CEAI, Vol.19, No.4 pp. 34-42, 2017 Printed in Romania

A Stochastic Single Machine-Infinite Bus System in Non-linear Filtering Contexts

Ravish H. Hirpara[*,**], **Shambhu N. Sharma**[**]

** PG Research Centre, Gujarat Technological University (GTU), Ahmedabad, Gujarat-382424, India*
*** S. V. National Institute of Technology, Surat-395007, Gujarat, India*
(e-mail: ravishhirpara@gmail.com, ap_ravish@gtu.edu.in, snsvolterra@gmail.com).

Abstract: In power systems, the single machine-infinite bus system is formalized as a second-order non-linear differential equation. After accomplishing the phase space formulation of the second-order non-linear swing equation, we are led to an ordinary differential equation. After accounting for the noise influence, the Single Machine-Infinite Bus (SMIB) system assumes the structure of a Stochastic Differential Equation (SDE). This paper revisits the noisy state vector of the stochastic SMIB system from non-linear filtering perspectives. In the Fokker-Planck setting, we consider process noise and ignore observation noise. On the other hand, the non-linear filtering perspective accounts for the process noise as well as observation noise correction terms. Notably, the exact non-linear filtering exploits an appealing stochastic integro-differential equation formalism. The stochastic partial differential equation arises in the higher-order and lower-order filtering. Since the SMIB accounts for greater order of non-linearities, we wish to estimate the states of the SMIB system using higher-order filter. Subsequently, we compare the filtered state trajectories with celebrated extended Kalman filtering. This paper fills a niche between non-linear filtering and power system dynamics as well as reveals a connection between the four formalisms. The four formalisms, which are the cornerstone in this paper, are non-linear ordinary differential equation, stochastic differential equation, stochastic partial-integro differential equation and stochastic partial differential equations.

Keywords: Non-linear filtering, Machine-Infinite Bus (SMIB) system, Itô stochastic differential equation, stochastic partial differential equations

1. INTRODUCTION

The Single Machine Infinite Bus system is well studied from the power system perspective. The stability of equilibrium point of the machine can be examined using swing equation in combination with the notion of the derivative of the Lyapunov function. The Lyapunov function is a scalar non-negative non-linear function of an $n-$ dimensional state vector. A second-order non-linear differential equation describes the machine swing equation. After accounting for correction terms stemming from the random perturbations associated with the single machine-infinite bus system, stochastic differential equation arises. The single machine-infinite bus system is influenced by several perturbations, i.e. renewable energy generations, random loads, the stochastic perturbations of rotor speed, rotor vibrations due to electrical harmonics, mechanical asymmetry and aging, damping coefficient, exogenous voltage flicker in the power system etc. The resulting stochastic system is described by a vector Stochastic Differential Equation (SDE). Generally, random perturbations are modelled as Gaussian white noise processes, generalized stochastic processes. (Mumford, 2000) argues stochastic differential equation descriptions in his philosophical paper in lieu of the deterministic setting for dynamical systems. The SDE description will refine stochastic stability conditions, estimation algorithms and control laws for dynamical systems. The white noise-driven stochastic SMIB system and effect of the noise intensity on power systems are nicely explained in (Wei and Luo, 2009) and see (Wang and Crow, 2013) as well. In their paper, first the Fokker-Planck equation for a stochastic SMIB system was developed and then numerical studies of the SMIB were demonstrated. The numerical experimentation is about the conditional probability density trajectory of the SMIB system, which utilizes a system of parameters and initial data. (Odun-ayo and Crow, 2013) analysed power system stability using stochastic energy function. (Qin and Li, 2014) added randomness in damping coefficient and explained a chaotic behaviour of power systems by choosing a set of noise intensities. To understand noise influence on the erosion of safe basins in power systems, (Wei et al., 2010) will be a good source, see (Zhang et al., 2012) as well. The voltage analysis of the stochastic SMIB system is available in (Ghanavati et al., 2013; Cotilla-Sanchez et al., 2012) which is relatively quite scarce. The non-linear perspective of the single machine-bus systems accounts for the process noise as well as observation noise correction terms. Note that the process noise as well as observation noise are independent random variables. On the other hand, the Fokker-Planck setting accounts for the process noise correction terms only.

The intent of this paper is to develop non-linear filtering equations of the single machine-infinite bus system. After accounting for random perturbations in deterministic dynamics of the SMIB system, we arrive at a randomly perturbed SMIB system. The randomly perturbed SMIB system in combination with the active and reactive power noisy measurement system forms a non-linear filtering model. In this paper, non-linear filtering perspectives of the single machine-infinite bus are the subject of investigations. Non-linear filtering perspectives of the multi-machine infinite bus system can be achieved using the component wise description involving coupling terms. Generally, there are three filtering models, which are popular in system theory: i.e. linear stochastic differential equation coupled with non-linear observation equation, non-linear stochastic differential equation coupled with linear observation equation and non-linear stochastic differential equation coupled with non-linear observation equation. In this paper, we exploit non-linear stochastic differential equation coupled with non-linear observation equation. The non-linear filtering of this paper hinges on the filtering density evolution equation, which assumes the structure of a non-linear stochastic partial integro differential equation. In (Ghahremani and Kamwa, 2011) dynamic state estimation of power system by applying discrete-discrete Extended Kalman filter for power system dynamics. In (Wang and Crow, 2013; Hirpara and Sharma 2015), the Fokker-Planck model is the subject of investigations. The Fokker-Planck model is a parabolic linear homogeneous equation of order two in partial differentiation for Markov processes. The Fokker-Planck equation is a special case of the filtering density evolution equation. Thus, the results of this paper are more general and sharper version of the available results in literature.

This paper is organized as follows: section 2 derives a theory of a stochastic single machine-infinite bus system. Section 3 discusses non-linear filtering equation for a stochastic SMIB system. Section 4 is about numerical simulations. Concluding remarks are given in section 5.

2. STOCHASTIC SINGLE MACHINE-INFINITE BUS (SMIB) SYSTEM

The swing equation of the single machine-infinite bus system is a second-order non-linear differential equation (Kundur 1994; Chen et al., 2005), i.e.

$$M\ddot{\delta} + D\dot{\delta} + \frac{VE'_a}{X}\sin\delta = P_m. \tag{1}$$

Note that the term V denotes the voltage magnitude of the infinite bus, M and D are the combined inertia constant and the damping coefficient of the generator and turbine respectively. The power system parameters E'_a and δ are the transient emf and the rotor angle of the generator respectively. The reactance X is the sum of the generator transient reactance X'_d and the line reactance X_l. The term P_m is the input mechanical power and $P_e = \frac{VE'_a}{X}\sin\delta$ is the electrical power of cylindrical-pole synchronous generator. The schematic diagram of equation (1) is illustrated in figure (1) of the paper.

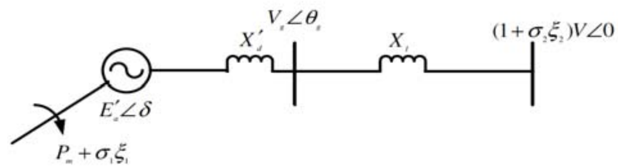

Fig. 1. A stochastic single machine-infinite bus system.

Here, we recast the swing equation that accounts for noise in the voltage magnitude of the infinite bus, $V(1+\sigma_2\xi_2)$, and the input mechanical power with additive noise $P_m + \sigma_1\xi_1$. Thus, we get

$$M\ddot{\delta} + D\dot{\delta} + \frac{(1+\sigma_2\xi_2)VE'_a}{X}\sin\delta = P_m + \sigma_1\xi_1, \tag{2}$$

where ξ_1 and ξ_2 are the independent white Gaussian noise processes. That are added to the mechanical power and the voltage magnitude of the infinite bus respectively. The parameters σ_1, σ_2 are the noise intensities of the white noise processes. The mechanical power P_m is assumed to be constant.

$$\ddot{\delta} = -\frac{D}{M}\dot{\delta} - \frac{VE'_a}{M\ X}\sin\delta + \frac{P_m}{M} + \frac{\sigma_1}{M}\dot{B}_1(t) - \frac{\sigma_2 VE'_a}{M\ X}\sin\delta\ \dot{B}_2(t).$$

For the notational clarity and consistence, we replace the terms ξ_1 and ξ_2 with $\dot{B}_1(t)$ and $\dot{B}_2(t)$ respectively. The terms $B_1(t)$ and $B_2(t)$ are two independent Brownian motion processes. In phase space formulations, we consider the state vector $y_t = (y_1, y_2)^T = (\delta, \omega)^T$, where $\omega = \dot{\delta}$ is the angular velocity of the rotor. Thus,

$$dy_1 = y_2 dt,$$

$$dy_2 = \left(-\frac{D}{M}y_2 - \frac{VE'_a}{M\ X}\sin y_1 + \frac{P_m}{M}\right)dt + \frac{\sigma_1}{M}dB_1(t) - \frac{\sigma_2 VE'_a}{M\ X}\sin y_1 dB_2(t).$$

Furthermore, the above system can be re-stated as

$$dy_t = f(y_t, t)dt + G(y_t, t)dB_t, \tag{3}$$

where,

$$f(y_t, t) = \begin{pmatrix} y_2 \\ -\frac{D}{M}y_2 - \frac{VE'_a}{M\ X}\sin y_1 + \frac{P_m}{M} \end{pmatrix},$$

$$G(y_t, t) = \begin{pmatrix} 0 & 0 \\ \frac{\sigma_1}{M} & -\frac{\sigma_2 VE'_a}{M\ X}\sin y_1 \end{pmatrix}, \quad B_t = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}^T.$$

Here, the term $f(y_t, t)$ is the system non-linearity and the $G(y_t, t)$ is the process noise coefficient matrix. We denote a vector Brownian motion using the notation B_t.

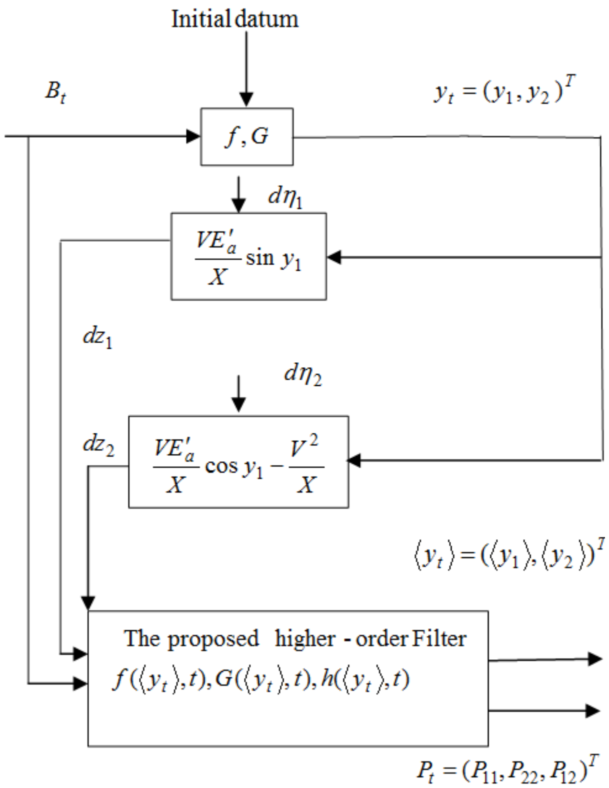

Fig. 2. An SMIB system filtering diagram.

Figure 2 shows the SMIB system filtering diagram. The parameters $D, M, \sigma_1, \sigma_2, X, E'_a, V, P_m$ are the system parameters of the stochastic system of the paper.

3. NON-LINEAR FILTERING EQUATIONS FOR A STOCHASTIC SMIB SYSTEM

This paper is intended to develop higher-order Kushner-Stratonovich filtering for the stochastic system considered here. The filtering method exploits observation equations. Here, we wish to develop 'filtering' by taking the active and reactive power measurements of the machine. Thus, the vector observation equation becomes

$$dz_t = h(y_t, t)dt + d\eta_t, \tag{4}$$

where the observation vectors

$$z_t = (z_1, z_2)^T,\ h(y_t, t) = \left(\frac{VE'_a}{X}\sin y_1,\ \frac{VE'_a}{X}\cos y_1 - \frac{V^2}{X}\right)^T,$$

$$d\eta(t) = \begin{pmatrix}\gamma\, d\eta_1 \\ \gamma\, d\eta_2\end{pmatrix},\ d\eta_1 d\eta_2 = 0,$$

$$\mathrm{var}(d\eta(t)) = \begin{pmatrix}\gamma^2 & 0 \\ 0 & \gamma^2\end{pmatrix} dt = \psi_\eta dt.$$

The terms $\eta(t)$ and ψ_η have interpretations as the observation noise and the intensity of the observation noise respectively. Note that equation (3) in combination with equation (4) becomes 'the filtering model' of the SMIB system. (Kalman, 1960) pioneered filtering for linear stochastic differential systems that shaped stochastic control theory. The Kalman's work on filtering theory was later extended to non-linear stochastic differential systems in combinations with non-linear observation equations. That is credited to (Kushner, 1967), see a notable work of On stochastic differential equations for the non linear filtering problem M. Fujisaki, G. Kallianpur and H. Kunita as well. The non-linear filtering theory hinges on the filtering density evolution, a non-linear stochastic partial integro-differential equation. In this context, (Pugachev and Sinitsyn, 1987, pp. 389) will be a good source, see (Jazwinski, 1970, pp. 178) as well, i.e.

$$dp = \pounds(p)dt + (h - \langle h\rangle)^T \psi_\eta^{-1}(t)(dz_t - \langle h\rangle dt)p, \tag{5}$$

where the filtering density $p = p(y, t|Y_t)$, $Y_t = \{z_\tau, t_0 \le \tau \le t\}$ and the operator $\pounds(.)$ denotes the *Kolmogorov-Fokker-Planck operator* (Sharma, 2008). H J Kushner developed the above density evolution equation in the Itô setting. R L Stratonovich developed the filtering density evolution using stochastic differential equation with $\frac{1}{2}-$differential. Note that stochastic differential equations with $\frac{1}{2}-$differential can be replaced with Itô equivalents. For this reason, the above filtering density evolution is usually known as the Kushner-Stratonovich equation. The term 'Kushner-Stratonovich equation' is coined in filtering theory, see (Pugachev and Sinitsyn, 1987, pp. 377) for a greater historical and mathematical remark. Notably, the 'Kushner-Stratonovich stochastic method' for dynamical systems exploits the Kushner-Stratonovich equation, the exact stochastic evolutions of conditional mean and conditional variance. Making the use of the Kushner-Stratonovich equation, the definition of the differential of the conditional expectation of the scalar function $d\varphi(y_t)$ of an $n-$dimensional state vector, we derive the exact stochastic evolutions of conditional mean and conditional variance, see (Jazwinski, 1970, pp. 182-184).

$$d\langle\varphi(y_t)\rangle = (\left\langle f^T \frac{\partial\varphi}{\partial y_t}\right\rangle + \frac{1}{2}\left\langle tr((GG)^T(y_t, t)\frac{\partial^2\varphi}{\partial y_t \partial y_t^T})\right\rangle)dt$$
$$+ (\langle\varphi h\rangle - \langle\varphi\rangle\langle h\rangle)^T \psi_\eta^{-1}(t)(dz_t - \langle h\rangle dt).$$

Since the procedure to arrive at the exact stochastic evolution is lengthy as well as that is quite known in stochastic processes literature, the details are omitted. The analytical and numerical solutions to exact stochastic evolutions are intractable. We explore higher-order Kushner-Stratonovich filtering for non-linear stochastic differential systems. Here, we develop higher-order filtering equations up to the third-order (Boutayeb et al. 1997; Sharma et al. 2006), i.e.

$$d\langle y_i\rangle = (f_i(\langle y_t\rangle,t) + \frac{1}{2}\sum_{p,q} P_{pq}\frac{\partial^2 f_i(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle})dt$$

$$+\sum_{\alpha,\beta}(\sum_p P_{ip}\frac{\partial h_\alpha(\langle y_t\rangle,t)}{\partial\langle y_p\rangle} + \frac{1}{2}\sum_{p,q,r} P_{ip}P_{qr}\frac{\partial^3 h_\alpha(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle\partial\langle y_r\rangle})$$

$$\times(\psi_\eta^{-1})_{\alpha\beta}(dz_\beta - (h_\beta(\langle y_t\rangle,t) + \frac{1}{2}\sum_{p,q} P_{pq}\frac{\partial^2 h_\beta(\langle y_t\rangle,t)}{\langle y_p\rangle\langle y_q\rangle})dt), \quad (6)$$

$$dP_{ij} = (\sum_p P_{ip}\frac{\partial f_j(\langle y_t\rangle,t)}{\partial\langle y_p\rangle} + \frac{1}{2}\sum_{p,q,r} P_{ip}P_{qr}\frac{\partial^3 f_j(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle\partial\langle y_r\rangle}$$

$$+\sum_p P_{jp}\frac{\partial f_i(\langle y_t\rangle,t)}{\partial\langle y_p\rangle} + \frac{1}{2}\sum_{p,q,r} P_{jp}P_{qr}\frac{\partial^3 f_i(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle\partial\langle y_r\rangle}$$

$$+(GG^T)_{ij}(\langle y_t\rangle,t) + \frac{1}{2}\sum_{p,q} P_{pq}\frac{\partial^2 (GG^T)_{ij}(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle}$$

$$-\sum_{\alpha,\beta}(\sum_p P_{ip}\frac{\partial h_\alpha(\langle y_t\rangle,t)}{\partial\langle y_p\rangle}$$

$$+\frac{1}{2}\sum_{p,q,r} P_{ip}P_{qr}\frac{\partial^3 h_\alpha(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle\partial\langle y_r\rangle})(\psi_\eta^{-1})_{\alpha\beta}$$

$$\times(\sum_p P_{jp}\frac{\partial h_\beta(\langle y_t\rangle,t)}{\partial\langle y_p\rangle} + \frac{1}{2}\sum_{p,q,r} P_{jp}P_{qr}\frac{\partial^3 h_\beta(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle\partial\langle y_r\rangle}))dt$$

$$+\sum_{\alpha,\beta}(\sum_{p,q} P_{ip}P_{jq}\frac{\partial^2 h_\beta(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle})(\psi_\eta^{-1})_{\alpha\beta}$$

$$\times(dz_\beta - (h_\beta(\langle y_t\rangle,t) + \frac{1}{2}\sum_{p,q} P_{pq}\frac{\partial^2 h_\beta(\langle y_t\rangle,t)}{\langle y_p\rangle\langle y_q\rangle})dt). \quad (7)$$

where

$$\hat{y}_i = \langle y_i\rangle = E(y_i(t)|Y_t),$$
$$P_{ij} = E((y_i(t) - E(y_i(t)|Y_t))(y_j(t) - E(y_j(t)|Y_t))|Y_t)$$

and the operator E is a conditional expectation operator, which is linear. Note that the proof of the above coupled filtering equations utilizes the 'Gaussianity' for the random state vector (Park and Scheeres, 2007). The Liptser-Shiryaev filtering approach, see (Liptser and Shiryavev, 1977) an alternative interpretation of non-linear filtering theory, hinges on the notion of the stochastic evolution of conditional characteristic function. The filtering density evolution equation is a consequence of the conditional characteristic function evolution equation. The *Appendix* of the paper discusses conditional characteristic function evolution equation of the SMIB system as well. From equations (6)-(7) and equations (3)-(4), we get the following Kushner-Stratonovich *power system* filtering equations, i.e. coupled conditional mean and conditional variance equations:

$$d\langle y_1\rangle = \langle y_2\rangle dt + \gamma^{-2}(1-\frac{P_{11}}{2})P_{11}\frac{VE_a'}{X}(\cos\langle y_1\rangle$$

$$\times(dz_1 - \frac{VE_a'}{X}\sin\langle y_1\rangle(1-\frac{P_{11}}{2})dt) - \sin\langle y_1\rangle$$

$$\times(dz_2 - (\frac{VE_a'}{X}\cos\langle y_1\rangle(1-\frac{P_{11}}{2}) - \frac{V^2}{X})dt)), \quad (8)$$

$$d\langle y_2\rangle = (-\frac{D}{M}\langle y_2\rangle + (\frac{P_{11}}{2}-1)\frac{VE_a'}{M\ X}\sin\langle y_1\rangle + \frac{P_m}{M})dt$$

$$+\gamma^{-2}(1-\frac{P_{11}}{2})P_{12}\frac{VE_a'}{X}(\cos\langle y_1\rangle$$
$$\times(dz_1 - \frac{VE_a'}{X}\sin\langle y_1\rangle(1-\frac{P_{11}}{2})dt) - \sin\langle y_1\rangle$$

$$\times(dz_2 - (\frac{VE_a'}{X}\cos\langle y_1\rangle(1-\frac{P_{11}}{2}) - \frac{V^2}{X})dt)), \quad (9)$$

$$dP_{11} = (2P_{12} - \gamma^{-2}P_{11}^2\frac{V^2E_a'^2}{X^2}(1-\frac{P_{11}}{2})^2)dt$$
$$-\gamma^{-2}P_{11}^2\frac{VE_a'}{X}\sin\langle y_1\rangle(dz_1 - \frac{VE_a'}{X}\sin\langle y_1\rangle(1-\frac{P_{11}}{2})dt)$$

$$-\gamma^{-2}P_{11}^2\frac{VE_a'}{X}\cos\langle y_1\rangle(dz_2 - (\frac{VE_a'}{X}\cos\langle y_1\rangle(1-\frac{P_{11}}{2}) - \frac{V^2}{X})dt), \quad (10)$$

$$dP_{22} = (2(-P_{12}\frac{VE_a'}{M\ X}\cos\langle y_1\rangle - P_{22}\frac{D}{M}) + \frac{\sigma_2^2V^2E_a'^2}{2M^2\ X^2} + \frac{\sigma_1^2}{M^2}$$

$$+(P_{11}-\frac{1}{2})\frac{\sigma_2^2V^2E_a'^2}{M^2\ X^2}\cos 2\langle y_1\rangle + P_{12}P_{11}\frac{VE_a'}{M\ X}\cos\langle y_1\rangle$$

$$-\gamma^{-2}P_{12}^2\frac{V^2E_a'^2}{X^2}(1-\frac{P_{11}}{2})^2)dt - \gamma^{-2}P_{12}^2\frac{VE_a'}{X}\sin\langle y_1\rangle$$
$$\times(dz_1 - \frac{VE_a'}{X}\sin\langle y_1\rangle(1-\frac{P_{11}}{2})dt) - \gamma^{-2}P_{12}^2\frac{VE_a'}{X}\cos\langle y_1\rangle$$

$$\times(dz_2 - (\frac{VE_a'}{X}\cos\langle y_1\rangle(1-\frac{P_{11}}{2}) - \frac{V^2}{X})dt), \quad (11)$$

$$dP_{12} = dP_{21} = (-P_{11}\frac{VE_a'}{M\ X}\cos\langle y_1\rangle - P_{12}\frac{D}{M} + P_{22}$$

$$+\frac{1}{2}P_{11}P_{11}\frac{VE'_a}{M\ X}\cos\langle y_1\rangle\ -\gamma^{-2}P_{11}P_{12}\frac{V^2E_a'^2}{X^2}(1-\frac{P_{11}}{2})^2)dt$$

$$-\gamma^{-2}P_{11}P_{12}\frac{VE'_a}{X}\sin\langle y_1\rangle(dz_1-\frac{VE'_a}{X}\sin\langle y_1\rangle(1-\frac{P_{11}}{2})dt)$$

$$-\gamma^{-2}P_{11}P_{12}\frac{VE'_a}{X}\cos\langle y_1\rangle$$

$$\times(dz_2-(\frac{VE'_a}{X}\cos\langle y_1\rangle(1-\frac{P_{11}}{2})-\frac{V^2}{X})dt). \qquad (12)$$

Extended Kalman Filtering (EKF)

The extended Kalman filtering is a non-linear filtering method as well as 'a special case' of the above higher-order coupled filtering equations. Importantly, extended Kalman filtering accounts for state-independent diffusion coefficients in the conditional variance evolution equation. The component wise descriptions of the extended Kalman filtering equations are

$$d\langle y_i\rangle = f_i(\langle y_t\rangle,t)dt+\sum_{\alpha,\beta}(\sum_p P_{ip}\frac{\partial h_\alpha(\langle y_t\rangle,t)}{\partial\langle y_p\rangle})(\psi_\eta^{-1})_{\alpha\beta}$$

$$\times(dz_\beta-h_\beta(\langle y_t\rangle,t)dt), \qquad (13)$$

$$dP_{ij}=(\sum_p P_{ip}\frac{\partial f_j(\langle y_t\rangle,t)}{\partial\langle y_p\rangle}+\sum_p P_{jp}\frac{\partial f_i(\langle y_t\rangle,t)}{\partial\langle y_p\rangle}$$

$$+(GG^T)_{ij}(t)-\sum_{\alpha,\beta}(\sum_p P_{ip}\frac{\partial h_\alpha(\langle y_t\rangle,t)}{\partial\langle y_p\rangle})(\psi_\eta^{-1})_{\alpha\beta}$$

$$\times(\sum_p P_{jp}\frac{\partial h_\beta(\langle y_t\rangle,t)}{\partial\langle y_p\rangle}))dt. \qquad (14)$$

Making the use of the above coupled extended Kalman filtering equations, i.e. equations (13)-(14), for the power system filtering model, i.e. equations (3)-(4), we get the following evolution equations:

$$d\langle y_1\rangle=\langle y_2\rangle dt+\gamma^{-2}P_{11}\frac{VE'_a}{X}(\cos\langle y_1\rangle$$

$$\times(dz_1-\frac{VE'_a}{X}\sin\langle y_1\rangle dt)\ -\sin\langle y_1\rangle$$

$$\times(dz_2-(\frac{VE'_a}{X}\cos\langle y_1\rangle-\frac{V^2}{X})dt)), \qquad (15)$$

$$d\langle y_2\rangle=(-\frac{D}{M}\langle y_2\rangle-\frac{VE'_a}{M\ X}\sin\langle y_1\rangle+\frac{P_m}{M})dt$$

$$+\gamma^{-2}P_{12}\frac{VE'_a}{X}(\cos\langle y_1\rangle\ (dz_1-\frac{VE'_a}{X}\sin\langle y_1\rangle dt)\ -\sin\langle y_1\rangle$$

$$\times(dz_2-(\frac{VE'_a}{X}\cos\langle y_1\rangle-\frac{V^2}{X})dt)), \qquad (16)$$

$$dP_{11}=(2P_{12}-\gamma^{-2}P_{11}^2\frac{V^2E_a'^2}{X^2})dt, \qquad (17)$$

$$dP_{22}=(2(-P_{12}\frac{VE'_a}{M\ X}\cos\langle y_1\rangle-P_{22}\frac{D}{M})+\frac{\sigma_2^2V^2E_a'^2}{M^2\ X^2}+\frac{\sigma_1^2}{M^2}$$

$$-\gamma^{-2}P_{12}^2\frac{V^2E_a'^2}{X^2})dt, \qquad (18)$$

$$dP_{12}=dP_{21}=(-P_{11}\frac{VE'_a}{M\ X}\cos\langle y_1\rangle-P_{12}\frac{D}{M}+P_{22}$$

$$-\gamma^{-2}P_{11}P_{12}\frac{V^2E_a'^2}{X^2})dt. \qquad (19)$$

Remark 1: Here, it is worth to remark about the exact, higher-order and lower-order filtering. The non-linear filtering theory is the consequence of the non-linear stochastic differential equation in combination with non-linear observation. The closed-form solution to non-linear filtering equations as well as their numerical solution become intractable.

The approximations to system-non-linearity, measurement non-linearity and process noise coefficient about the conditional expectation as well as nearly Gaussian assumption are led to convenient forms of higher-order and lower-order non-linear filtering equations. For the general case, exact filtering equations assume the structure of non-linear stochastic integro-differential equations. Furthermore, after approximations and assumptions, we are led to non-linear partial stochastic differential equations. In this regard, Liptser and Shiryaev (1977) would be a good source.

4. NUMERICAL SIMULATIONS

We consider the following *first* set of initial conditions and system parameters (Ghanavati et al., 2013) for the numerical simulations of the stochastic SMIB system:

$V=1.0$ pu, $E'_a=1.2$ pu, $D=0.03$ pu/rad/sec, $X'_d=0.15$ pu,

$H=4$ MW/MVA, $X_l=0.1$ pu, $X=0.25$ pu, $\sigma_1=0.08$,

$\omega_s=2\pi\times 60=376.8$ rad/sec, $M=\frac{2H}{\omega_s}=0.02123, \sigma_2=0.06$,

$\langle y_1(0)\rangle=1$ rad, $\langle y_2(0)\rangle=2$ rad/sec, $P_m=1$ pu, $\gamma=1000$,

$P_{11}(0)=1\text{ rad}^2$, $P_{12}(0)=0\text{ rad}^2/\text{sec}$, $P_{22}(0)=1\text{ rad}^2/\text{sec}^2$.

Figures (3)-(4) demonstrate numerical simulations of the unperturbed and filtered trajectories of the rotor angle and angular velocity of machine. The solid line denotes the unperturbed trajectories of the rotor angle and angular velocity. The dotted line denotes higher-order 'filtered' state trajectories. These two solid and dotted line trajectories utilize the above set of system parameters and initial conditions with a relatively smaller damping parameter, $D = 0.03$ pu/rad/sec . The filtered state trajectories account for the random forcing term in the non-linear stochastic swing equation. The filtered state trajectories are the most probable trajectories that account for observation corrections as well. On the other hand, the unperturbed trajectory does not. The filtered state trajectories are the consequence of the numerical simulation of stochastic differential equations (8)-(12) that can be regarded as the filtering equations of the non-linear stochastic swing equation. As a result of these, the filtered state trajectory becomes closer to the actual state trajectory and respects stochasticity, a reality of dynamical systems. Importantly, the difference between the filtered and the unperturbed trajectories is attributed to the observation noise and process noise correction terms.

The extended Kalman filtering trajectories are displayed via the solid line and higher-order filtering equations are demonstrated in figures (5)-(6) via the dotted line.

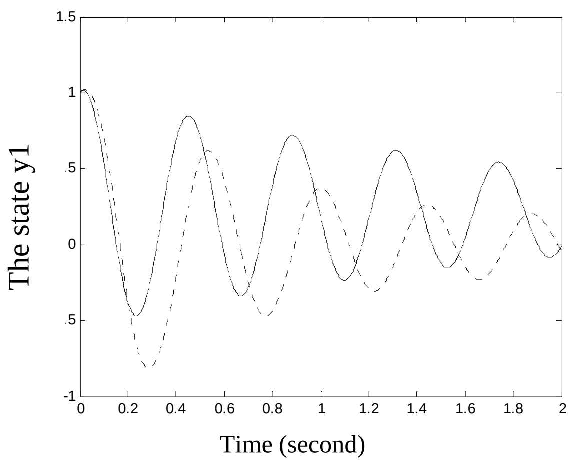

Fig. 3. A comparison between unperturbed and filtered state trajectories.

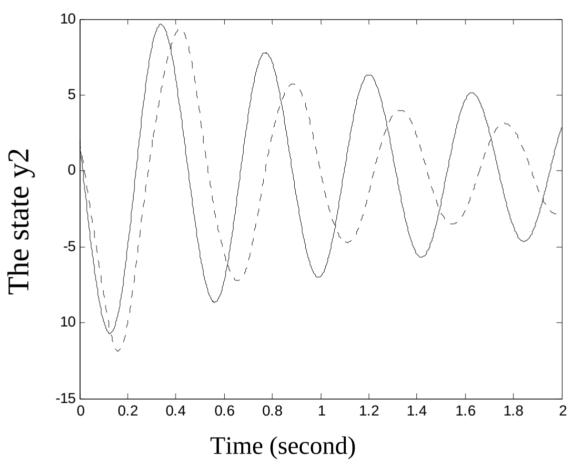

Fig. 4. A comparison between unperturbed and filtered state trajectories.

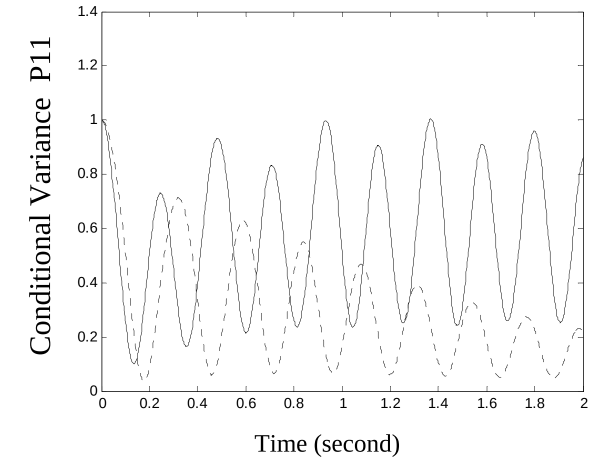

Fig. 5. Conditional variance trajectories using the EKF and higher-order non-linear filtering.

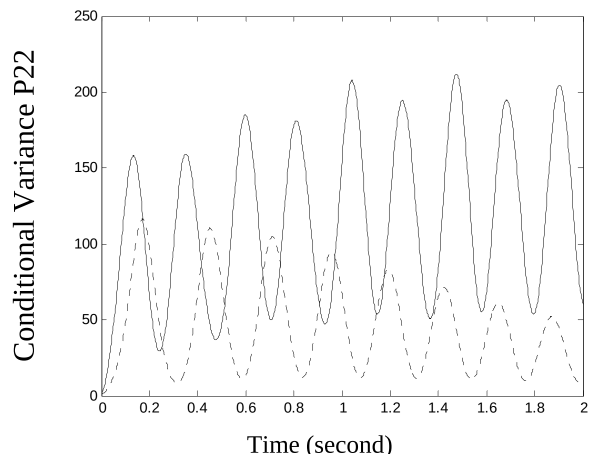

Fig. 6. Conditional variance trajectories using the EKF and higher-order non-linear filtering.

Now, we state the following *second* set of data:

$V = 1.0$ pu, $E'_a = 1.2$ pu, $D = 0.1$ pu/rad/sec, $X'_d = 0.15$ pu,

$H = 4$ MW/MVA, $X_l = 0.1$ pu, $X = 0.25$ pu, $\sigma_1 = 0.08$,

$\omega_s = 2\pi \times 60 = 376.8$ rad/sec, $M = \dfrac{2H}{\omega_s} = 0.02123, \sigma_2 = 0.06$,

$\langle y_1(0) \rangle = 1$ rad, $\langle y_2(0) \rangle = 2$ rad/sec, $P_m = 1$ pu, $\gamma = 1000$,

$P_{11}(0) = 1\text{ rad}^2$, $P_{12}(0) = 0\text{ rad}^2/\text{sec}$, $P_{22}(0) = 1\text{ rad}^2/\text{sec}^2$.

Note that the second set of data is different from the first set of data in the sense that the second set accounts for the larger damping parameter $D = 0.1$ pu/rad/sec in contrast to the first set. The intent of choosing the second set of data is to achieve extensive numerical simulations in lieu of scanty numerical simulations of non-linear filtering equations. Figures (7)-(8)

reveal that the difference between the state trajectories, which are the consequence of deterministic swing equation and stochastic swing equation, are larger. The difference between the two trajectories are attributed to the larger damping parameter. This suggests the fact that stochasticity considerations for the machine swing equation describing the larger damping machine dynamics are imperative.

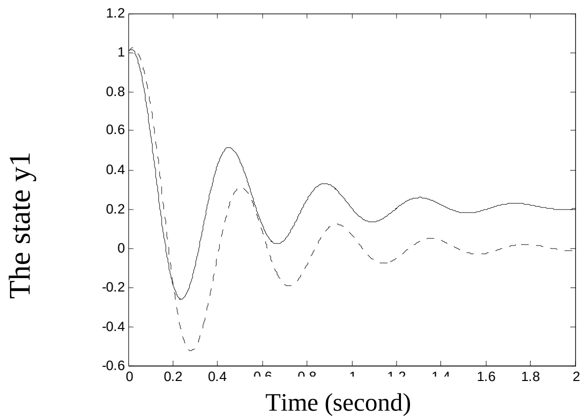

Fig. 7. A comparison between unperturbed and filtered state trajectories.

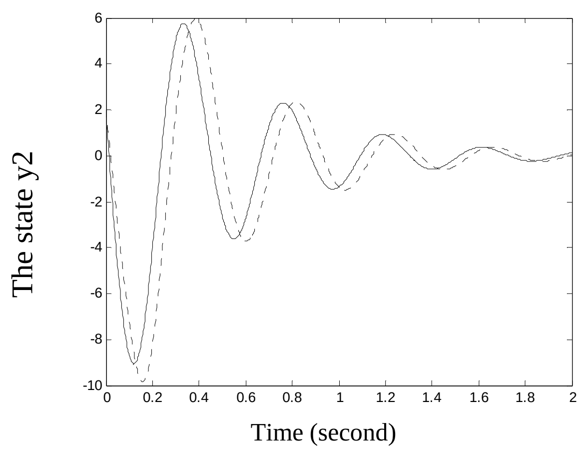

Fig. 8. A comparison between unperturbed and filtered trajectories.

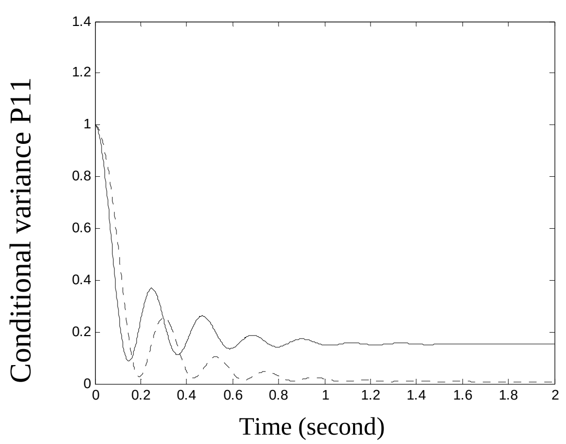

Fig. 9. Conditional variance trajectories using the EKF and higher-order non-linear filtering.

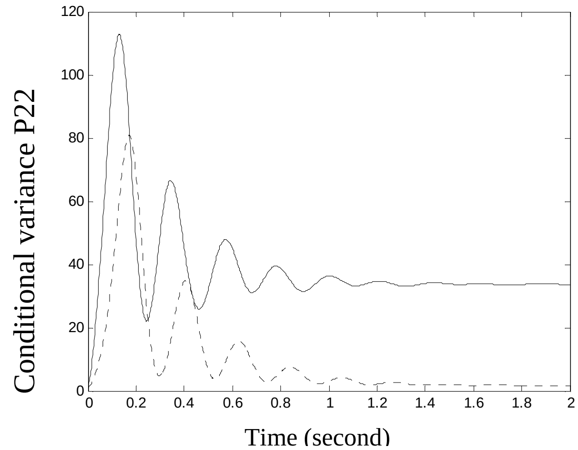

Fig. 10. Conditional variance trajectories using the EKF and higher-order non-linear filtering.

The numerical simulations demonstrated in figures (5)-(6) and figures (9)-(10) suggest the superiority of non-linear filtering equations (8)-(12) in comparison to EKF equations (15)-(19). Generally, the filtering efficacy is adjudged by exploiting system parameters leading to bounded state trajectories. As a result of this, the variance trajectories will be bounded for the specific set of parameters. The variance trajectories of the both filters are bounded under two different sets of data. Thus, the both filters for the non-linear stochastic swing equation are useful. Furthermore, the proposed filtering of the non-linear swing equation is superior, since it offers less variance at time instants. This paper recommends the non-filtering equations stated in (8)-(12) for analysing the stochasticity of the SMIB system in the conditional variance sense. Note that evolution of conditional variance, the non-linear filtering equation (7), accounts for the observation noise correction term coupled with higher-order measurement non-linearities. On the other hand, evolution of conditional variance, EKF equation (14), accounts for the observation noise correction term coupled with lower-order measurement non-linearities.

Remark 2: The universality of the method of the paper is attributed to four formalisms of applied mathematics, ordinary differential equations, and stochastic differential equations, a parabolic linear partial differential equation, stochastic integro-differential equation and stochastic partial differential equation. Here, the Authors explain how these four formalisms arise in the paper. The phase space formulation of the deterministic non-linear stochastic swing equation assumes the ordinary differential equation formalism. After accounting for the random forcing term, we are led to a specific case of the non-linear Itô stochastic differential equation.

Since this paper intends to achieve non-linear filtering of the stochastic swing equation that can be achieved using equation (5) of the paper as well as the succeeding equation. Equation (5) and the succeeding equation assume the structure of non-linear stochastic integro-differential equation. Furthermore, equation (7) of the paper is a consequence of equation (5) of the paper. Notably, equation (7) of the paper assumes the structure of a stochastic partial differential equation in the general setting. In a specific setting, equation (7) for the non-linear swing equation results coupled non-linear stochastic differential equations, i.e.(8)-(12). Thus, the above explains a connection between the four formalisms.

5. CONCLUSION

The main achievement of this paper is to embed non-linear filtering perspectives into power system engineering. The non-linear filtering equations of the paper hinges on the Kushner-Stratonovich filtering for non-linear stochastic differential equations. The Kushner-Stratonovich filtering has ability to preserve qualitative characteristics of non-linear stochastic systems as well as non-linear observation equations.

For the first-time, this paper achieves the non-linear filtering of the SMIB system by taking active and reactive power measurements of the synchronous generator in lieu of linear observation equations. As a result of this, we are led to a system of two non-linear filtering observation equations. That embeds stochastic corrections in the conditional variance evolution equations. On the other hand, the stochastic correction term in the conditional variance equation vanishes for linear observation equations. Thus, the results of this paper are sharper and refined.

Finally, this paper demonstrates a connection between the notion of conditional characteristic function of stochastic processes and the parameters of the SMIB system in the sense of *non-linear filtering* equation, see the *appendix* of the paper.

REFERENCES

Boutayeb, M., Darouach, M., and Frank, P. M. (1997). High order filtering for non-linear dynamical systems, *Proceedings of the American Control Conference*, Albuquerque, New Mexico, pp. 2177-2179.

Chen, H. K., Lin, T. N., and Chen, J. H. (2005). Dynamic analysis, controlling chaos and chaotification of SMIB power system, *Chaos, Solutions and Fractals*, vol.24, pp.1307-1315.

Cotilla-Sanchez, E., Hines, P.D.H., and Danforth, C. (2012). Predicting critical transitions from time series synchrophasor data, *IEEE Transactions on Smart Grid*, vol. 3, no. 4, pp. 1832-1840.

Ghahremani, E. and Kamwa, I., (2011). Dynamic state estimation in power system by applying the extended Kalman filter with unknown inputs to phasor measurements, *IEEE Transactions on Power Systems*, vol. 26, no. 4, pp. 2556-2566.

Ghanavati, G., Hines, P. D. H., Lakoba, T., and Cotilla-Sanchez, E., (2013). Calculation of the autocorrelation function of the stochastic single machine infinite bus system, *North American Power Symposium*, Manhattan, pp.1-6.

Hirpara, R. H. and Sharma, S. N. (2015). The Fokker-Planck equation for a stochastic single machine-infinite bus (SMIB) system, *Journal of Control Engineering and Applied Informatics (CEAI)*, vol. 17, no. 2, pp. 55-63.

Jazwinski, A. H. (1970). *Stochastic Processes and Filtering Theory*, Academic Press, New York and London.

Kalman, R. E. (1960). A new approach to linear filtering and prediction problems, *The ASME Transactions, Journal of Basic Engineering*, vol. 82, pp. 34-45.

Kloeden, P. E. and Platen, E. (1999). *Numerical Solution of Stochastic Differential Equation*, Springer-Verlag, Berlin, Germany.

Kundur, P. (1994). *Power System Stability and Control*, Mc-Graw-Hill, New York.

Kushner, H. J. (1967). Approximations to optimal non-linear filters, *IEEE Transactions on Automatic Control*, vol. 12, no. 5, pp. 546-556.

Liptser, R. S. and Shiryaev, A. N. (1977). *Statistics of Random Processes I*, Springer-Verlag, Heidelberg New York and Berlin.

Mumford, D. (2000). The dawning of the age of stochasticity in Mathematics: Fronteiers and Perspectives, Arnold, V., Atiyah, M., Lax, P., Mazur, B., Ed. *American Mathematical Society*, pp. 197-218.

Odun-ayo, T. and Crow, M. L. (2013). An analysis of power system transient stability using stochastic energy functions, *International Transactions on Electrical Energy Systems*, vol. 23, pp. 151-165.

Park, R. S. and Scheeres, D. J. (2007). Nonlinear mapping of Gaussian statistics: theory and applications to spacecraft trajectory design, *Journal of Guidance, Control and Dynamics*, vol. 29, no. 6, pp. 1367-1375.

Pugachev, V. S. and Sinitsyn, I. N. (1987). *Stochastic Differential Systems (analysis and filtering)*, John Wiley and Sons, Chichester and New York.

Qin, Y. H. and Li, J. C. (2014). Random parameters induce chaos in power systems, *Nonlinear Dynamics*, vol. 77, pp. 1609-1615, doi: 10.1007/s11071-014-1403-4.

Sharma, S. N., Parthasarathy, H., and Gupta, J. R.P. (2006). Third-order approximate Kushner filter for a non-linear dynamical system, *International Journal of Control*, vol. 79, no. 9, pp. 1096-1106.

Sharma, S. N. (2008). A Kolmogorov-Fokker-Planck approach for a stochastic Duffing-van der Pol system, *Differential Equations and Dynamical Systems (An International Journal for Theory, Application, Computer Simulations)*, vol. 16, no. 4, pp.351-377.

Wang, K. and Crow, M. L. (2013). The Fokker-Planck equation for power system stability probability density function evolution, *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2994-3001.

Wei, D. Q. and Luo, X. S. (2009). Noise-induced chaos in single-machine infinite-bus power systems, *EPL A Letters Journal Exploring the Frontiers and Physics*, pp.50008-p1-50008-p6.

Wei, D. Q., Zhang, B., Qiu, D. Y., and Luo, X. S. (2010). Effect of noise on erosion of safe basin in power systems, *Nonlinear Dynamics*, vol. 61, pp. 477-482.

Zhang, J. Y., Zu, P., Yu, Y. P., and Wu, F. (2012). Responses and stability of power system under small Gauss-type random excitation," *Science China Technological Sciences*, vol. 55, no. 7, pp. 1873-1880.

APPENDIX

Since this paper is about the non-linear filtering of the stochastic single machine-infinite bus system, it is worthwhile to write the conditional characteristic function evolution equation in the 'non-filtering sense'. The notion of characteristic function has found applications to sketch the proof of the Fokker-Planck equation as well as non-linear filtering density evolution equation. The stochastic evolution $d\langle\varphi(y_t)\rangle$ of conditional moment becomes

$$d\langle\varphi(y_t)\rangle = (\left\langle f^T(y_t,t)\frac{\partial\varphi(y_t)}{\partial y_t}\right\rangle$$
$$+\frac{1}{2}\left\langle tr((GG^T)(y_t,t)\frac{\partial^2\varphi(y_t)}{\partial y_t\partial y_t^T})\right\rangle)dt$$
$$+(\langle\varphi h^T\rangle-\langle\varphi\rangle\langle h^T\rangle)\psi_n^{-1}(dz_t-\langle h\rangle dt).$$

The above stochastic evolution of the conditional expectation of the scalar function is the central result of exact non-linear filtering. In the component-wise description, the above evolution is recast as

$$d\langle\varphi(y_t)\rangle = \left\langle\sum_i f_i(y_t,t)\frac{\partial\varphi(y_t)}{\partial y_i}\right.$$
$$+\frac{1}{2}\sum_i (GG^T)_{ii}(y_t,t)\frac{\partial^2\varphi(y_t)}{\partial y_i^2}$$
$$\left.+\sum_{i\langle j}(GG^T)_{ij}(y_t,t)\frac{\partial^2\varphi(y_t)}{\partial y_i\partial y_j}\right\rangle dt$$
$$+\sum_{\alpha,\beta}(\langle\varphi h_\alpha\rangle-\langle\varphi\rangle\langle h_\alpha\rangle)\psi_{\alpha\beta}^{-1}(dz_\beta-\langle h_\beta\rangle dt). \qquad (A.1)$$

Suppose $\varphi(y_t)=e^{s^T y_t}$, equation (A.1) becomes the stochastic evoluition of the conditional characteristic function $\varphi(y_t)=e^{s^T y_t}$, i.e.

$$d\langle e^{s^T y_t}\rangle = \left\langle\sum_i f_i(y_t,t)s_i e^{s^T y_t}+\frac{1}{2}\sum_i (GG^T)_{ii}(y_t,t)s_i^2 e^{s^T y_t}\right.$$
$$\left.+\sum_{i\langle j}(GG^T)_{ij}(y_t,t)s_i s_j e^{s^T y_t}\right\rangle dt$$
$$+\sum_{\alpha,\beta}(\langle e^{s^T y_t}h_\alpha\rangle-\langle e^{s^T y_t}\rangle\langle h_\alpha\rangle)\times\psi_{\alpha\beta}^{-1}(dz_\beta-\langle h_\beta\rangle dt).$$

Since the conditional expectation operator $\langle\ \rangle$ is a linear operator, we get

$$d\langle e^{s^T y_t}\rangle = (\left\langle\sum_i f_i(y_t,t)s_i e^{s^T y_t}\right\rangle$$
$$+\frac{1}{2}\left\langle\sum_i (GG^T)_{ii}(y_t,t)s_i^2 e^{s^T y_t}\right\rangle)dt$$
$$+\sum_{\alpha,\beta}(\langle e^{s^T y_t}h_\alpha\rangle-\langle e^{s^T y_t}\rangle\langle h_\alpha\rangle)\times\psi_{\alpha\beta}^{-1}(dz_\beta-\langle h_\beta\rangle dt).$$

$$(A.2)$$

For the stochastic SMIB system of the paper, the stochastic evolution of conditional characteristic function becomes a special case of equation (A.2), i.e.

$$d\langle e^{s^T y_t}\rangle = d\langle e^{s_1\delta+s_2\omega}\rangle = (\langle y_2 s_1 e^{s^T y_t}\rangle$$
$$+\left\langle(-\frac{D}{M}y_2-\frac{VE'_a}{M\ X}\sin y_1+\frac{P_m}{M})s_2 e^{s^T y_t}\right\rangle$$
$$+\frac{1}{2}\left\langle(\frac{\sigma_1^2}{M^2}+\frac{V^2E'^2_a}{M^2\ X^2}\sin^2 y_1)s_2^2 e^{s^T y_t}\right\rangle)dt$$
$$+(\left\langle\frac{VE'_a}{X}\sin y_1 e^{s^T y_t}\right\rangle-\left\langle\frac{VE'_a}{X}\sin y_1\right\rangle\langle e^{s^T y_t}\rangle)$$
$$\times\psi_{11}^{-1}(dz_1-\frac{VE'_a}{X}\sin\langle y_1\rangle dt)$$
$$+(\left\langle(\frac{VE'_a}{X}\cos y_1-\frac{V^2}{X})e^{s^T y_t}\right\rangle-\left\langle\frac{VE'_a}{X}\cos y_1-\frac{V^2}{X}\right\rangle\langle e^{s^T y_t}\rangle)$$
$$\times\psi_{22}^{-1}(dz_2-(\frac{VE'_a}{X}\cos\langle y_1\rangle-\frac{V^2}{X})dt)$$
$$=(\langle\omega s_1 e^{s_1\delta+s_2\omega}\rangle$$
$$+\left\langle(-\frac{D}{M}\omega-\frac{VE'_a}{M\ X}\sin\delta+\frac{P_m}{M})s_2 e^{s_1\delta+s_2\omega}\right\rangle$$
$$+\frac{1}{2}\left\langle(\frac{\sigma_1^2}{M^2}+\frac{V^2E'^2_a}{M^2\ X^2}\sin^2\delta)s_2^2 e^{s_1\delta+s_2\omega}\right\rangle)dt$$
$$+(\left\langle\frac{VE'_a}{X}\sin\delta e^{s_1\delta+s_2\omega}\right\rangle-\left\langle\frac{VE'_a}{X}\sin\delta\right\rangle\langle e^{s_1\delta+s_2\omega}\rangle)$$
$$\times\psi_{\delta\delta}^{-1}(dz_1-\frac{VE'_a}{X}\sin\langle\delta\rangle dt)$$
$$+(\left\langle(\frac{VE'_a}{X}\cos\delta-\frac{V^2}{X})e^{s_1\delta+s_2\omega}\right\rangle$$
$$-\left\langle\frac{VE'_a}{X}\cos\delta-\frac{V^2}{X}\right\rangle\langle e^{s_1\delta+s_2\omega}\rangle)$$
$$\times\psi_{\omega\omega}^{-1}(dz_2-(\frac{VE'_a}{X}\cos\langle\delta\rangle-\frac{V^2}{X})dt).$$