INVARIANT SETS WITH ARBITRARY TIME-DEPENDENCE IN THE DYNAMICS OF LINEAR SYSTEMS WITH INTERVAL-TYPE UNCERTAINTIES

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Abstract: *The paper develops an extensive analysis of the time-dependent sets that are positively invariant with respect to the dynamics of interval matrix systems. The interval matrix systems may be described by discrete- or continuous-time equations. For the invariant sets, the time dependence is considered arbitrary, and the shape is defined in terms of Hőlder vector p-norms,* $1 \le p \le \infty$. *The key instrument in this study is a matrix test function (appropriately defined for discrete- and continuous-time dynamics) which is applied to a matrix majorizing the interval matrix.*

Keywords: *parameter uncertainties, interval dynamical systems, discrete-time systems, continuous-time systems, invariant sets, Hőlder norms.*

1. INTRODUCTION

The exploration of (positively) invariant sets stimulated a great deal of research in the field of linear dynamical systems as resulting from the survey paper [2] and the papers cited therein. Within this context it is worth noticing that the literature on invariant sets with arbitrary time dependence is rather scarce compared with the studies devoted to constant or exponentially contractive sets. Therefore the interest of the current paper focuses on the first type of invariant sets which are investigated for interval dynamical systems. While most of the existing works analyze the time-dependent invariant sets with respect to completely known dynamics [6], [7], [11], [14], [16]-[19],our emphasis is now placed on uncertain systems, by intending to expand our previous results in [13], [10].

Let us consider a linear system with parameter uncertainties of interval-type, also called *interval system* (IS), with *discrete-time* (DT) dynamics, described by

$$x(t+1) = Ax(t),\ x(t_0) = x_0 \in \mathbb{R}^n,\quad t,\ t_0 \in \mathbb{Z}_+,\ t \ge t_0,\ A \in \mathcal{A}^I, \qquad (1)$$

or *continuous-time* (CT) dynamics, described by

$$\dot{x}(t) = Ax(t),\ x(t_0) = x_0 \in \mathbb{R}^n,\quad t,\ t_0 \in \mathbb{R}_+,\ t \ge t_0,\ A \in \mathcal{A}^I. \tag{2}$$

In (1) and (2) $\mathcal{A}^I$ is the interval matrix

$$\mathcal{A}^I = \{A \in \mathbb{R}^{n\times n} \mid A^- \le A \le A^+\} \tag{3}$$

defined by the componentwise inequalities $a_{ij}^- \le a_{ij} \le a_{ij}^+$, $i,j = 1,\dots,n$, where a_{ij}^-, a_{ij}, a_{ij}^+ represent the generic elements of matrices A^-, A and A^+ respectively. For $\mathcal{A}^I$, we define the majorant matrix $U = [u_{ij}] \in \mathbb{R}^{n\times n}$ as follows:

DT case: $u_{ij} = \max\{|a_{ij}^-|, |a_{ij}^+|\},\quad i,j = 1,\dots,n$, (4)

CT case:

$$\begin{cases} u_{ii} = a_{ii}^+,\ i = 1,\dots,n, \\ u_{ij} = \max\{|a_{ij}^-|, |a_{ij}^+|\},\ i \ne j,\ i,j = 1,\dots,n. \end{cases} \tag{5}$$

It is worth pointing out that many papers use matrix U in the stability analysis of IS (1) or (2), mainly for formulating sufficient criteria [3], ([1], Theorem 3), ([16], Corollary 1.2), ([8], Theorem 3.4.17) (*DT case*) and [9], ([4], Theorem 3.1), ([16], Corollary 2.2) (*CT case*). Also, for some particular classes of ISs, necessary and sufficient conditions are obtained ([16], Corollary 1.3), ([1], Corollary 1), ([8], Lemma 3.4.18) (*DT case*) and ([16], Corollary 2.3), ([20], Theorem 1) (*CT case*).

Our recent researches [12] and [13] show that the majorant matrix U can be used for the characterization of time-dependent, rectangular sets which are positively invariant with respect to the state-space trajectories of IS (1) or (2). These results encouraged our initiative of searching for new connections between the majorant matrix U and the invariance properties exhibited by IS (1) or (2). Thus, the current paper explores time-dependent invariant sets with general forms defined by Hölder p-norms ($1 \le p \le \infty$) and proves that the time-dependent rectangular boxes studied in [12] and [13] represent the particular case of $p = \infty$.

The remainder of the text is organized as follows. Section 2 provides notation and nomenclature. The time-dependent invariant sets for ISs with DT and CT dynamics are separately analyzed by the sections 3 and 4, respectively. Section 5 formulates some concluding remarks on the contributions of our work.

2. NOTATIONS AND NOMENCLATURE

Throughout the paper we use the following notations. For a vector $x \in \mathbb{R}^n$: $\|x\|_p$ is the Hölder vector p-norm defined for $1 \le p \le \infty$. For a square matrix $M \in \mathbb{R}^{n\times n}$: $\|M\|_p$ is the matrix norm induced by the vector norm $\| \ \|_p$; $m_p(M) = \lim_{\theta \downarrow 0}(\|I + \theta M\|_p - 1)/\theta$ is a matrix measure ([5], pp. 29) associated with the matrix norm $\| \ \|_p$.

Let $\mathbb{T}$ be the set in which the time parameter t in equations (1) and (2) takes values, namely $\mathbb{T} = \mathbb{Z}_+$ in the DT case and $\mathbb{T} = \mathbb{R}_+$ in the CT case. Let $h_i(t): \mathbb{T} \to \mathbb{R}_+^*$, $i = 1,\dots,n$, be positive functions, which, in the CT case, are also continuously differentiable, and construct the diagonal matrix function

$$H(t) = \mathrm{diag}\{h_1(t),\cdots,h_2(t)\}. \tag{6}$$

Given a Hölder p-norm, $1 \le p \le \infty$, use $H(t)$ for defining the *time-dependent set*

$$S_{p,H(t)} = \left\{x \in \mathbb{R}^n \mid \|H^{-1}(t)x\|_p \le 1\right\},\ t \in \mathbb{T}. \tag{7}$$

Definition 1. The set $S_{p,H(t)}$ defined by (7) is said to be *positively invariant with respect to* (abbreviated as PI w.r.t.) IS (1) or (2), if any trajectory initialized at t_0 inside $S_{p,H(t_0)}$ remains inside the set $S_{p,H(t_1)}$ at any $t_1 > t_0$. If $x(t; t_0, x_0)$ denotes a state-space trajectory of IS (1) or (2) initialized in $x(t_0) = x_0$, this condition can be formally written as:

$$\forall t_0 \in \mathbb{T}, \forall x_0 \in S_{p,H(t_0)} \Rightarrow \Rightarrow x(t_1; t_0, x_0) \in S_{p,H(t_1)},\ \forall t_1 > t_0. \quad \blacksquare \tag{8}$$

Remark 1. If the set $S_{p,H(t)}$ is PI w.r.t. IS (1) or (2), then, due to the linearity of the dynamics, all the sets $S_{p,cH(t)}$ defined with arbitrary $c > 0$ are PI w.r.t. IS (1) or (2). ■

3. INTERVAL SYSTEMS WITH DT DYNAMICS

Given an arbitrary p-norm ($1 \le p \le \infty$) and a diagonal matrix function $H(t)$ (6), let us introduce the test function:

$$\nu_{p,H(t)}(M) = \| H^{-1}(t+1) M H(t) \|_p , \quad (9)$$

where $M \in \mathbb{R}^{n\times n}$ is an arbitrary square matrix. The following result shows that the invariance of $S_{p,H(t)}$ (7) w.r.t. the trajectories of the DT IS (1) can be explored by evaluating the above test function for the majorant matrix U (4).

Theorem 1. a) Let $p = 1,\infty$. The set $S_{p,H(t)}$ is PI w.r.t. IS (1) if and only if $\nu_{p,H(t)}(U) \le 1$.

b) Let $1 < p < \infty$. The set $S_{p,H(t)}$ is PI w.r.t. IS (1) if $\nu_{p,H(t)}(U) \le 1$.

Proof: The proof has three parts:

Part I. First, we consider a time-invariant system with DT dynamics described by

$$x(t+1) = Mx(t),\ x(t_0) = x_0 \in \mathbb{R}^n, \quad t, t_0 \in \mathbb{Z}_+,\ t \ge t_0, \quad (10)$$

and characterize the positive invariance of time-dependent sets $S_{p,H(t)}$ w.r.t. system (10) by proving the equivalence between the following statements for arbitrary $1 \le p \le \infty$:

(i) the set $S_{p,H(t)}$ is PI w.r.t. system (10);

(ii) the function

$$W(x,t): \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}_+, W(x,t) = \| H^{-1}(t) x \|_p \quad (11)$$

is nonincreasing along each trajectory of system (10);

(iii) $\nu_{p,H(t)}(M) \le 1$.

(i) $\Rightarrow$ (ii): Take arbitrary $t_0 \in \mathbb{Z}_+$ and $x_0 \in \mathbb{R}^n$, $x_0 \ne 0$, and let $c = \| H^{-1}(t_0) x_0 \|_p > 0$. According to Remark 1, the set $S_{p,cH(t)}$ is also PI w.r.t. system (10); consequently, for the trajectory $x(t) = x(t; t_0, x_0)$ of (10), we can write $\| (cH(t))^{-1} x(t) \|_p \le 1$, $\forall t > t_0$. This is equivalent to $\| H^{-1}(t) x(t) \|_p \le c = \| H^{-1}(t_0) x(t_0) \|_p$, $\forall t > t_0$, which shows that the function $W(x(t), t)$ is nonincreasing along the considered trajectory. Obviously, for $x_0 = 0$, $x(t; t_0, 0) = 0$, $\forall t > t_0$, so that $W(x(t), t) = 0$ along the considered trajectory. The proof is completed, since t_0 and x_0 are arbitrary.

(ii) $\Rightarrow$ (i): Assume, by contradiction, that $S_{p,H(t)}$ is not PI w.r.t. system (10). This means there exists a trajectory $\tilde{x}(t)$ of system (10), which is initialized inside $S_{p,H(t)}$, but leaves $S_{p,H(t)}$. Thus, we can find a time instant $t^* \in \mathbb{Z}_+$ such that $\|H^{-1}(t^*) \tilde{x}(t^*)\|_p \le 1$, whereas, for some $t > t^*$, $\|H^{-1}(t) \tilde{x}(t)\|_p > 1$. This fact contradicts the assumption "the function $W(x(t), t)$ is non-increasing along each trajectory of system (10)".

(iii) $\Rightarrow$ (ii): Consider $x(t)$ an arbitrary trajectory of system (10). For any time instant $t \in \mathbb{Z}_+$ we can write

$$W(x(t+1), t+1) = \| H^{-1}(t+1) x(t+1) \|_p = \| H^{-1}(t+1) M x(t) \|_p = \| \Gamma_{H(t)}(M) H^{-1}(t) x(t) \|_p, \quad (12)$$

where

$$\Gamma_{H(t)}(M) = H^{-1}(t+1) M H(t). \quad (13)$$

Hence, $W(x(t+1), t+1) - W(x(t), t) \le$
$\| \Gamma_{H(t)}(M) \|_p \| H^{-1}(t) x(t) \|_p - W(x(t), t) =$
$\left[\nu_{p,H(t)}(M) - 1\right] W(x(t), t) \le 0.$

This ensures that $W(x(t), t)$ is nonincreasing along each trajectory of system (10).

(ii) $\Rightarrow$ (iii): Taking (13) into account, we get

$$\| \Gamma_{H(t)}(M) \|_p = \sup_{\|H^{-1}(t)x(t)\|_p = 1} \| \Gamma_{H(t)}(M) [H^{-1}(t) x(t)] \|_p = \sup_{\|H^{-1}(t)x(t)\|_p = 1} \| H^{-1}(t+1) x(t+1) \|_p .$$

Since, on the one hand, "$W(x,t)$ is non-increasing along each trajectory" implies $\| H^{-1}(t+1) x(t+1) \|_p \le \| H^{-1}(t) x(t) \|_p$, and, on the other hand, $\| H^{-1}(t) x(t) \|_p = 1$, we finally obtain $\nu_{p,H(t)}(M) = \| \Gamma_{H(t)}(M) \|_p \le 1$.

Part II. Next, we consider the IS (1) defined by the interval matrix A^I introduced in (3) and prove that for all $1 \le p \le \infty$, the majorant matrix U (4) satisfies

$$\nu_{p,H(t)}(A) \le \nu_{p,H(t)}(U), \ \forall A \in \mathcal{A}^I, \quad (14)$$

where $\nu_{p,H(t)}$ is the test function defined by (9).

For a real square matrix $Q=[q_{ij}]\in\mathbb{R}^{n\times n}$, denote by $\overline{Q}=[\overline{q}_{ij}]\in\mathbb{R}^{n\times n}$, the nonnegative matrix built from Q as follows

$$\overline{q}_{ij}=|q_{ij}|, \ i,j=1,\ldots,n. \quad (15)$$

According to Lemma 4 in [15], we have $\|Q\|_p \le \|\overline{Q}\|_p$ for all $1\le p\le\infty$. If P is a real square matrix fulfilling the componentwise inequality $\overline{Q}\le P$, by the same technique as in the proof of Lemma 4 in [15], we can show that $\|\overline{Q}\|_p \le \|P\|_p$ for all $1\le p\le\infty$.

On the other hand, for any matrix $A\in\mathcal{A}^I$ we can write $A\le\overline{A}\le U$. By using the notation (13) introduced above, we get the componentwise matrix inequality $\Gamma_{H(t)}(A)\le\Gamma_{H(t)}(\overline{A})\le\Gamma_{H(t)}(U)$. Therefore, $\|\Gamma_{H(t)}(A)\|_p\le\|\Gamma_{H(t)}(U)\|_p$ for all $1\le p\le\infty$, which completes the proof of (14).

Part III. Finally, we prove the statements in Theorem 1.
a) *Sufficiency*, and b): For all $1\le p\le\infty$, whenever $\nu_{p,H(t)}(U)\le 1$ inequality (14) allows concluding that $\nu_{p,H(t)}(A)\le 1$ for all $A\in\mathcal{A}^I$. According to Part I in the proof of Theorem 1, this implies that $S_{p,H(t)}$ is PI w.r.t. all systems of form (10) with $M=A\in\mathcal{A}^I$, i.e. $S_{p,H(t)}$ is PI w.r.t. IS (1).

a) *Necessity*: There exits a matrix $A^*\in\mathcal{A}^I$ such that $\overline{A^*}=U$, where the operator $\overline{\circ}$ acts according to (15).
For $p=1,\infty$, $\nu_{p,H(t)}(A^*)=\nu_{p,H(t)}(\overline{A^*})$, leading to $\nu_{p,H(t)}(A^*)=\nu_{p,H(t)}(U)$. Moreover, "$S_{p,H(t)}$ is PI w.r.t. IS (1)" implies "$S_{p,H(t)}$ is PI w.r.t. system (10) with $M=A^*$". According to Part I in the proof of Theorem 1, we get $\nu_{p,H(t)}(A^*)\le 1$ and, hence, $\nu_{p,H(t)}(U)\le 1$, which completes the proof. ■

Corollary 1. Let $1\le p\le\infty$. Assume that $U\in\mathcal{A}^I$ or $-U\in\mathcal{A}^I$. The set $S_{p,H(t)}$ is PI w.r.t. IS (1) if and only if $\nu_{p,H(t)}(U)\le 1$.

Proof: Necessity. "$S_{p,H(t)}$ is PI w.r.t. IS (1)" implies "$S_{p,H(t)}$ is PI w.r.t. all systems of form (10) with $M=A\in\mathcal{A}^I$" If $U\in\mathcal{A}^I$, then part I in the proof of Theorem 1 implies $\sigma_{p,H(t)}(U)\le 1$. If $-U\in\mathcal{A}^I$, then, by the same reason, $\sigma_{p,H(t)}(-U)\le 1$, and finally $\sigma_{p,H(t)}(U)=\sigma_{p,H(t)}(-U)\le 1$. *Sufficiency.* It is ensured by Theorem 1. ■

Remark 2. The hypothesis on the majorant matrix U used in Corollary 1 is satisfied by any interval matrix for which the mean matrix $A_m=\frac{1}{2}(A^-+A^+)$ is nonnegative or nonpositive.■

Remark 3. The necessary and sufficient condition of set invariance provided by Theorem 1 for the particular case of $p=\infty$ can be equivalently written as the difference inequality $Uh(t)\le h(t+1)$, where the vector function $h(t):\mathbb{Z}_+\to\mathbb{R}^n_+$ is defined by the positive functions $h_i(t):\mathbb{Z}_+\to\mathbb{R}^*_+$, $i=1,\ldots,n$, used in (6) The results of papers [12] and [13] (considering the invariance problem only for rectangular boxes) are presented in terms of the above difference inequality. ■

4. INTERVAL SYSTEMS WITH CT DYNAMICS

Given a diagonal matrix function $H(t)$ (6) and an arbitrary p-norm ($1\le p\le\infty$), let us introduce the test function:

$$\mu_{p,H(t)}(M)=m_p\left(H^{-1}(t)MH(t)-H^{-1}(t)\dot{H}(t)\right), \quad (16)$$

where $M\in\mathbb{R}^{n\times n}$ is an arbitrary square matrix. The following result shows that the invariance of $S_{p,H(t)}$ (7) w.r.t. the trajectories of the CT IS (2) can be explored by evaluating the above test function for the majorant matrix U (5).

Theorem 2. a) Let $p=1,\infty$. The set $S_{p,H(t)}$ is PI w.r.t. IS (2) if and only if $\mu_{p,H(t)}(U)\le 0$.

b) Let $1<p<\infty$. The set $S_{p,H(t)}$ is PI w.r.t. IS (1) if $\mu_{p,H(t)}(U)\le 0$.

Proof: Similarly to the DT case, the proof has three parts:

Part I. First, we consider a time-invariant system with CT dynamics described by

$$\dot{x}(t)=Mx(t),\ x(t_0)=x_0\in\mathbb{R}^n,\ t,t_0\in\mathbb{R}_+,\ t\ge t_0, \quad (17)$$

and characterize the positive invariance of time-dependent sets $S_{p,H(t)}$ w.r.t. system (17) by proving the equivalence between the following statements for arbitrary $1\le p\le\infty$:

(i) the set $S_{p,H(t)}$ is PI w.r.t. system (17);

(ii) the function

$$W(x,t):\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+,\ W(x,t)=\|H^{-1}(t)x\|_p \quad (18)$$

is nonincreasing along each trajectory of system (17);

(iii) $\mu_{p,H(t)}(M)\le 0$.

(i) $\Rightarrow$ (ii): Take arbitrary $t_0\in\mathbb{R}_+$ and $x_0\in\mathbb{R}^n$, $x_0\ne 0$, and let $c=\|H^{-1}(t_0)x_0\|_p>0$. According to Remark 1, the set $S_{p,cH(t)}$ is also PI w.r.t. system (17); consequently, for the trajectory $x(t)=x(t;t_0,x_0)$ of (17), we can write $\|(cH(t))^{-1}x(t)\|_p\le 1$, $\forall t>t_0$. This is equivalent to $\|H^{-1}(t)x(t)\|_p\le c=\|H^{-1}(t_0)x(t_0)\|_p$, $\forall t>t_0$, which shows that the function $W(x(t),t)$ is nonincreasing along the considered trajectory. Obviously, for $x_0=0$, $x(t;t_0,0)=0$, $\forall t>t_0$, so that $W(x(t),t)=0$ along the considered trajectory. The proof is completed, since t_0 and x_0 are arbitrary.

(ii) $\Rightarrow$ (i): Assume, by contradiction, that $S_{p,H(t)}$ is not PI w.r.t. system (17). This means there exists a trajectory $\tilde{x}(t)$ of system (17), which is initialized inside $S_{p,H(t)}$, but leaves $S_{p,H(t)}$. Thus, we can find a time instant $t^*\in\mathbb{R}_+$ such that $\|H^{-1}(t^*)\tilde{x}(t^*)\|_p=1$, whereas, for some $t>t^*$, $\|H^{-1}(t)\tilde{x}(t)\|_p>1$. This fact contradicts the assumption "the function $W(x(t),t)$ is nonincreasing along each trajectory of system (17)".

(iii) $\Rightarrow$ (ii): Consider $x(t)$ an arbitrary trajectory of system (17). For any time instant $t\in\mathbb{R}_+$ we can write

$$H^{-1}(t+\theta)x(t+\theta)=H^{-1}(t)x(t)+\theta\frac{d}{dt}\left(H^{-1}(t)x(t)\right)+$$
$$\theta O(\theta),\ \text{with}\ \lim_{\theta\downarrow 0}\|O(\theta)\|_p=0.$$

By rewriting the second term as

$$\theta\frac{d}{dt}\left(H^{-1}(t)x(t)\right)=\theta\left(H^{-1}(t)MH(t)-H^{-1}(t)\dot{H}(t)\right)\cdot\left(H^{-1}(t)x(t)\right),$$

we have:

$$W(x(t+\theta),(t+\theta))=\|\left(I+\theta\Phi_{H(t)}(M)\right)\left(H^{-1}(t)x(t)\right)+\theta O(\theta)\|_p, \quad (19)$$

where

$$\Phi_{H(t)}(M)=H^{-1}(t)MH(t)-H^{-1}(t)\dot{H}(t). \quad (20)$$

Hence

$$W(x(t+\theta),(t+\theta))\le\|I+\theta\,\Phi_{H(t)}(M)\|_p\,W(x(t),t)+\theta\|O(\theta)\|_p$$ and

$$D_t^+W(x(t),t)=\lim_{\theta\downarrow 0}[(W(x(t+\theta),t+\theta)-W(x(t),t))/\theta]\le$$
$$\lim_{\theta\downarrow 0}\frac{\|I+\theta\,\Phi_{H(t)}(M)\|_p-1}{\theta}W(x(t),t)+\lim_{\theta\downarrow 0}O(\theta)=$$
$$=m_p(\Phi_{H(t)}(M))W(x(t),t)=\mu_{p,H(t)}(M)W(x(t),t)\le 0$$

.

This ensures that $W(x(t),t)$ is nonincreasing along each trajectory of system (17).

(ii) $\Rightarrow$ (iii): Taking (20) into account, we get

$$\mu_{p,H(t)}(M)=m_p\left(\Phi_{H(t)}(M)\right)=$$
$$\lim_{\theta\downarrow 0}\frac{1}{\theta}\left[\sup_{\|H^{-1}(t)x(t)\|_p=1}\|(I+\theta\Phi_{H(t)}(M))H^{-1}(t)x(t)\|_p-1\right]=$$
$$\lim_{\theta\downarrow 0}\left[\frac{1}{\theta}\sup_{\|H^{-1}(t)x(t)\|_p=1}\left(\|H^{-1}(t)x(t)+\theta\Phi_{H(t)}(M)H^{-1}(t)x(t)\|_p\right)-\frac{1}{\theta}\right]=$$
$$\lim_{\theta\downarrow 0}\left[\frac{1}{\theta}\sup_{\|H^{-1}(t)x(t)\|_p=1}\left(\|H^{-1}(t+\theta)x(t+\theta)-\theta O(\theta)\|_p\right)-\frac{1}{\theta}\right]\le$$
$$\lim_{\theta\downarrow 0}\left[\frac{1}{\theta}\sup_{\|H^{-1}(t)x(t)\|_p=1}\left(\|H^{-1}(t+\theta)x(t+\theta)\|_p+\theta\|O(\theta)\|_p\right)-\frac{1}{\theta}\right].$$

Since "$W(x,t)$ is nonincreasing along each trajectory of system (17)" implies $\| H^{-1}(t+\theta)x(t+\theta)\|_p \le \| H^{-1}(t)x(t)\|_p$, and, on the other hand, $\| H^{-1}(t)x(t)\|_p = 1$, we obtain $\mu_{p,H(t)}(M) \le \lim_{\theta\downarrow 0}\left(\frac{1}{\theta} + \| O(\theta)\|_p - \frac{1}{\theta}\right) = 0$.

Part II. Next, we consider the IS (2) defined by the interval matrix $\mathcal{A}^I$ introduced in (3) and prove that for all $1 \le p \le \infty$, the majorant matrix U (5) satisfies

$$\mu_{p,H(t)}(A) \le \mu_{p,H(t)}(U),\ \forall A \in \mathcal{A}^I, \tag{21}$$

where $\mu_{p,H(t)}$ is the test function defined by (16).

For a real square matrix $Q = [q_{ij}] \in \mathbb{R}^{n\times n}$, denote by $\overline{Q} = [\overline{q}_{ij}] \in \mathbb{R}^{n\times n}$, the nonnegative matrix built from Q as follows

$$\begin{aligned} &\overline{q}_{ii} = q_{ii},\ i = 1,\ldots,n; \\ &\overline{q}_{ij} = |q_{ij}|,\ i \ne j,\ i,j = 1,\ldots,n. \end{aligned} \tag{22}$$

According to Lemma 4 in [15], we have $m_p(Q) \le m_p(\overline{Q})$ for all $1 \le p \le \infty$. If P is a real square matrix fulfilling the componentwise inequality $\overline{Q} \le P$, by the same technique as in the proof of Lemma 4 in [15], we can show that $m_p(\overline{Q}) \le m_p(P)$ for all $1 \le p \le \infty$.

On the other hand, for any matrix $A \in \mathcal{A}^I$ we can write $A \le \overline{A} \le U$. By using the notation (20) introduced above, we get the componentwise inequality $\Phi_{H(t)}(A) \le \Phi_{H(t)}(\overline{A}) \le \Phi_{H(t)}(U)$. Thus, $m_p(\Phi_{H(t)}(A)) \le m_p(\Phi_{H(t)}(U))$ for all $1 \le p \le \infty$, which completes the proof of (21).

Part III. Finally, we prove the statements in Theorem 2.

a) *Sufficiency*, and b): For all $1 \le p \le \infty$, whenever $\mu_{p,H(t)}(U) \le 0$ inequality (21) allows concluding that $\mu_{p,H(t)}(A) \le 0$ for all $A \in \mathcal{A}^I$. According to Part I in the proof of Theorem 2, this implies that $S_{p,H(t)}$ is PI w.r.t. all systems of form (17) with $M = A \in \mathcal{A}^I$, i.e. $S_{p,H(t)}$ is PI w.r.t. IS (2).

a) *Necessity*: There exits a matrix $A^* \in \mathcal{A}^I$ such that $\overline{A^*} = U$, where the operator $\overline{\circ}$ acts according to (22). For $p = 1,\infty$, $\mu_{p,H(t)}(A^*) = \mu_{p,H(t)}(\overline{A^*})$, which leads to $\mu_{p,H(t)}(A^*) = \mu_{p,H(t)}(U)$. Moreover, "$S_{p,H(t)}$ is PI w.r.t. IS (2)" implies "$S_{p,H(t)}$ is PI w.r.t. system (17) with $M = A^*$". According to Part I in the proof of Theorem 2, we get $\mu_{p,H(t)}(A^*) \le 0$ and, hence, $\mu_{p,H(t)}(U) \le 0$, which completes the proof. ■

Corollary 2. Let $1 \le p \le \infty$. Assume that $U \in \mathcal{A}^I$. The set $S_{p,H(t)}$ is PI w.r.t. IS (2) if and only if $\mu_{p,H(t)}(U) \le 0$.

Proof: Necessity. "$S_{p,H(t)}$ is PI w.r.t. IS (1)" implies "$S_{p,H(t)}$ is PI w.r.t. all systems of form (10) with $M = A \in \mathcal{A}^I$" If $U \in \mathcal{A}^I$, then part I in the proof of Theorem 2 implies $\mu_{p,H(t)}(U) \le 0$. *Sufficiency.* It is ensured by Theorem 2. ■

Remark 4. The hypothesis on the majorant matrix U used in Corollary 2 is satisfied by any interval matrix for which the mean matrix $A_m = \frac{1}{2}(A^- + A^+)$ is essentially nonnegative. ■

Remark 5. The necessary and sufficient condition of set invariance provided by Theorem 1 for the particular case of $p = \infty$ can be equivalently written as the differential inequality $Uh(t) \le \dot{h}(t)$, where the vector function $h(t): \mathbb{R}_+ \to \mathbb{R}_+^n$ is defined by the positive vector functions $h_i(t): \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\ldots,n$, used in (6) The results of papers [12] and [13] (considering the invariance problem only for rectangular boxes) are presented in terms of the above differential inequality. ■

5. CONCLUSIONS

Our results open a larger perspective on the existence of positively invariant sets with arbitrary time dependence in the dynamics of interval systems. Thus, we are able to generalize, for $1 \le p \le \infty$ norms, the analysis developed by some previous works only for rectangular boxes ($p = \infty$) depending on time. The whole approach shows that the considered majorant matrices play an important role in revealing the properties of various families of time-dependent invariant sets.

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