alpha))$ stable because $K(\alpha)$ is a strongly stabilizing solution. $M(\alpha)$ admitting a matrix sign function, we apply a matrix sign function to the latter equation. Thanks to the property (14), that yields:

$$\left[\begin{array}{c:c} W_{11}(\alpha) & W_{12}(\alpha) \\ \hdashline W_{21}(\alpha) & W_{22}(\alpha)\end{array}\right]\begin{bmatrix} I_n \\ K(\alpha)\end{bmatrix} = -\begin{bmatrix} I_n \\ K(\alpha)\end{bmatrix}. \tag{20}$$

A simple rewriting leads to the over-determined system (18). The uniqueness of the solution is a straightforward consequence of the uniqueness of the strongly stabilizing solution and ends the proof. ■

Theorem 6. Let consider the admissible set of parameters $\mathscr{U}$ for which the characteristic matrix $M(\alpha)$ admits a matrix sign function $\mathrm{sign}\,(M(\alpha))$, denoted by (16). Consider the restriction $\mathscr{U}_1 \subset \mathscr{U}$ for which $\mathrm{Trace}(\mathrm{sign}\,(M(\alpha))) = p - n$. Denote also the subset $\mathscr{U}_2 \subset \mathscr{U}_1$ related to parameters α such that $\mathrm{rank}\left(\begin{bmatrix} W_{12}(\alpha) \\ W_{22}(\alpha)+I_p \end{bmatrix}\right) = p$. Over $\mathscr{U}_2$, the over-determined system

$$\begin{bmatrix} W_{12}(\alpha) \\ W_{22}(\alpha)+I_p \end{bmatrix} K(\alpha) = -\begin{bmatrix} W_{11}(\alpha)+I_n \\ W_{21}(\alpha) \end{bmatrix} \qquad (21)$$

admits one and only one solution $K(\alpha)$. This parameter dependent matrix is the strongly stabilizing solution to PNARE (1).

□

Proof: The trace condition implies that the characteristic matrix $M(\alpha)$ and $\mathrm{sign}\,(M(\alpha))$ have exactly n stable eigenvalues and p antistable eigenvalues. The over-determined system (21) admits a solution if and only if all the columns of $\begin{bmatrix} W_{11}(\alpha)+I_n \\ W_{21}(\alpha) \end{bmatrix}$ depend linearly on the columns of matrix $\begin{bmatrix} W_{12}(\alpha) \\ W_{22}(\alpha)+I_p \end{bmatrix}$, or in other words

$$\begin{aligned} &\mathrm{rank}\left(\begin{bmatrix} W_{12}(\alpha) \\ W_{22}(\alpha)+I_p \end{bmatrix}\right) \\ &= \mathrm{rank}\left(\left[\begin{array}{c:c} W_{11}(\alpha)+I_n & W_{12}(\alpha) \\ W_{21}(\alpha) & W_{22}(\alpha)+I_p \end{array}\right]\right) \\ &= \mathrm{rank}\,(\mathrm{sign}\,(M(\alpha))+I_{n+p}) \\ &= p. \end{aligned}$$

The latter equality holds because the spectrum is (n,p)-splited. Moreover the matrix $\begin{bmatrix} W_{12}(\alpha) \\ W_{22}(\alpha)+I_p \end{bmatrix}$ is thus full column ranked, which concludes the uniqueness of the solution. ■

Theorem 5 emphasizes the fact that the strongly stabilizing solution over any set of admissible parameters can be obtained thanks to Theorem 6. Based on Theorem 6, it is easy to build a numerical algorithm to solve the strongly stabilizing solution to a PNARE (1).

The following subsection will be dedicated to use this material in order to obtain solutions of NAREs of other types than the strongly stabilizing one.

4.2 Computation of other solutions

A specific application of Theorem 6 allows to use the same approach for computing other solutions of a NARE as well:

- Let $\tilde{M}$ be a dichotomically separable matrix and consider the PNARE defined by the characteristic matrix $M(\alpha) = -\tilde{M} + \alpha I_{n+p}$, with
$$\mathscr{U} = \{\alpha \in \mathbb{R},\ \mathrm{Re}\left(\lambda_p(\tilde{M})\right) < \alpha < \mathrm{Re}\left(\lambda_{p+1}(\tilde{M})\right)\}.$$
The strongly stabilizing solution of the PNARE (1) exists and it is precisely the dichotomic solution related to $\tilde{M}$.
- Let $\tilde{M}$ be a reverse dichotomically separable matrix and consider the PNARE defined by the characteristic matrix $M(\alpha) = \tilde{M} - \alpha I_{n+p}$, with
$$\mathscr{U} = \{\alpha \in \mathbb{R},\ \mathrm{Re}\left(\lambda_n(\tilde{M})\right) < \alpha < \mathrm{Re}\left(\lambda_{n+1}(\tilde{M})\right)\}.$$
The strongly stabilizing solution of the PNARE (1) exists and it is the reverse dichotomic solution related to $\tilde{M}$. An alternative point of view is provided in (Jungers, 2014) by rewriting the NARE

$$\begin{aligned} &\tilde{M}_{21} + (\tilde{M}_{22} - \alpha I_p)K - K(\tilde{M}_{11} - \alpha I_n) - K\tilde{M}_{12}K \\ &= \tilde{M}_{21} + \tilde{M}_{22}K - K\tilde{M}_{11} - K\tilde{M}_{12}K. \qquad (22) \end{aligned}$$

4.3 Discrete-time PNAREs

The discrete-time parameterized NARE can be defined as the algebraic quadratic equation over $\alpha \in \mathscr{U}$, $(n,p \in \mathbb{N}^*)$

$$\begin{aligned} K(\alpha) &= N_{22}(\alpha) \\ &+ N_{12}(\alpha)K(\alpha)(I_n + N_{11}(\alpha)K(\alpha))^{-1}N_{21}(\alpha), \qquad (23) \end{aligned}$$

in the unknown $K(\alpha) \in \mathbb{R}^{p\times n}$, where $N_{11}(\alpha) \in \mathbb{R}^{n\times p}$, $N_{21}(\alpha) \in \mathbb{R}^{n\times n}$, $N_{12}(\alpha) \in \mathbb{R}^{p\times p}$ and $N_{22}(\alpha) \in \mathbb{R}^{p\times n}$.

Equation (23) is associated with the characteristic matrix pencil, or first degree polynomial in $z \in \mathbb{C}$,

$$\begin{aligned} N_2(\alpha) - zN_1(\alpha) &= \left[\begin{array}{c:c} N_{21}(\alpha) & 0_{n\times p} \\ \hdashline N_{22}(\alpha) & -I_p \end{array}\right] \\ &- z\left[\begin{array}{c:c} I_n & N_{11}(\alpha) \\ \hdashline 0_{p\times n} & -N_{12}(\alpha) \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+p)}, \end{aligned} \qquad (24)$$

and can be equivalently rewritten as

$$\begin{aligned} &N_2(\alpha)\begin{bmatrix} I_n \\ K(\alpha) \end{bmatrix} \\ &= N_1(\alpha)\begin{bmatrix} I_n \\ K(\alpha) \end{bmatrix}(I_n + N_{11}(\alpha)K(\alpha))^{-1}N_{21}(\alpha). \qquad (25) \end{aligned}$$

Under the assumption stating that the square pencil $N_2(\alpha) - zN_1(\alpha)$ in (24) is regular (square and non vanishing determinant) and has no infinite eigenvalues, the matrices $N_1(\alpha)$ and $N_{12}(\alpha)$ are both invertible, while the eigenvalue problem for the pencil $N_2(\alpha) - zN_1(\alpha)$ reduces to that for the matrix $N_1^{-1}(\alpha)N_2(\alpha)$.

In discrete-time domain, a strongly stabilizing solution of Equation (23) is defined as a solution $K(\alpha)$ such that

$$\Lambda((I_n + N_{11}(\alpha)K(\alpha))^{-1}N_{21}(\alpha)) \subset \mathbb{D}, \qquad (26)$$

$$\Lambda(-(N_{12}^{-1}(\alpha) + K(\alpha)N_{11}(\alpha)N_{12}^{-1}(\alpha))) \subset \mathbb{C}\backslash\mathbb{D}. \qquad (27)$$

A strongly stabilizing solution exists if and only if

$$|\lambda_n(N_2(\alpha),N_1(\alpha))| < 1 < |\lambda_{n+1}(N_2(\alpha),N_1(\alpha))|, \qquad (28)$$

where $\lambda_i(N_2(\alpha),N_1(\alpha))$, $i \in \{1,\cdots,n+p\}$ denotes the (finite) eigenvalues of the pencil $N_2(\alpha) - zN_1(\alpha)$, that are ordered such that their modulus form a nondecreasing sequence,

$$\begin{aligned} &|\lambda_1(N_2(\alpha),N_1(\alpha))| \le |\lambda_2(N_2(\alpha),N_1(\alpha))| \\ &\le \cdots \le |\lambda_{n+p}(N_2(\alpha),N_1(\alpha))|. \qquad (29) \end{aligned}$$

By applying the Cayley transform (see (Bini et al., 2012)), the discrete-time strongly stabilizing solution of Equation (23) associated with the parameterized pencil $N_2(\alpha) - zN_1(\alpha)$ coincides with the strongly stabilizing solution to the continuous-time PNARE (1) associated with the characteristic matrix

$$M(\alpha) = (N_2(\alpha) - N_1(\alpha))^{-1}(N_2(\alpha) + N_1(\alpha)).$$

This property allows to use the method presented above.

5. NUMERICAL EXAMPLES

We give here several numerical examples to demonstrate the efficiency of our matrix sign function approach.

Example 1: Let in continuous-time domain with $n = 1$, $p = 2$

$$M(\alpha) = \left[\begin{array}{c|cc} 1+\alpha+\alpha^2 & 0 & \alpha \\ \hline \cos(\alpha)-1-\alpha-\alpha^2 & \cos(\alpha) & -\alpha \\ -\alpha^2(1+\alpha) & 0 & 1-\alpha^2 \end{array}\right],\ \alpha \in \mathbb{R}.$$

We solve, with the Matrix Sign Approach, the parametric Non-symmetric Algebraic Riccati Equation:

$$\begin{bmatrix} \cos(\alpha)-1-\alpha-\alpha^2 \\ -\alpha^2(1+\alpha) \end{bmatrix} + \begin{bmatrix} \cos(\alpha) & -\alpha \\ 0 & 1-\alpha^2 \end{bmatrix} K(\alpha)$$
$$-K(\alpha)\left[\,1+\alpha+\alpha^2\,\right] - K(\alpha)\left[\,0\ \ \alpha\,\right]K(\alpha) = 0_{2\times 1},\ \forall \alpha \in \mathbb{R}. \tag{30}$$

The matrix $\operatorname{sign}(M(\alpha))$ is well posed for $\alpha \in \mathscr{U}$, where $\mathscr{U} = \mathbb{R}/(\{-1\}\cup\{\pi k,\ k \in \mathbb{N}\})$, and is given by

$$\begin{bmatrix} m_1(\alpha) & 0 & \operatorname{sign}(1+\alpha)-1 \\ m_2(\alpha) & \operatorname{sign}(\cos(\alpha)) & 1-\operatorname{sign}(1+\alpha) \\ m_3(\alpha) & 0 & 1-\alpha(\operatorname{sign}(1+\alpha)-1) \end{bmatrix},$$

with

$$m_1(\alpha) = (1+\alpha)\operatorname{sign}(1+\alpha)-\alpha,$$
$$m_2(\alpha) = -(1+\alpha)\operatorname{sign}(1+\alpha)+\operatorname{sign}(\cos(\alpha))+\alpha,$$
$$m_3(\alpha) = -\alpha(1+\alpha)(\operatorname{sign}(1+\alpha)-1).$$

Then we have Trace(sign($M(\alpha)$))=$\operatorname{sign}(1+\alpha)+\operatorname{sign}(\cos(\alpha))+1$=1=$n\times(-1)+p\times(+1)$ for $\alpha > -1$ and $\cos(\alpha) < 0$, or for $\alpha < -1$ and $\cos(\alpha) > 0$.

In the case where $\alpha > -1$ and $\cos(\alpha) < 0$, we have

$$\operatorname{sign}(M(\alpha)) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is independent of α and does not admit a graph invariant subspace related to the value -1. Therefore, there is no solution to the NARE for these parameters.

In the case where $\alpha < -1$ and $\cos(\alpha) > 0$, we have

$$\operatorname{sign}(M(\alpha)) = \begin{bmatrix} -1-2\alpha & 0 & -2 \\ 2(1+\alpha) & 1 & 2 \\ 2\alpha(1+\alpha) & 0 & 1+2\alpha \end{bmatrix}.$$

The overdetermined system verified by $K(\alpha)$ is thus

$$\begin{bmatrix} 0 & -2 \\ 2 & 2 \\ 0 & 2+2\alpha \end{bmatrix} K(\alpha) = -\begin{bmatrix} -2\alpha \\ 2(1+\alpha) \\ 2\alpha(1+\alpha) \end{bmatrix}$$

yielding $K(\alpha) = \begin{bmatrix} -1 \\ -\alpha \end{bmatrix}$ the unique strongly stabilizing solution of the PNARE (30), for which $M_{11}(\alpha)+M_{12}(\alpha)K(\alpha) = 1+\alpha < 0$.

Example 2: Consider now the example in (Lin, 1998) to solve the parameterized ARE

$$-K(\alpha)A - A'K(\alpha) + K(\alpha)BB'K(\alpha) - Q(\alpha) = 0_2,$$

with

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ Q = \alpha I_2,\ \forall \alpha \in (0,1].$$

The induced matrix $M(\alpha)$ is defined by

$$M(\alpha) = \left[\begin{array}{cc|cc} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ \hline -\alpha & 0 & 1 & 1 \\ 0 & -\alpha & -1 & -1 \end{array}\right],\quad \forall \alpha \in (0,1].$$

The characteristic polynomial of $M(\alpha)$ is $\det(xI_4 - M(\alpha)) = x^4 - \alpha x^2 + 2\alpha = 0$ and has exactly two stable roots for any $\alpha \in (0,1]$. After a simple calculus, we obtain

$$\begin{aligned}
&\det(y^2 I_4 + M^2(\alpha)) \\
&\quad = (y^4+\alpha y^2+2\alpha)^2, \\
&\quad = \left(y^2 - y\sqrt{2\sqrt{2\alpha}-\alpha}+\sqrt{2\alpha}\right)^2 \\
&\qquad \times \left(y^2 + y\sqrt{2\sqrt{2\alpha}-\alpha}+\sqrt{2\alpha}\right)^2,
\end{aligned}$$

and

$$M(\alpha)\left(y^2 I_4 + M^2(\alpha)\right)^{-1} = \frac{1}{y^4+\alpha y^2+2\alpha}\times$$
$$\begin{bmatrix} -y^2-\alpha & y^2 & -1 & -1 \\ -y^2+\alpha & y^2 & -1 & -y^2-1 \\ -\alpha(y^2+\alpha+2) & 2\alpha & y^2+\alpha & y^2-\alpha \\ 2\alpha & -\alpha(y^2+2) & -y^2 & -y^2 \end{bmatrix}.$$

Moreover, denoting for sake of clarity $\chi = \sqrt{\alpha+2\sqrt{2\alpha}}$ and $\xi = \frac{1}{\sqrt{2\alpha}}$, and thanks to polynomial factorization one gets

$$\frac{2}{\pi}\int_0^{+\infty} \frac{ay^2+b}{y^4+\alpha y^2+2\alpha}\mathrm{d}y = \frac{a+b\xi}{\chi},\quad \forall \alpha \in (0,1],$$

implying that $\operatorname{sign}(M(\alpha))$ is given by

$$\frac{1}{\chi}\begin{bmatrix} -1-\alpha\xi & 1 & -\xi & -\xi \\ -1+\alpha\xi & 1 & -\xi & -1-\xi \\ -\alpha-(2\alpha+\alpha^2)\xi & 2\alpha\xi & 1+\alpha\xi & 1-\alpha\xi \\ 2\alpha\xi & -\alpha-2\alpha\xi & -1 & -1 \end{bmatrix}.$$

We can verify that $\operatorname{Trace}(\operatorname{sign}(M(\alpha))) = 0$ which is in accordance with $M(\alpha)$ having exactly two stable eigenvalues for $\alpha \in (0,1]$. We would like to choose the solution $K(\alpha)$ to stabilize $M_{11}(\alpha)+M_{12}(\alpha)K(\alpha) = A - BB'K(\alpha)$. The overdetermined linear system (18) becomes:

$$\begin{bmatrix} -\xi & -\xi \\ -\xi & -1-\xi \\ 1+\alpha\xi+\chi & 1-\alpha\xi \\ -1 & -1+\chi \end{bmatrix} K(\alpha)$$
$$= -\begin{bmatrix} -1-\alpha\xi+\chi & 1 \\ -1+\alpha\xi & 1+\chi \\ -\alpha-(2\alpha+\alpha^2)\xi & 2\alpha\xi \\ 2\alpha\xi & -\alpha-2\alpha\xi \end{bmatrix},$$

resulting into the symmetric and positive definite matrix

$$K(\alpha) = \begin{bmatrix} \chi(1+\sqrt{2\alpha}-\chi) & \sqrt{2\alpha}-\chi \\ \sqrt{2\alpha}-\chi & \chi \end{bmatrix},\quad \forall \alpha \in (0,1].$$

We recognize the analytical solution provided in (Lin, 1998).

Example 3: Let us consider $n = 2$, $p = 1$ and

$$\tilde{M} = \left[\begin{array}{cc|c} -1 & -2 & 6 \\ 6 & 7 & -18 \\ \hline 3 & 2 & -4 \end{array}\right],$$

for which the eigenvalues are $\{-1,1,2\}$. For $\alpha \in (-1,1)$, we have

$$\operatorname{sign}\left(-\tilde{M}+\alpha I_3\right) = \left[\begin{array}{cc:c} 2s_1-3s_2+2s_3 & s_1-s_2 & 2s_3-2s_1 \\ 12s_2-6s_1-6s_3 & 4s_2-3s_1 & 6s_1-6s_3 \\ \hdashline 3s_2-2s_1-s_3 & s_2-s_1 & 2s_1-s_3 \end{array}\right] = \left[\begin{array}{cc:c} 3 & 2 & -4 \\ -12 & -7 & 12 \\ \hdashline -4 & -2 & 3 \end{array}\right],$$

with

$$s_1 = \operatorname{sign}(1+\alpha) = 1,$$
$$s_2 = \operatorname{sign}(-1+\alpha) = -1,$$
$$s_3 = \operatorname{sign}(-2+\alpha) = -1.$$

The overdetermined linear system leads to $K(\alpha) = \left[1 \;\; \frac{1}{2}\right]$, which is the dichotomic solution related to $\tilde{M}$ because the eigenvalues of $\tilde{M}_{11}+\tilde{M}_{12}K(\alpha)$ are $\{1,2\}$.

To compute the dichotomic solution, let us select $\alpha \in (1,2)$ and get

$$\operatorname{sign}\left(\tilde{M}-\alpha I_3\right) = \left[\begin{array}{cc:c} 2s_4-3s_5+2s_6 & s_4-s_5 & 2s_6-2s_4 \\ 12s_5-6s_4-6s_6 & 4s_5-3s_4 & 6s_4-6s_6 \\ \hdashline 3s_5-2s_4-s_6 & s_5-s_4 & 2s_4-s_6 \end{array}\right] = \left[\begin{array}{cc:c} 3 & 0 & 4 \\ -12 & -1 & -12 \\ \hdashline -2 & 0 & -3 \end{array}\right],$$

with

$$s_4 = \operatorname{sign}(-1-\alpha) = -1,$$
$$s_5 = \operatorname{sign}(1-\alpha) = -1,$$
$$s_6 = \operatorname{sign}(2-\alpha) = 1.$$

The overdetermined linear system leads to $K(\alpha) = [\,-1 \;\; 0\,]$, which is the reverse dichotomic solution related to $\tilde{M}$ because the eigenvalues of $\tilde{M}_{11}+\tilde{M}_{12}K(\alpha)$ are $\{-1,1\}$.

Example 4: We illustrate the computation of the reverse dichotomic and dichotomic solutions to the NARE. Let $n=2$, $p=4$ and the characteristic matrix

$$\tilde{M} = \left[\begin{array}{cc:cccc} 1 & 3 & 1 & 0 & 1 & 4 \\ 2 & 1 & 3 & 2 & -1 & -3 \\ \hdashline 1 & 0 & -2 & 0 & 0 & 0 \\ 2 & 1 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 0 \\ 3 & 1 & 0 & 0 & 0 & -3 \end{array}\right],$$

with the eigenvalues

$$\Lambda(\tilde{M}) = \left\{\begin{array}{r} 4.5535 \\ 0 \\ -1.9991+0.2001i \\ -1.9991-0.2001i \\ -3.1037 \\ -5.4516 \end{array}\right\}.$$

The matrix $\tilde{M}$ is singular (and therefore $\operatorname{sign}(\tilde{M})$ does not exist), dichotomic separable, and reverse dichotomic separable.

To compute the reverse dichotomic solution, choose $\alpha = \frac{1}{2}(\operatorname{Re}\left(\lambda_2(\tilde{M})+\lambda_3(\tilde{M})\right) = -2.5514$ and apply the method described in subsection 4.2, which leads to the reverse dichotomic solution

$$K = \begin{bmatrix} -0.2332 & 0.0974 \\ -0.8568 & -0.7678 \\ 11.7004 & 20.9855 \\ -4.5335 & -6.1135 \end{bmatrix}.$$

As a verification, we have

$$\Lambda(\tilde{M}_{11}+\tilde{M}_{12}K) = \{-5.4516; -3.1037\},$$

which are the expected values $\lambda_1(\tilde{M})$ and $\lambda_2(\tilde{M})$.

To compute the dichotomic solution, choose the shift $\alpha = \frac{1}{2}\operatorname{Re}\left(\lambda_4(\tilde{M})+\lambda_5(\tilde{M})\right)$ and apply the method to get the dichotomic solution

$$K = \begin{bmatrix} 0.2464 & -0.1690 \\ 0.3521 & 0.0681 \\ 0.1628 & -0.5581 \\ 0.4786 & -0.0143 \end{bmatrix}.$$

We verify that

$$\Lambda(\tilde{M}_{11}+\tilde{M}_{12}K) = \{0.0000; 4.5535\} = \{\lambda_5(\tilde{M}); \lambda_6(\tilde{M})\}.$$

6. CONCLUSION

The parameterized continuous–time non-symmetric algebraic Riccati equation has been investigated and a method to obtain the parameterized strongly stabilizing solution has been provided by means of computing the matrix sign function approach via its integral formulation. The approach exhibits the advantages to determine explicitly the parameter dependent solution which leads to the possibility of computing as well various specific solutions of the NARE. These techniques may be also applied to the discrete–time framework.

REFERENCES

Abou-Kandil, H., Freiling, G., Ionescu, V., and Jank, G. (2003). *Matrix Riccati Equations in Control and Systems Theory*. Birkhäuser.

Anderson, B. and Moore, J.B. (1989). *Optimal control: linear quadratic methods*. Prentice-Hall International Editions.

Apkarian, P. and Gahinet, P. (1995). A convex characterization of gain-scheduled $\mathscr{H}_\infty$ controllers. *IEEE Transactions on Automatic Control*, 40(5), 853–864.

Bai, Z.Z., Guo, X.X., and Xu, S.F. (2006). Alternately linearized implicit iteration methods for the minimal nonnegative solutions of the nonsymmetric algebraic Riccati equations. *Numerical Linear Algebra with Applications*, 13, 655–674.

Barraud, A. (1979). Investigation autour de la fonction signe d'une matrice, application à l'équation de Riccati. *R.A.I.R.O. Automatique*, 13, 335–368.

Başar, T. and Olsder, G.J. (1995). *Dynamic Noncooperative Game Theory*. SIAM.

Benner, P. and Byers, R. (2006). An arithmetic for matrix pencils: theory and new algorithms. *Numerische Mathematik*, 103(4), 539–573.

Bini, D.A., Iannazzo, B., and Meini, B. (2012). *Numerical Solution of Algebraic Riccati Equations*. Number 9 in Fundamentals of Algorithms. SIAM.

Bittanti, S., Laub, J.A., and Willems, J.C. (1991). *The Riccati Equation*. Springer Verlag.

Byers, R. (1987). Solving the algebraic Riccati equation with the matrix sign function. *Linear Algebra and its Applications*, 85, 267–280.

Denman, E. and Beavers, A. (1976). The matrix sign function and computations in systems. *Applied Mathematics and Computation*, 2(1), 63–94.

Dorato, P., Abdallah, C.T., and Cerone, V. (1995). *Linear-Quadratic Control, An Introduction*. Ed. Prentice-Hall.

Engwerda, J.C. (2005). *LQ Dynamic Optimization and Differential Games*. Wiley.

Faibusovich, L.E. (1986). Algebraic Riccati equation and symplectic algebra. *International Journal of Control*, 43(3), 781–792.

Freiling, G., Jank, G., and Abou-Kandil, H. (1999). Discrete-time Riccati equations in open-loop Nash and Stackelberg games. *European Journal of Control*, 5, 56–66.

Gardiner, J.D. and Laub, A.J. (1986). A generalization of the matrix-sign-function solution for algebraic Riccati equations. *International Journal of Control*, 44, 823–832.

Guerra, J., Yagoubi, M., and Chevrel, P. (2012). A Matrix Sign Function framework for robust stability analysis and parameter-dependent Lyapunov and Riccati equalities. In *Proceedings of the 51st IEEE Conference on Decision and Control*, 3490–3495. Maui, Hawai, USA.

Guerra, J., Yagoubi, M., and Chevrel, P. (2015). Mixed matrix sign function /DFT inversion method for solving parameter-dependent Riccati equation. *Journal of Franklin Institute*, 352(12), 5611–5626.

Guo, C.H. (2001). Nonsymmetric algebraic Riccati equations and Wiener-Hopf factorization for *m*-matrices. *SIAM J. Matrix Anal. & Appl.*, 23(1), 225–242.

Guo, C.H. and Laub, A.J. (2000). On the iterative solution of a class of nonsymmetric algebraic Riccati equation. *SIAM J. Matrix Anal. & Appl.*, 22(2), 376–391.

Guo, X.X. and Bai, Z.Z. (2005). On the minimal nonnegative solution of nonsymmetric algebraic Riccati equation. *Journal of Computational Mathematics*, 23(3), 305–320.

Ho, Y.C. (1970). Survey paper: Differential games, dynamic optimization and generalized control theory. *Journal of Optimization Theory and Applications*, 6(3), 179–209.

Ionescu, V. and Oară, C. (1996). Generalized discrete-time Riccati theory. *SIAM J. Control Optim.*, 34, 601–619.

Ionescu, V., Oară, C., and Weiss, M. (1999). *Generalized Riccati theory and robust control: a Popov function approach*. Chichester, New York, John Wiley.

Ionescu, V. and Weiss, M. (1993). Continuous and discrete-time Riccati theory: a Popov-function approach. *Linear Algebra and its Applications*, 193, 173–209.

Jamshidi, M., D'Ans, G., and Kokotović, P. (1970). Application of a parameter-imbedded Riccati equation. *IEEE Transactions on Automatic Control*, 15(6), 682–683.

Juang, J. (1995). Existence of algebraic matrix Riccati equations arising in transport theory. *Linear Algebra and its Applications*, 230, 89–100.

Jungers, M. (2008). On Stackelberg linear quadratic games with time preference rate. *IEEE Transactions on Automatic Control*, 53(2), 621–625.

Jungers, M. (2014). Computing (reverse) dichotomic solutions of non-symmetric algebraic Riccati equations via the Matrix Sign Function. In *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 1645–1650. University of Groningen, Groningen, The Netherlands.

Jungers, M. (2017). Historical perspectives of the Riccati equations. In *The 20th World Congress of the International Federation of Automatic Control*, Invited session "300 years developments for an essential tool in control system theory: the Riccati equations", 9945–9956.

Jungers, M. and Oară, C. (2012). Continuous-time non-symmetric algebraic Riccati theory: A matrix pencil approach. *European Journal of Control*, 18(1), 74–81.

Jungers, M., Oară, C., Abou-Kandil, H., and Ştefan, R. (2009). General matrix pencil techniques for solving non–symmetric algebraic Riccati equations. *SIAM, J. Matrix Anal. & Appl.*, 31(3), 1257–1278.

Karow, M. and Kressner, D. (2014). On a perturbation bound for invariant subspaces of matrices. *SIAM J. Matrix Anal. Appl.*, 35(2), 599–618.

Kenney, C.S. and Laub, A.J. (1995). The matrix sign function. *IEEE Transactions on Automatic Control*, 40(8), 1330–1348.

Kremer, D. (2003). *Non-symmetric Riccati theory and noncooperative games*. Ph.D. thesis, RWTH Aachen.

Kučera, V. (1972). A contribution to matrix quadratic equations. *IEEE Transactions on Automatic Control*, 344–347.

Lancaster, P. and Rodman, L. (1995). *Algebraic Riccati equations*. Oxford science publications.

Laub, A.J. (1979). A Schur method for solving the algebraic Riccati equation. *IEEE Transactions on Automatic Control*, 30, 97–108.

Laub, A.J. and Meyer, K.R. (1974). Canonical forms for Hamiltonian and symplectic matrices. *Celestial Mechanics*, 9, 213–238.

Lin, Z. (1998). Global control of linear systems with saturating actuators. *Automatica*, 34(7), 897–905.

Lu, L.Z. (2005). Newton iterations for a non-symmetric algebraic Riccati equation. *Numerical Linear Algebra with Applications*, 12, 191–200.

Medanic, J. (1982). Geometric properties and invariant manifolds of the Riccati equation. *IEEE Transactions on Automatic Control*, 27(3), 670–677.

Radon, J. (1927). Über die Oszillationstheoreme der konjugierten Punkte beim Problem von Lagrange. (german). *Münchener Sitzungsberichte*, 57, 243–257.

Ran, A.C.M. and Rodman, L. (1988). On parameter dependence of solutions of algebraic Riccati equations. *Mathematics of Control, Signals, and Systems*, 1(3), 269–284.

Ran, A.C.M. and Rodman, L. (1992). Stable Hermitian solutions of discrete algebraic Riccati equations. *Mathematics of Control, Signals, and Systems*, 5, 165–193.

Roberts, J. (1980). Linear model reduction and solution of the algebraic Riccati equation by use of the sign function. *International Journal of Control*, 32, 677–687.

Sandell, N.R. (1974). On Newton's method for Riccati equation solution. *IEEE Transactions on Automatic Control*, 19(3), 254–255.

Teel, A.R. (1995). Linear systems with input nonlinearities: Global stabilization by scheduling a family of $\mathscr{H}_\infty$-type controllers. *International Journal of Robust and Nonlinear Control*, 5(5), 399–411.

Trentelman, H.L. (1987). Families of Linear-Quadratic problems: Continuity properties. *IEEE Transactions on Automatic Control*, 32(4), 323–329.

Van Dooren, P.M. (1981). A generalized eigenvalue approach for solving Riccati equations. *SIAM J. Sci. Stat. Comput.*, 2, 121–135.

Weng, P.C.Y., Fan, H.Y., and Chu, E.K.W. (2012). Low-rank approximation to the solution of a nonsymmetric algebraic Riccati equation from transport theory. *Applied Mathematics and Computation*, 219, 729–740.