# Adaptive Iterating learning sliding mode control for output tracking of incommensurate fractional-order systems

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**Abstract:** This paper develops a novel controller called adaptive iterative learning sliding mode (AILSM) to control linear and nonlinear incommensurate fractional-order systems. By definition of weighting parameter, the proposed controller can have both adaptive and iterative learning control structure in order to use the advantages of both controllers simultaneously and therefore, it achieves better control performance. This controller is designed in a way to be robust against the external disturbance. It also estimates unknown parameters of fractional-order systems. The proposed controller, unlike the conventional iterative learning control, does not need to apply direct control input to the output of the system and can also be implemented for incommensurate fractional-order systems. It is shown that the controller performs well for two cases. First, all state variables exist in output dynamic and second, only some of the state variables exist in the output dynamic. Illustrative examples verify the performance of the proposed control in presence of unknown disturbances and model uncertainties.

*Keywords:* Incommensurate Fractional system; Iterative learning control; Adaptive control; Sliding mode control; Lyapunov theory; Partial and complete observability.

## 1. INTRODUCTION

Iterative Learning Control (ILC) is one of the recent topics in control theories. ILC, which belongs to the intelligent control methodology, it is an approach to improve the transient performance of systems that operate repetitively over a fixed time interval. In detail, they apply a fixed-length input signal on a certain system. After the complete input is applied, the system returns to the same initial state and the output trajectory that is resulted from the applied input is compared with the desired reference. The possible error is used to construct a new input signal of the same length that is applied to the next iteration. The aim of the ILC algorithm is to continue the trial so that as more trials are executed, the output would approach the desired trajectory more (Li et al., 2011a).

Recently, an advanced calculus called fractional order calculus is applied in many controllers (Hosseinnia et al., 2010, 2013, 2014), as well as ILC to improve their performance. Fractional calculus is an old mathematical operation with a 300-year-old history (Podlubny, 1999). For many years, this branch of science has been considered as a purely mathematical and theoretical discipline with nearly no application (Gutierrez et al., 2010). It also has been found that the behavior of many physical systems can be properly described by using the fractional-order system theory (EI-Saye, 1996; Jenson and Jeffreys, 1997; Kusnezov, 1999). In fact, most of the real-world processes can better be modeled by fractional-order systems (Torvik, 1984) and the fractional controller has shown better performance for such a system (Razmju et al., 2010; Önder, 2011; Yin et al., 2012; Wang et al., 2013; Yin et al., 2013; Bigdeli, 2015; Bouarroudj et al., 2015; Rebai et al., 2015; Kumar et al., 2016 ).

In an article by (Li et al., 2011a), the authors have defined $D^{\alpha}$-type ILC algorithm for linear fractional-order systems. In this paper, the designed controller is not robust against the external disturbance and can be implemented only for linear systems. The system dynamic is also without disturbance, the parameters of the system assumed to be fully known and the fractional system was considered in commensurate order. The process can only track the desired output if the control signal is directly applied to the output. $PD^{\alpha}$-type and $P$-type ILC algorithms are designed for nonlinear fractional-order systems in the articles by (Li et al., 2011b; Lan, 2012) respectively. In both papers, the designed controllers are not robust against the external disturbance, system dynamic has no unknown parameters and can be implemented only for a special class of nonlinear commensurate fractional-order systems. For the first time, the robust controller with ILC structure for a fractional system with $D^{\alpha}$-type and adaptive $P$-type based have been introduced by (Lan and Zhou, 2013) and (Li et al., 2014), respectively. While the introduced methods were robust to an external disturbance, there were some restrictions. Firstly, for the desired output tracking process the input controller must be fed by output dynamic. Secondly, the system states dynamic is assumed without any unknown parameters.

In Recent papers, (Razmjou et al., 2018a, 2018b), the authors considered an only fractional system in the commensurate order form.

The structure of the proposed controller in this paper is adaptive iterative learning sliding mode (AILSM) and its basic idea is illustrated in Fig. 1, where $u_{kitr}$ and $y_{kitr}$ are, respectively, the system input and output in the $kth$ iteration, $\hat{\theta}_{kitr}$ and $\hat{\theta}_{(k+1)itr}(t)$ are the adaptive recursive control part of the $kth$ and $(k+1)th$ trial, that are used to learn the unknown parameters, $r(t)$ is the given desired output and $d(t)$ is the output disturbance.

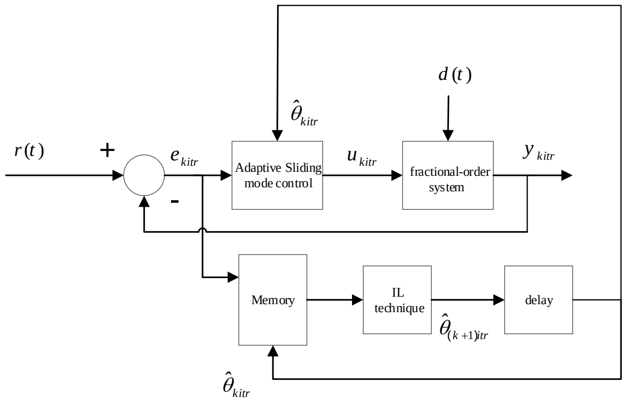


Fig. 1. The basic scheme of AILSM control.

The goal of control is that $\lim_{kitr\to\infty} y_{kitr}(t) = r(t)$ for all $t \in [0,T]$, where $T$ is a fixed constant. The advantages of using sliding mode controller are being robust in the presence of disturbance and also has a similar structure for linear and nonlinear systems. So far, ILC has been applied in proportional plus integral and derivative (PID) to control fractional-order systems (Li et al., 2011a; Li et al., 2011b; Lan, 2012; Lan and Zhou, 2013; Li et al., 2014), in which convergence is a key component of the design. Furthermore, this condition is different in linear and nonlinear systems that make use of such structure impossible to control both linear and nonlinear system simultaneously and is not also robust in presence of an unknown disturbance. These are the limitations that will be overcome by our proposed method. The mentioned controller in this paper guarantees the convergence of tracking error using adaptive iterative learning algorithm in the presence of unknown parameters at system dynamic. By choosing the proper weighting parameter, we can take advantages of both adaptive control and IL algorithm. These are the prominences of the proposed method in comparison with the referenced PID controllers. In addition, model reference adaptive control (MRAC) is applied in the sliding mode control structure. By using that, any disturbance will be rejected and the system will be kept at the desired sliding surface without having any information on disturbance dynamic. In addition to the mentioned advantages, unlike the conventional ILC-based control the proposed control, such as PIDILC (Li et al., 2011a; Li et al., 2011b; Lan, 2012; Lan and Zhou, 2013; Li et al., 2014) does not require to be applied directly to the output of the system while guaranteeing the desired performance. The proposed controller in this paper unlike the controllers introduced in past articles with PIDIL or ILSM structure can also control incommensurate fractional order systems. This paper is organized as follows: in Section 2, basic definitions of fractional calculus are presented. System description and its assumptions are expressed in Section 3. AILSM scheme as well as the convergent condition for incommensurate fractional-order systems, are discussed in Section 4. MATLAB/SIMULINK results for two benchmarks (Duffing and Chaotic oscillator systems) are shown in Section 5. Finally, some conclusions are drawn in Section 6.

## 2. PRELIMINARIES

There are several definitions for fractional calculus, but in the rest of this paper, the notation $D^{\alpha}(.)$ indicates the Riemann–Liouville derivative of order $\alpha$.

**Definition** (Tavazoei and Haeri, 2008)**.** Consider the Eq. (1):

$$D^{\alpha_i} x_i = f_i(x_1, x_2, \dots x_n) \qquad i = 1, 2, \dots, n \tag{1}$$

This system is called commensurate order system if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ and is called incommensurate order if $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_n$ .

**Lemma1**(Deng et al., 2007)**:** Consider the following n-dimensional linear incommensurate fractional-order system:

$$\begin{cases} D^{\alpha_1} x_1 = a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ D^{\alpha_2} x_2 = a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ \vdots \\ D^{\alpha_n} x_n = a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n \end{cases} \tag{2}$$

where all $\alpha_i$ are rational numbers between 0 and 1. Assume $W$ be the lowest common multiple of the denominators $z_i$ of $\alpha_i$ , where $\alpha_i = \upsilon_i / z_i$, $(z_i, \upsilon_i) = 1$, $z_i, \upsilon_i \in Z^+$, for $i = 1, 2, \dots, n$ define:

$$\Delta(\lambda) = \begin{bmatrix} \lambda^{Wa_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{Wa_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{Wa_n} - a_{nn} \end{bmatrix} \tag{3}$$

Then the zero solution of system (2) is globally asymptotically stable in the Lyapunov sense if all roots of the equation $\det(\Delta(\lambda)) = 0$ satisfy:

$$\min \left|\arg(\lambda_i)\right| > \pi / 2W. \tag{4}$$

$\Delta(s)$ is called the characteristic matrix and $\det(\Delta(s))$ is called the characteristic polynomial of the system (2).

## 3. SYSTEM DESCRIPTION

Consider the following higher-order single-input and single-output linear/nonlinear incommensurate fractional-order dynamic system described by:

$$
\begin{aligned}
&D^{\alpha_i} x_i(t) = x_{i+1}(t), \quad i = 1, 2, \dots, n-1 \\
&D^{\alpha_n} x_n(t) = F(x,t) + \theta^T \xi(x) + b(x,t)u(t) + d(t) \\
&0 < \alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_n < 1 \\
&y(t) = \begin{bmatrix} c_1 & c_2 & \cdots & c_q \end{bmatrix} X(t) \\
&X(0) = X_0
\end{aligned} \tag{5}
$$

where $X(t) = [x_1, x_2, ..., x_n]^T$ is the measurable state variables, $u(t)$ is a control input, $y(t)$ is a system output, $F(x,t)$ is a linear/nonlinear certain dynamic function, $b(x,t)$ is a known non-zero function and the variable $d(t)$ represents the disturbance with unknown dynamics. $\theta$ is an unknown and constant vector with $l \times 1$ dimension to be learned, $\xi(x)$ is a function of state variables with $l \times 1$ dimension and $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_q \end{bmatrix}$ is a constant row vector $(c_i \in R)$ and assumes that $q$ is the maximum value of $i$ so that $c_i \neq 0$ (for $i = 1, 2, \dots, n$ ).

**Assumption 1.** The unknown disturbance variable $d(t)$ is bounded such that:

$|d(t)| \leq M \quad \forall t \in [0, T]$

where $M$ is a known positive constant.

**Assumption 2.** The desired and real outputs $r(t), y(t)$ are bounded and differentiable with the respect to time $t$ up to the $n$th order on a finite time interval [0, $T$] and all of the fractional derivatives are available and bounded.

From assumption 2 follows:

$$
\begin{aligned}
&D^{\alpha_i} r_i(t) = r_{i+1}(t) \qquad r_1(t) = r(t) \\
&D^{\alpha_i} y_i(t) = y_{i+1}(t) \qquad y_1(t) = y(t)
\end{aligned} \tag{6}
$$

**Remark1:** According to the definition of system output in (5) the error equation is defined as follows:

$$
e_1(t) = r_1(t) - y_1(t) = r(t) - \sum_{i=1}^{q} c_i x_i(t) \tag{7}
$$

From (6), the error equation is obtained as follows:

$$
\begin{aligned}
e_2(t) &= r_2(t) - y_2(t) = D^{\alpha_1} r_1(t) - D^{\alpha_1} y_1(t) \\
&= D^{\alpha_1} r_1(t) - D^{\alpha_1} \sum_{i=1}^{q} c_i x_i(t) \\
&= D^{\alpha_1} r_1(t) - c_1 x_2(t) - D^{\alpha_1} \sum_{i=1}^{q-1} c_{i+1} x_{i+1}(t) \\
e_3(t) &= r_3(t) - y_3(t) = D^{\alpha_2} r_2(t) - D^{\alpha_2} y_2(t) \\
&= D^{\alpha_2} r_2(t) - c_1 x_3(t) - D^{\alpha_1 + \alpha_2} \sum_{i=1}^{q-1} c_{i+1} x_{i+1}(t)
\end{aligned} \tag{8}
$$

**Remark2:** According to the definition of error in (7), the error equation for the $n$th order will be as follows:

$$
\begin{aligned}
e_{n+1}(t) &= r_{n+1}(t) - y_{n+1}(t) \\
&= D^{\alpha_n} r_n(t) - c_1 D^{\alpha_n} x_n(t) - D^{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{q-1} c_{i+1} x_{i+1}(t)
\end{aligned} \tag{9}
$$

Replacing $D^{\alpha_n} x_n(t)$ from (5) into (9) we have:

$$
\begin{aligned}
e_{n+1}(t) = &\, D^{\alpha_n} r_n(t) - D^{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{q-1} c_{i+1} x_{i+1}(t) \\
&- c_1 (F(x,t) + \theta^T \xi(x) + b(x,t)u(t) + d(t))
\end{aligned} \tag{10}
$$

From (6), (8) and (10) error equations can be taken as follows:

$$
\begin{aligned}
&D^{\alpha_1} e_1(t) = e_2(t) \\
&D^{\alpha_2} e_2(t) = e_3(t) \\
&\vdots \\
&D^{\alpha_n} e_n(t) = D^{\alpha_n} r_n(t) - D^{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{q-1} c_{i+1} x_{i+1}(t) \\
&\qquad - c_1 (F(x,t) + \theta^T \xi(x) + b(x,t)u(t) + d(t))
\end{aligned} \tag{11}
$$

**Assumption 3.** The initial conditions are $D^{-1} e_1(0) = D^{\alpha-1} e_1(0) = D^{\alpha-1} e_2(0) = \cdots = D^{\alpha-1} e_n(0) = 0$ at any iteration $\forall t \in [0, T]$ .

## 4. MAIN RESULT

This paper mainly aims to design a robust controller for the system (5) so that the output $y(t)$ tracks a time variable reference signal $r(t)$ and error asymptotically tends to zero. To do so, a new design of AILSM control is proposed. This new algorithm uses a combined time-domain and iteration-domain adaptation law allows to guarantee the boundedness of the tracking error and the control input, in the sense of the infinity norm, as well as the convergence of the tracking error to zero, without any a priori knowledge of fractional system parameters. Also, the mentioned controller is robust in the presence of external disturbance without knowing the details of the disturbance dynamic.

### *4.1. Sliding surface design*

For the considered system (5), an integral sliding function dynamic is chosen as follows:

$$
\begin{aligned}
S(t) = &\, k_0 D^{-1} e_1(t) + k_1 D^{\alpha_1 - 1} e_1(t) \\
&+ k_2 D^{\alpha_2 - 1} e_2(t) + \cdots + k_{n-1} D^{\alpha_{n-1} - 1} e_{n-1}(t) + D^{\alpha_n - 1} e_n(t)
\end{aligned} \tag{12}
$$

From assumption 3 the sliding surface $S(0) = 0$ .

Maintaining the system's states on the sliding surface $S(t) = 0$ results as:

$$
\begin{aligned}
S(t) = 0 \Rightarrow\ & k_0 D^{-1} e_1(t) + k_1 D^{\alpha_1 - 1} e_1(t) \\
& + k_2 D^{\alpha_2 - 1} e_2(t) + \cdots + k_{n-1} D^{\alpha_{n-1} - 1} e_{n-1}(t) + D^{\alpha_n - 1} e_n(t) = 0 \\
\Rightarrow\ & D^{\alpha_n - 1} e_n(t) = -k_0 D^{-1} e_1(t) \\
& - \sum_{i=1}^{n-1} k_i D^{\alpha_i - 1} e_i(t)
\end{aligned}
\tag{13}
$$

By applying $D^1$ operator in both sides of equation (13) we have:

$$
\begin{aligned}
D^{\alpha_n} e_n(t) &= -k_0 e_1(t) - \sum_{i=1}^{n-1} k_i D^{\alpha_i} e_i(t) \\
&= -\sum_{i=0}^{n-1} k_i e_{i+1}(t)
\end{aligned}
\tag{14}
$$

From (8) and (9) the following relations can be obtained:

$$
\begin{aligned}
& D^{\alpha_1} e_1(t) = e_2(t) \\
& D^{\alpha_2} e_2(t) = e_3(t) \\
& \vdots \\
& D^{\alpha_n} e_n(t) = e_{n+1}(t)
\end{aligned}
\tag{15}
$$

where $e_{n+1}(t)$ is viewed as a control input, the task is to design $e_{n+1}(t)$ to stabilize the origin (equilibrium point) of the system (15). This task may be achieved by choosing:

$$
\begin{aligned}
& e_{n+1}(t) = -(k_0 e_1(t) + k_1 e_2(t) + \cdots + k_{n-1} e_n(t)) \\
& \Rightarrow e_{n+1}(t) = -\sum_{i=0}^{n-1} k_i e_{i+1}(t)
\end{aligned}
\tag{16}
$$

By substituting (16) in (15), the following formula will be given:

$$
\begin{aligned}
& D^{\alpha_i} E_i(t) = A E_i(t) \\
& E_i(t) = [e_1(t)\, e_2(t) \cdots e_n(t)]^T \\
& A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \ddots & & \\ -k_0 & -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix}
\end{aligned}
\tag{17}
$$

where $k_0$ to $k_{n-1}$ are chosen so that the eigenvalues of the matrix $A$ satisfy the stability condition of the Lemma1. In this situation, the linear fractional-order error system (17), is asymptotically stable and the components of the error vector decay toward zero. So, in (12), it's obvious that for $S(t) = 0$ the error vector is toward zero and therefore, output tracking of the time-varying reference signal is achieved.

Taking derivatives with respect to time $t$ on both sides of (12) where $k_n = 1$ it is obtained:

$$
\begin{aligned}
\dot{S}(t) &= k_0 e_1(t) + k_1 D^{\alpha_1} e_1(t) \\
&+ k_2 D^{\alpha_2} e_2(t) + \cdots + D^{\alpha_n} e_n(t)
\end{aligned}
\tag{18}
$$

Considering (10) and the fact that $e(t) = y(t) - r(t)$, (18) can be further expanded:

$$
\begin{aligned}
\dot{S}(t) &= \sum_{i=0}^{n-1} k_i e_{i+1}(t) + D^{\alpha_n} r_n(t) - D^{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{q-1} c_{i+1} x_{i+1}(t) \\
&- c_1 (F(x,t) + \theta^T \xi(x) + b(x,t) u(t) + d(t))
\end{aligned}
\tag{19}
$$

By taking $c_1 = 1$, the above equation is as below:

$$
\begin{aligned}
\dot{S}(t) &= \sum_{i=0}^{n-1} k_i e_{i+1}(t) + D^{\alpha_n} r_n(t) - D^{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{q-1} c_{i+1} x_{i+1}(t) \\
&- F(x(t),t) - \theta^T \xi(x(t)) - b(x(t),t) u(t) - d(t)
\end{aligned}
\tag{20}
$$

The above equation can be interpreted as the sliding variable dynamics. The condition $S(t) = 0$ defines the motion of the system on the sliding surface.

*4.2. AILSM controller configuration*

The control signal $u(t)$ is defined as an iterative and continuous control input signal. Therefore, an AILSM control at *kth* iteration is designed as follows:

$$
u_{kitr}(t) = b^{-1}(x_{kitr}, t) \begin{pmatrix} \sum_{i=0}^{n-1} k_i e_{i+1,kitr}(t) + D^{\alpha_n} r_n(t) \\ -D^{\sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{q-1} c_{i+1} x_{i+1,kitr}(t) \\ -F(x_{kitr}(t), t) - \hat{\theta}_{kitr}^T \xi(x_{kitr}(t)) \\ +\psi_{kitr}(t) \operatorname{sgn}(S_{kitr}(t)) \end{pmatrix}
\tag{21}
$$

where *kitr* indicates the number of iterations, sgn is the signum function, $\hat{\theta}(t)$ is the adaptive recursive control part that is used to learn the unknown parameter $\theta(t)$ and generated by the following update law:

$$
(1-\gamma) \dot{\hat{\theta}}_{kitr}(t) = -\gamma \hat{\theta}_{kitr}(t) + \gamma \hat{\theta}_{(k-1)itr}(t) - \xi(x_{kitr}) \beta S_{kitr}(t) \tag{22}
$$

where $\gamma \in (0,1), \beta > 0$ are defined as the weighting and learning gain respectively. In general, (22) will become a pure adaptive law if $\gamma = 0$, or a pure iteration learning law if $\gamma = 1$. Therefore, update the law in (22) defines AIL mechanism. The initial value of the parameter vector is set to $\hat{\theta}_{kitr}(0) = \hat{\theta}_{(k-1)itr}(T)\ \forall k = 1, 2, \ldots$, and the initial parameter profile for *kitr=0* is chosen arbitrarily as $\hat{\theta}_{0itr}(t) = \theta_{0itr}\ \ \forall t \in [0, T]$, where $\theta_{0itr}$ is a constant parameter vector. The reason to choose this initial condition for the parameter update mechanism in (22) is that a constant parameter will hold the same value at $t$=0 and $t$=$T$. If $\hat{\theta}_{kitr}(0) \neq \hat{\theta}_{(k-1)itr}(T)$, it would be meaningless to apply the consecutive initial condition. The consecutive initial condition is applicable to different types of updating mechanism, only if we have additional knowledge that $\hat{\theta}_{kitr}(0) = \hat{\theta}_{(k-1)itr}(T)$ (Xu and Xu, 2004).

The variable $\psi_{kitr}(t)$ is used to attenuate the effect of the unknown disturbance. This variable is defined as below:

$$\dot{\psi}_{kitr}(t) = \rho\left|S_{kitr}(t)\right| \qquad \psi_{kitr}(0) = \hat{\psi} > M \tag{23}$$

where $\rho$ is a constant positive parameter and is effective to eliminating speed of unknown disturbance and $M$ is upper bound of uncertainty that was said in assumption 1. As indicated in equation (23), $\psi_{kitr}(t)$ becomes constant only when $S_{kitr}(t) = 0$. Therefore, the sliding variable dynamic (19) can be simplified by inserting the AILSM law (21):

$$\dot{S}_{kitr}(t) = \phi^T_{kitr}(t)\xi(x_{kitr}) - \psi_{kitr}(t)\,\mathrm{sgn}(S_{kitr}(t)) - d(t) \tag{24}$$

where $\phi_{kitr}(t) = \hat{\theta}_{kitr}(t) - \theta$ is the parametric estimation error.

### *4.3. Convergence of the output-tracking error*

To study the stability and convergence, we use the concept of $L_{pe}[0,T]$ in the subsequent discussions to denote the set of Lebesgue measurable (or piecewise continuous) real-valued (vector) functions with $L_{pe}$ norm (Tayebi, 2004):

$$\|x(t)\|_{pe} = \begin{cases} \left(\int_0^T \|x(t)\|^p\, dt\right)^{1/p} & if\ p \in [1,\infty) \\ \sup\limits_{0\le t\le T} \|x(t)\| & if\ p = \infty \end{cases}$$

We say that $x(t) \in L_{pe}[0,T]$ when $\|x(t)\|_{pe}$ exists and is finite.

We will prove that the convergence of the sliding surface and output tracking error dynamics with the new method is presented blow:

**Theorem 1** (Appendix A)**.** Consider the fractional order system (5) under the adaptive robust control laws (21), (23) and parameter AIL law (22). If assumptions (A1)-(A3) are satisfied, then we have $S_{1itr}(t), \hat{\theta}_{1itr}(t) \in L_{\infty e}[0,T]$ and bounded.

**Theorem 2.** Consider the same problem set up in Theorem 1. If assumptions (A1)–(A3) are satisfied, then we have $S_{kitr}(t), \hat{\theta}_{kitr}(t) \in L_{\infty e}[0,T]$ for all $kitr \in Z^+$ and $\lim\limits_{kitr\to\infty} \left|S_{kitr}(t)\right| = 0 \ \ \forall t \in [0,T]$.

**Proof.** To evaluate the convergence property cited in Theorem 2, we define the following composite energy function at *k*th iteration:

$$W_{kitr}(T) = \int_0^T \frac{\gamma}{2\beta}\phi^T_{kitr}(\tau)\phi_{kitr}(\tau)d\tau + \frac{1-\gamma}{2\beta}\phi^T_{kitr}(T)\phi_{kitr}(T) \tag{25}$$

The difference between $W_{kitr}(T)$ and $W_{(k-1)itr}(T)$ can be derived as follows:

$$\begin{aligned}\Delta W_{kitr}(T) &= W_{kitr}(T) - W_{(k-1)itr}(T) \\ &= \int_0^T \frac{\gamma}{2\beta}\left(\phi^T_{kitr}(\tau)\phi_{kitr}(\tau) - \phi^T_{(k-1)itr}(\tau)\phi_{(k-1)itr}(\tau)\right. \\ &\quad + \frac{1-\gamma}{2\beta}\phi^T_{kitr}(T)\phi_{kitr}(T) \\ &\quad - \frac{1-\gamma}{2\beta}\phi^T_{(k-1)itr}(T)\phi_{(k-1)itr}(T)\end{aligned} \tag{26}$$

Considering the fact $\phi_{kitr}(0) = \phi_{(k-1)itr}(T)$, and the integral form of $\frac{1-\gamma}{2\beta}\phi^T_{kitr}(T)\phi_{kitr}(T)$, (26) will be defined as:

$$\begin{aligned}\Delta W_{kitr}(T) = &\int_0^T \frac{\gamma}{2\beta}\left(\phi^T_{kitr}(\tau)\phi_{kitr}(\tau) - \phi^T_{(k-1)itr}(\tau)\phi_{(k-1)itr}(\tau)\right)d\tau \\ &+ \frac{1-\gamma}{\beta}\int_0^T \phi^T_{kitr}(\tau)\dot{\phi}_{kitr}(\tau)d\tau \\ &+ \frac{1-\gamma}{2\beta}\phi^T_{kitr}(0)\phi_{kitr}(0) \\ &- \frac{1-\gamma}{2\beta}\phi^T_{(k-1)itr}(T)\phi_{(k-1)itr}(T)\end{aligned} \tag{27}$$

**Remark3:** Because of $\hat{\theta}_{kitr}(0) = \hat{\theta}_{(k-1)itr}(T)$, therefore $\phi_{kitr}(0) = \hat{\theta}_{kitr}(0) - \theta = \hat{\theta}_{(k-1)itr}(T) - \theta = \phi_{(k-1)itr}(T)$ :

$$\begin{aligned}\Delta W_{kitr}(T) = &\int_0^T \frac{\gamma}{2\beta}\left(\phi^T_{kitr}(\tau)\phi_{kitr}(\tau) - \phi^T_{(k-1)itr}(\tau)\phi_{(k-1)itr}(\tau)\right)d\tau \\ &+ \frac{1-\gamma}{\beta}\int_0^T \phi^T_{kitr}(\tau)\dot{\phi}_{kitr}(\tau)d\tau\end{aligned} \tag{28}$$

By using AIL law (22), above equation is simplified to:

$$\begin{aligned}\Delta W_{kitr}(T) = &\int_0^T \frac{\gamma}{2\beta}\left(\phi^T_{kitr}(\tau)\phi_{kitr}(\tau) - \phi^T_{(k-1)itr}(\tau)\phi_{(k-1)itr}(\tau)\right)d\tau \\ &+ \int_0^T \frac{\gamma}{\beta}\left(-\phi^T_{kitr}(\tau)\phi_{kitr}(\tau) + \phi^T_{kitr}(\tau)\phi_{(k-1)itr}(\tau)\right)d\tau \\ &- \int_0^T \phi^T_{kitr}(\tau)\xi(x_{kitr})S_{kitr}(\tau)\,d\tau \\ = &-\int_0^T \frac{\gamma}{2\beta}\left(\phi_{kitr}(\tau) - \phi_{(k-1)itr}(\tau)\right)^T\left(\phi_{kitr}(\tau) - \phi_{(k-1)itr}(\tau)\right)d\tau \\ &- \int_0^T \phi^T_{kitr}(\tau)\xi(x_{kitr})S_{kitr}(\tau)\,d\tau\end{aligned} \tag{29}$$

Now, define a positive definite function $U_{kitr}(t)$ as:

$$U_{kitr}(t) = \frac{1}{2}S^2_{kitr}(t) + \frac{1}{2\rho}(\psi_{kitr}(t) - \hat{\psi})^2 \tag{30}$$

According to the equations (23) and (24), the time derivative of $U_{kitr}(t)$ with respect to time *t* will satisfy:

$$\dot{U}_{kitr}(t) \le \left|S_{kitr}(t)\right|\left(M - \hat{\psi}\right) + \phi^T_{kitr}(t)\xi(x_{kitr})S_{kitr}(t) \tag{31}$$

Integrating (31) from 0 to $T$ gives out:

$$U_{kitr}(T)-U_{kitr}(0)\le\int_0^T\left|S_{kitr}(\tau)\right|\left(M-\hat{\psi}\right)d\tau + \int_0^T\phi_{kitr}^T(\tau)\xi(x_{kitr})S_{kitr}(\tau)d\tau \quad (32)$$

By applying the fact $U_{kitr}(0)=0$, due assumption 3 and definition (23) implies that:

$$-\int_0^T\phi_{kitr}^T(\tau)\xi(x_{kitr})S_{kitr}(\tau)d\tau\le\int_0^T\left|S_{kitr}(\tau)\right|(M-\hat{\psi})d\tau - U_{kitr}(T) \quad (33)$$

According to the result of (29) and (33), we will have:

$$\begin{aligned}\Delta W_{kitr}(T)&\le -U_{kitr}(T)-\int_0^T\left|S_{kitr}(\tau)\right|(\hat{\psi}-M)d\tau\\&-\int_0^T\frac{\gamma}{2\beta}\left(\phi_{kitr}(\tau)-\phi_{(k-1)itr}(\tau)\right)^T\left(\phi_{kitr}(\tau)-\phi_{(k-1)itr}(\tau)\right)d\tau\\ \Delta W_{kitr}(T)&=W_{kitr}(T)-W_{(k-1)itr}(T)\le -U_{kitr}(T)\\&-\int_0^T\left|S_{kitr}(\tau)\right|(\hat{\psi}-M)d\tau\le 0\end{aligned} \quad (34)$$

Since $W_{1itr}(T)$ is bounded as proved in appendix A section, bounded of $W_{kitr}(T)$ and hence, $\int_0^T\phi_{kitr}^T(\tau)\phi_{kitr}(\tau)d\tau$ as a result, $\int_0^t\phi_{kitr}^T(\tau)\phi_{kitr}(\tau)d\tau \ \ \forall t\in[0,T]$ and $\phi_{kitr}^T(T)\phi_{kitr}(T)$ are guaranteed $kitr\in Z^+$.

Consider the time derivative of a Lyapunov-like function in (63). By integrating (63) from 0 to $t$ for any $t\in[0,T]$, we have:

$$V_{kitr}(t)\le V_{kitr}(0)+\int_0^t\frac{\gamma}{4\beta}\phi_{(k-1)itr}^T(\tau)\phi_{(k-1)itr}(\tau)d\tau \quad (35)$$

Since $V_{kitr}(0)=\dfrac{1-\gamma}{2\beta}\phi_{kitr}^T(0)\phi_{kitr}(0)=\dfrac{1-\gamma}{2\beta}\phi_{(k-1)itr}^T(T)\phi_{(k-1)itr}(T)$ and $\int_0^t\phi_{(k-1)itr}^T(\tau)\phi_{(k-1)itr}(\tau)d\tau \ \ \forall t\in[0,T]$ are bounded for all $kitr\in Z^+$, therefore left-hand side of inequality (35) also bounded $(V_{kitr}(t)\in L_{\infty e}[0,T])$ and consequently $S_{kitr}(t),\hat{\theta}_{kitr}(t)\in L_{\infty e}[0,T]$.

(34) implies that:

$$U_{kitr}(T)+\int_0^T\left|S_{kitr}(\tau)\right|(\hat{\psi}-M)d\tau\le W_{(k-1)itr}(T)-W_{kitr}(T)\le W_{1itr}(T) \quad (36)$$

Note that (34) also gives out:

$$W_{kitr}(T)\le W_{1itr}(T)-\sum_{jitr=2}^{kitr}U_{jitr}(T)-\sum_{jitr=2}^{kitr}\int_0^T\left|S_{jitr}(\tau)\right|(\hat{\psi}-M)d\tau \quad (37)$$

Or equivalently:

$$\begin{aligned}&\left(\sum_{jitr=2}^{kitr}U_{jitr}(T)+\sum_{jitr=2}^{kitr}\int_0^T\left|S_{jitr}(\tau)\right|(\hat{\psi}-M)d\tau\right)\\&\le W_{1itr}(T)-W_{kitr}(T)\\&\le W_{1itr}(T)\end{aligned} \quad (38)$$

Hence, from (36) and (38) we conclude that:

$$\lim_{kitr\to\infty}\int_0^T\left|S_{kitr}(\tau)\right|d\tau=0 \quad (39)$$

From the above equation, it is obvious that $\lim_{kitr\to\infty}\left|S_{kitr}(t)\right|=0 \ \ \forall t\in[0,T]$. This completes the proof.

From the result of Theorem 2, we can conclude that the sliding function dynamics $S_{kitr}(t)$ converges to the origin. Since the parameters of sliding function dynamics are chosen so that it satisfies the stability condition of the Lemma1, then the output tracking error is convergent and finally, the output tracking is satisfied.

## 5. SIMULATION RESULTS

In this section, our goal is to achieve AILSM control by applying the method on two different incommensurate fractional-order systems:

**Example 1.** Consider the fractional order Chaotic Oscillator, which is written as (Ahmad and Sprott, 2003):

$$\begin{cases}D^{\alpha_1}x_1(t)=x_2(t)\\ D^{\alpha_2}x_2(t)=x_3(t)\\ D^{\alpha_3}x_3(t)=-\theta(x_1(t)+x_2(t)+x_3(t))\\ \qquad +\operatorname{sgn}(x_1(t))+u(t)+d(t)\end{cases} \quad (40)$$

where from equations (5) and (40), $(\alpha_1,\alpha_2,\alpha_3)=(0.98,0.9,0.95)$, $\theta=0.4$, $\xi(x)=(x_1+x_2+x_3)$, $F(x,t)=\operatorname{sgn}(x_1(t))$ and $d(t)=0.1\sin(t)$. The output of the above system is considered as follows:

$$y(t)=x_1(t)$$

As it is clear from the above equation, one state variable exists in the output dynamic and the system has one unknown parameter $(a)$.

The goal of control is that to design a robust controller for the system (40) so that the output $y(t)$ tracks a time variable reference signal $r(t)=\sin(t)$ and error asymptotically tends to zero.

To achieve that, by considering (12) and (21), the sliding function and control law is given:

$$S_{kitr}(t)=k_0D^{-1}e_{1,kitr}(t)+k_1D^{\alpha_1-1}e_{1,kitr}(t)+k_2D^{\alpha_2-1}e_{2,kitr}(t)+D^{\alpha_3-1}e_{3,kitr}(t) \quad (41)$$

$$\begin{aligned} u_{kitr}(t) &= \sum_{i=0}^{2} k_i e_{i+1,kitr}(t) + D^{\alpha_3} r_3(t) - \operatorname{sgn}(x_{1,kitr}(t)) \\ &\quad + \hat{a}_{kitr}(t)(x_{1,kitr}(t) + x_{2,kitr}(t) + x_{3,kitr}(t)) \\ &\quad + \psi_{kitr}(t)\operatorname{sgn}(S_{kitr}(t)) \end{aligned} \tag{42}$$

where $k_0 = 5, k_1 = 2, k_2 = 4$. To choose these coefficients and from equations (15) and (16) and according to Lemma1, $\Delta(\lambda)$ is calculated as follows:

$$\Delta(\lambda) = \begin{bmatrix} \lambda^{98} & -1 & 0 \\ 0 & \lambda^{90} & -1 \\ 5 & 2 & \lambda^{95} + 4 \end{bmatrix}$$

As a result, the equation $\det(\Delta(\lambda)) = 0$ is obtained:

$$\lambda^{283} + 4\lambda^{188} + 2\lambda^{98} + 5 = 0$$

Finally from the above equation $\min|\arg(\lambda)| = 0.7507 > \pi / 200 = 0.0157$. Therefore from the Lemma1 and (17) stability of error dynamic will be established.

According to (22), AIL mechanism is defined:

$$\begin{aligned} (1-\gamma)\dot{\hat{a}}_{kitr}(t) &= -\gamma \hat{a}_{kitr}(t) + \gamma \hat{a}_{(k-1)itr}(t) \\ &\quad + (x_{1,kitr}(t) + x_{2,kitr}(t) + x_{3,kitr}(t))\beta S_{kitr}(t) \end{aligned} \tag{44}$$

where $\gamma = 0.6, \beta = 2$. The simulation results are shown, when in (23) $\rho = 0.5$ and the initial value of reach gain is $\psi(0) = 0.8$. From assumption 3, the initial values of state variables are zero.

We applied the proposed method on Chaotic Oscillator system in 10 iterations. The simulation results are demonstrated with using the Matlab software in figures 2-6. Dormand- Prince method was used to solve differential equations in all numerical simulations. Also, Crone approximation was used to estimate fractional derivatives.

In many of the last published articles (Wang et al., 2012; Yin et al., 2012; Yin et al., 2013) adaptive sliding mode (ASM) control was used to control fractional-order systems. Fig. 2. shows that the root mean squares (RMS) of the output error, after 10 iterations with ASM and AILSM control, gradually decreases. According to this obvious figure by using the structure of AILSM control in comparison with ASM control, the action of reference output tracking was done with better accuracy and speed. In Fig. 3, output variables for different iterations are shown. From this figure, it’s obvious that through increasing the number of iteration the process of desired output tracking improves.

Fig. 4 displays that the output of the system converges to the desired trajectory at the 10th iteration. Fig. 5 indicates the resulting control input signal. Chattering phenomena has been created due to switching operation that is shown in Fig.5. Fig. 6 shows that the sliding function variations.

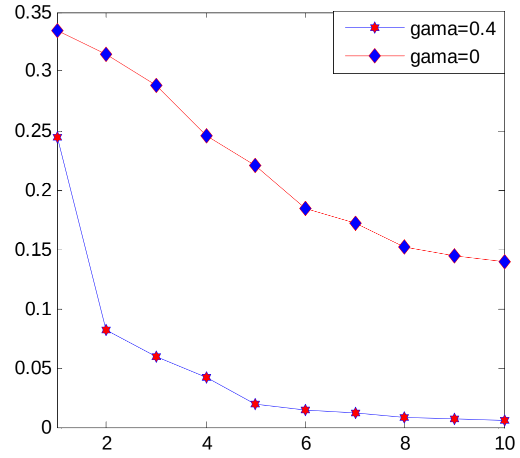


Fig. 2. RMS of output error for 10 iterations with ASM $(\gamma = 0)$ and AILSM $(\gamma = 0.4)$ control.

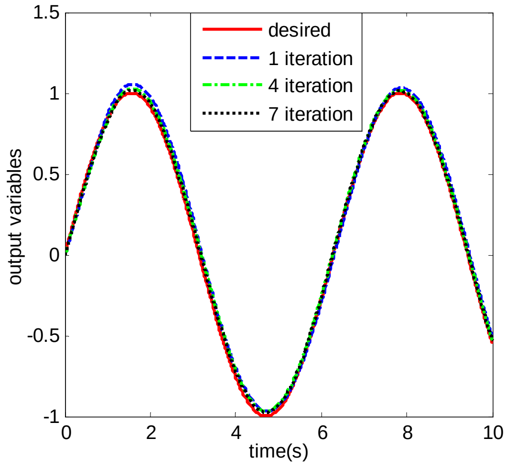


Fig. 3. output variables for different iterations.

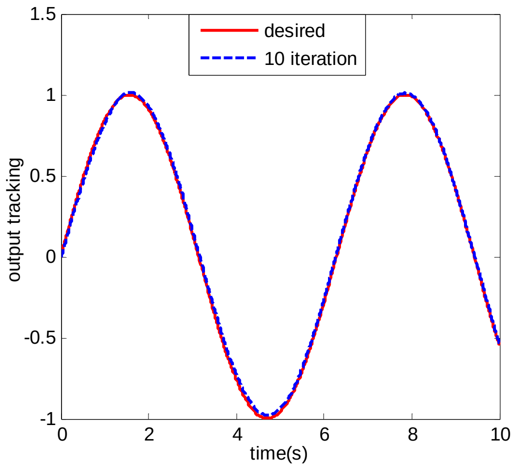


Fig. 4. Real output (cross line) and the desired trajectory (line) at the 10th iteration.

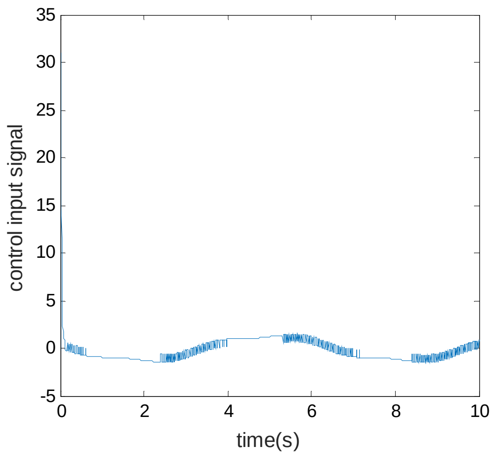


Fig. 5. Control input at the 10th iteration.

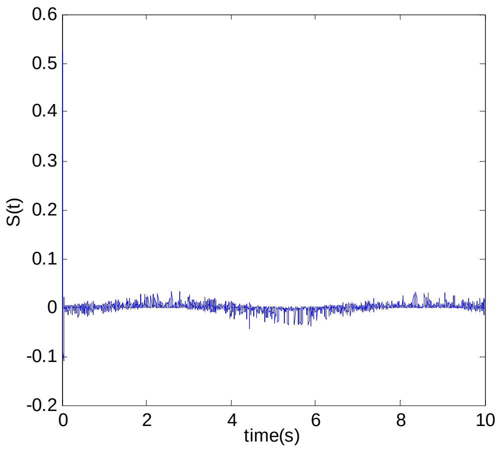


Fig. 6. Variations of sliding function at the 10th iteration.

**Example 2.** Consider the fractional order Duffing system, which is expressed as (Ge and Ou, 2008):

$$\begin{cases} D^{\alpha_1}x_1(t) = x_2(t) \\ D^{\alpha_2}x_2(t) = -\theta_1 x_2(t) + \theta_2 x_1(t) - x_1^3(t) \\ \qquad + 0.3\cos(t) + u(t) + d(t) \end{cases} \tag{45}$$

where from equations (5) and (45), $(\alpha_1, \alpha_2) = (0.98, 0.9)$, $(\theta_1, \theta_2)^T = (0.5, 1)$, $\xi^T(x) = (x_2, x_1)$, $F(x,t) = -x_1^3 + 0.3\cos(t)$ and $d(t) = 0.1\sin(t)$. The output of the above system is considered as follows:

$$y(t) = x_1(t) + x_2(t) \tag{46}$$

It is obvious that all of the state variables exist in the output dynamic and the system has two unknown parameters $(a, b)$.

The goal of control is that to design a robust controller for the system (45) so that the output $y(t)$ tracks a time variable reference signal $r(t) = \sin(t)$ and error asymptotically tends to zero.

To achieve that, by considering (12) and (21), the sliding function and control law is given:

$$\begin{aligned} S_{kitr}(t) &= k_0 D^{-1} e_{1,kitr}(t) + k_1 D^{\alpha_1 - 1} e_{1,kitr}(t) \\ &+ D^{\alpha_2 - 1} e_{2,kitr}(t) \end{aligned} \tag{47}$$

$$\begin{aligned} u_{kitr}(t) &= \sum_{i=0}^{1} k_i e_{i+1,kitr}(t) + D^{\alpha_2} r_2(t) - D^{\sum_{i=1}^{2}\alpha_i} x_{2,kitr}(t) \\ &+ x_{1,kitr}^3(t) - 0.3\cos t + \hat{a}_{kitr}(t) x_{2,kitr}(t) \\ &- \hat{b}_{kitr}(t) x_{1,kitr}(t) + \psi_{kitr}(t)\operatorname{sgn}(S_{kitr}(t)) \end{aligned} \tag{48}$$

where $k_0 = 5, k_1 = 2$. By substituting these coefficients in the equations (15) and (16) and according to the Lemma1, $\Delta(\lambda)$ is calculated as follows:

$$\Delta(\lambda) = \begin{bmatrix} \lambda^{49} & -1 \\ 5 & \lambda^{45} + 2 \end{bmatrix}$$

As a result, the equation $\det(\Delta(\lambda)) = 0$ is obtained:

$$\lambda^{94} + 2\lambda^{49} + 5 = 0$$

Finally from the above equation $\min|\arg(\lambda)| = 0.7767 > \pi/100 = 0.0314$. Therefore from the Lemma1 and (17) stability of error dynamic will be established

According to (22), AIL mechanisms are defined:

$$\begin{aligned} (1-\gamma)\dot{\hat{b}}_{kitr}(t) &= -\gamma\hat{b}_{kitr}(t) + \gamma\hat{b}_{(k-1)itr}(t) - x_{1,kitr}(t)\beta S_{kitr}(t) \\ (1-\gamma)\dot{\hat{a}}_{kitr}(t) &= -\gamma\hat{a}_{kitr}(t) + \gamma\hat{a}_{(k-1)itr}(t) + x_{2,kitr}(t)\beta S_{kitr}(t) \end{aligned} \tag{49}$$

where $\gamma = 0.4, \beta = 1$. In the simulations we assume $\rho = 0.3$ and $\psi(0) = 0.3$. From assumption 3, the initial values of state variables are zero.

Duffing system is utilized in 10 iterations. The simulation results are shown in figures 7-11. The simulation results are demonstrated with using the Matlab software in figures 7-11. Dormand- Prince method was used to solve differential equations in all numerical simulations. Also, Crone approximation was used to estimate fractional derivatives. Fig. 7. shows that the root mean squares (RMS) of the output error, after 10 iterations with ASM and AILSM control, gradually tends to zero. According to this figure, it is obvious that by using the structure of AILSM control in comparison of ASM control the action of reference output tracking was done with better accuracy and speed. In Fig. 8, output variables for different iterations are shown. According to this figure, it's obvious that by increasing the number of iteration, the process of desired output tracking improves. Fig. 9, displays that the output of the system converges to the desired trajectory at the 10th iteration. Fig. 10, indicates the resulting control input signal. Chattering phenomena has been created due to switching operation that is shown in Fig.10. Fig. 11 shows that the variations of the sliding function.

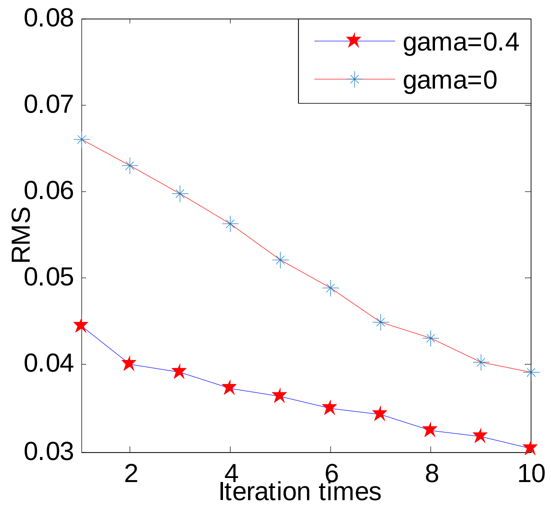


Fig. 7. RMS of output error for 10 iterations with ASM ($\gamma = 0$) and AILSM ($\gamma = 0.4$) control.

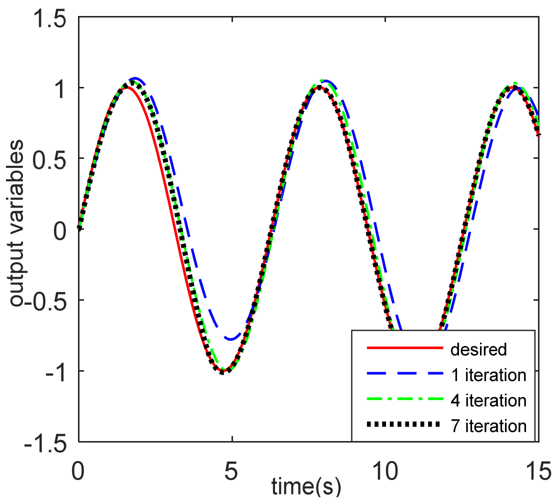


Fig. 8. Output variables for different iterations.

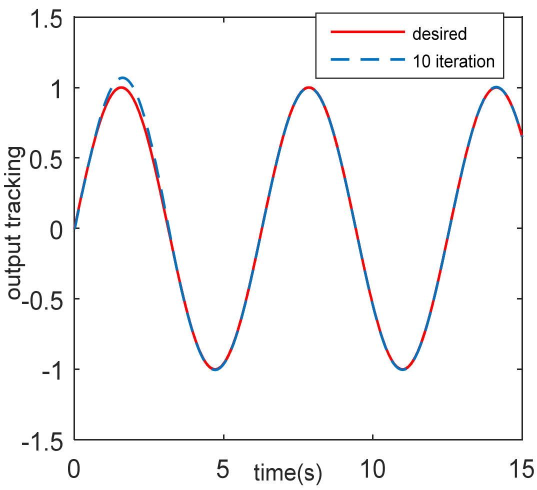


Fig. 9. Real output (cross line) and the desired trajectory (line) at the 10th iteration.

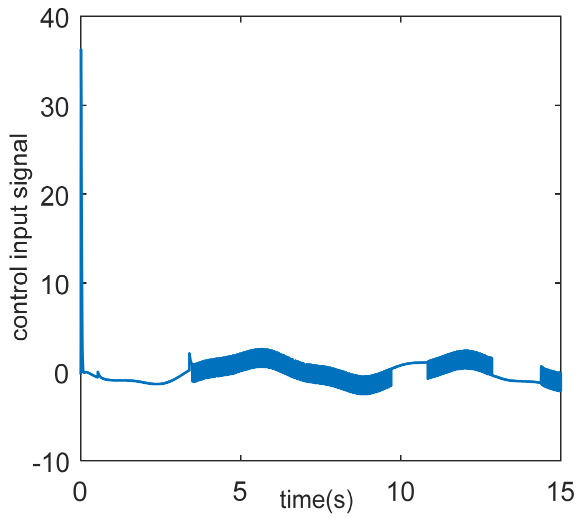


Fig. 10. Control input at the 10th iteration.

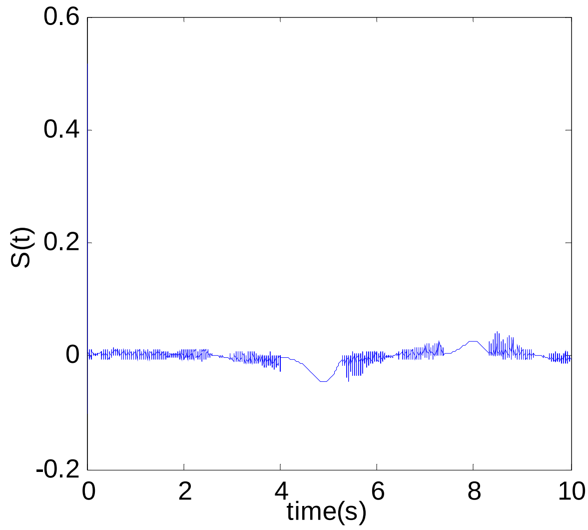


Fig. 11. Variations of sliding surface at the 10th iteration.

## 6. CONCLUSION

In this paper, we proposed a new controller with AILSM structure for tracking output linear/nonlinear incommensurate fractional-order systems. Designed controller is a robust controller, mixed time-domain and iteration-domain adaptation law. The integral sliding function of the sliding mechanism is defined to attenuate the effect of the disturbance. In addition, the proposed controller is designed in a way that without applying it directly to the system output, the process of tracking is well-performed. A rigorous proof, via a Lyapunov-like approach and composite energy reduction in each iteration, is given to show the finiteness of tuning control parameters, rejection of the disturbance effect and the asymptotic error convergence along the iteration axis. The simulation results have clearly exhibited the excellent output tracking performance by the proposed robust AILSM controller.

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## APPENDIXES

### Appendix A: Boundary in the first iteration

**Proof.** Let us consider the following Lyapunov-like positive definite function:

$$V_{kitr}(t)=\frac{1}{2}S_{kitr}^{2}(t)+\frac{1-\gamma}{2\beta}\phi_{kitr}^{T}(t)\phi_{kitr}(t)+\frac{1}{2\rho}(\psi_{kitr}(t)-\hat{\psi})^{2} \tag{50}$$

The derivation of (50) with respect to time $t$ and by substituting (24), can be computed as follows:

$$\begin{aligned}\dot{V}_{kitr}(t)&=S_{kitr}(t)\begin{pmatrix}\phi_{kitr}^{T}(t)\xi(x_{kitr})\\-\psi_{kitr}(t)\operatorname{sgn}(S_{kitr}(t))-d(t)\end{pmatrix}\\&+\frac{1-\gamma}{\beta}\phi_{kitr}^{T}(t)\dot{\phi}_{kitr}(t)+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{51}$$

Given that $\dot{\theta}=0\Rightarrow\dot{\phi}_{kitr}(t)=\dot{\hat{\theta}}_{kitr}(t)$, we will have:

$$\begin{aligned}\dot{V}_{kitr}(t)&=S_{kitr}(t)\begin{pmatrix}\phi_{kitr}^{T}(t)\xi(x_{kitr})\\-\psi_{kitr}(t)\operatorname{sgn}(S_{kitr}(t))-d(t)\end{pmatrix}\\&+\frac{1-\gamma}{\beta}\phi_{kitr}^{T}(t)\dot{\hat{\theta}}_{kitr}(t)+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{52}$$

Substituting the AIL term proposed in (22) in (52), (53) is obtained:

$$\begin{aligned}\dot{V}_{kitr}(t)&=S_{kitr}(t)\begin{pmatrix}\phi_{kitr}^{T}(t)\xi(x_{kitr})\\-\psi_{kitr}(t)\operatorname{sgn}(S_{kitr}(t))-d(t)\end{pmatrix}\\&+\frac{1}{\beta}\phi_{kitr}^{T}(t)\begin{pmatrix}-\gamma\hat{\theta}_{kitr}(t)+\gamma\hat{\theta}_{(k-1)itr}(t)\\-\xi(x_{kitr})\beta S_{kitr}(t)\end{pmatrix}\\&+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{53}$$

The above equation can be simplified as follows:

$$\begin{aligned}\dot{V}_{kitr}(t)&=S_{kitr}(t)\begin{pmatrix}\phi_{kitr}^{T}(t)\xi(x_{kitr})\\-\psi_{kitr}(t)\operatorname{sgn}(S_{kitr}(t))-d(t)\end{pmatrix}\\&+\frac{1}{\beta}\phi_{kitr}^{T}(t)(-\gamma\hat{\theta}_{kitr}(t)+\gamma\hat{\theta}_{(k-1)itr}(t))\\&-\phi_{kitr}^{T}(t)\xi(x_{kitr})S_{kitr}(t)+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{54}$$

In (54) the term of $\phi_{kitr}^{T}(t)\xi(x_{kitr})S_{kitr}(t)$ can be reduced:

$$\begin{aligned}\dot{V}_{kitr}(t)&=-\psi_{kitr}(t)\operatorname{sgn}(S_{kitr}(t))S_{kitr}(t)-d(t)S_{kitr}(t)\\&+\frac{1}{\beta}\phi_{kitr}^{T}(t)(-\gamma\hat{\theta}_{kitr}(t)+\gamma\hat{\theta}_{(k-1)itr}(t)-\gamma\theta+\gamma\theta)\\&+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{55}$$

Considering the fact that $S_{kitr}(t)\operatorname{sgn}(S_{kitr}(t))=|S_{kitr}(t)|$ and defining it, $\phi_{kitr}(t)=\hat{\theta}_{kitr}(t)-\theta$, (54) is simplified to:

$$\begin{aligned}\dot{V}_{kitr}(t)&=-\psi_{kitr}(t)|S_{kitr}(t)|-d(t)S_{kitr}(t)\\&-\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{kitr}(t)+\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{(k-1)itr}(t)\\&+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{56}$$

From assumption 1, the following inequality is obtained:

$$\begin{aligned}\dot{V}_{kitr}(t)&\le-\psi_{kitr}(t)|S_{kitr}(t)|+M\,|S_{kitr}(t)|-\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{kitr}(t)\\&+\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{(k-1)itr}(t)+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{57}$$

Positive and negative of $\hat{\psi}|S_{kitr}(t)|$ are added in (57):

$$\begin{aligned}\dot{V}_{kitr}(t)&\le-\psi_{kitr}(t)|S_{kitr}(t)|+M\,|S_{kitr}(t)|-\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{kitr}(t)\\&+\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{(k-1)itr}(t)+\hat{\psi}|S_{kitr}(t)|-\hat{\psi}|S_{kitr}(t)|\\&+\frac{1}{\rho}(\psi_{kitr}(t)-\hat{\psi})\dot{\psi}_{kitr}(t)\end{aligned} \tag{58}$$

**Remark4:** The terms of $|S_{kitr}(t)|$ and $(\psi_{kitr}(t)-\hat{\psi})$ can be factored:

$$\begin{aligned}\dot{V}_{kitr}(t)&\le(M-\hat{\psi})|S_{kitr}(t)|-\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{kitr}(t)\\&+\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{(k-1)itr}(t)\\&+(\psi_{kitr}(t)-\hat{\psi})\left(\frac{1}{\rho}\dot{\psi}_{kitr}(t)-|S_{kitr}(t)|\right)\end{aligned} \tag{59}$$

By inserting adaptive law (23) in (59), one can derive:

$$\begin{aligned}\dot{V}_{kitr}(t)&\le(M-\hat{\psi})|S_{kitr}(t)|-\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{kitr}(t)\\&+\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{(k-1)itr}(t)\end{aligned} \tag{60}$$

Since $(M-\hat{\psi})|S_{kitr}(t)|<0$, it is obvious that:

$$\dot{V}_{kitr}(t)\le-\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{kitr}(t)+\frac{\gamma}{\beta}\phi_{kitr}^{T}(t)\phi_{(k-1)itr}(t) \tag{61}$$

Inequality in (61) could also be expressed in the form below:

$$\begin{aligned}\dot{V}_{kitr}(t)&\le\frac{\gamma}{4\beta}\phi_{(k-1)itr}^{T}(t)\phi_{(k-1)itr}(t)\\&-\frac{\gamma}{\beta}\left(\phi_{kitr}(t)-\frac{1}{2}\phi_{(k-1)itr}(t)\right)^{T}\left(\phi_{kitr}(t)-\frac{1}{2}\phi_{(k-1)itr}(t)\right)\end{aligned} \tag{62}$$

It is straightforward to verify that:

$$\dot{V}_{kitr}(t)\le\frac{\gamma}{4\beta}\phi_{(k-1)itr}^{T}(t)\phi_{(k-1)itr}(t) \tag{63}$$

Now, consider the first iteration of *kitr*=1. As already mentioned $\hat{\theta}_{0itr}(t)$ in the AIL law (22) is chosen as a constant vector $\theta_{0itr}\ \forall t \in [0,T]$, then we have $\phi_{0itr}(t) = \hat{\theta}_{0itr}(t) - \theta = \theta_{0itr} - \theta \equiv \bar{\theta}_{0itr}$ and $\phi_{1itr}(0) = \hat{\theta}_{1itr}(0) - \theta = \hat{\theta}_{0itr}(T) - \theta = \theta_{0itr} - \theta \equiv \bar{\theta}_{0itr}$. This implies that according to assumption 3 and $\psi_{kitr}(0) = \hat{\psi}$ the initial condition of Lyapunov-like function as shown in the following equation is bounded:

$$\begin{aligned} V_{1itr}(0) &= \frac{1}{2}S_{1itr}^{2}(0) + \frac{1-\gamma}{2\beta}\phi_{1itr}^{T}(0)\phi_{1itr}(0) \\ &\quad + \frac{1}{2\rho}(\psi_{1itr}(0) - \hat{\psi})^{2} \\ &\equiv \frac{1-\gamma}{2\beta}\bar{\theta}_{0itr}^{T}\bar{\theta}_{0itr} \end{aligned} \tag{64}$$

The derivative of Lyapunov-like function at the first iteration will satisfy:

$$\dot{V}_{1itr}(t) \le \frac{\gamma}{4\beta}\phi_{0itr}^{T}(t)\phi_{0itr}(t) \equiv \frac{\gamma}{4\beta}\bar{\theta}_{0itr}^{T}\bar{\theta}_{0itr} \tag{65}$$

Which concludes $V_{1itr}(t) \in L_{\infty e}[0,T]$ and therefore $S_{1itr}(t)$, $\hat{\theta}_{1itr}(t)$ and $\psi_{1itr}(t)$ are bounded.