A New Approach of State Estimation of Linear Discrete Systems

E. Aljuwaiser*, R. Badr*, M. F. Hassan**

**Electronics and Electrical Communication Engineering Department, Cairo University, Giza, Egypt;*
Email : elhamalj@gmail.com.
***Electrical Engineering Deparment, Kuwait University, P.O. Box:5969, Safat 13060, Kuwait;*
Email : m.f.hassan47@gmail.com.

Abstract: In this paper a new technique is developed to estimate the states of either deterministic or stochastic discrete-time linear systems. The proposed approach is based on pole placement technique in which a set of inequality constraints is imposed on the output estimation error. For the stochastic case, the proposed approach can deal with either certain or uncertain systems, Gaussian or non-Gaussian input and output noise signals with known or unknown statistical data. The main feature of the proposed observer is that it leads to an estimation error which approaches the desired zero steady state value very shortly without showing the undesired large overshoots. The stability of the estimation error is rigorously analysed for both the deterministic and the stochastic cases. Illustrative examples are presented to show the effectiveness of the developed technique in solving estimation problems for both deterministic and stochastic discrete-time linear systems. Moreover, comparative studies are performed with other techniques to show the superiority of the developed observer.

Keywords: State estimation, Discrete systems, Constrained estimator, Stability analysis, Pole placement.

1. INTRODUCTION

Most, if not all real systems are represented by incomplete state measurements because some of the states are either very expensive and/or impossible to measure. Examples of these systems are pollution in streams (Mahmoud et al., 1985), jet engines (Maggiore et al., 2003) … etc. Knowledge of the performance of the state variables is necessary in order to know the performance of the system and/or to generate state feedback control strategies.

Luenberger observer is normally used to estimate the states of linear deterministic systems for which the systems dynamics and the measurement models are known. The estimated error resulting from this technique showed to be stable and goes to zero as the time goes to infinity (Phillips and Nagle, 2007). As an alternative approach, and due to the duality principle between estimation and control (Goodwin et al., 2001), linear quadratic regulator (LQR) technique can be applied to design the observer gain (Zhong, 2004). For these types of estimators, the poles of the observer are usually chosen to be near the origin of the z-plane in order to achieve fast response of the estimation error. However, this may lead to large overshoot of the estimator at the start of the estimation process.

In real life, systems are corrupted with both input and output noise vectors. Moreover, they can be uncertain due to the impact of the environment, aging … etc. (Williams, 2010; Åström, 2012; Brittain et al., 2010). In such a situation, observer gains which are designed to allocate the closed loop poles far away from those of the original system lead to large overshoot at the start of the estimation process. In such a case, the system many be saturated or damaged if the estimated states are used to generate closed loop control strategies. On the other hand, for linear systems with certain parameters and uncorrelated input and output white Gaussian noise vectors with known statistics, Kalman filter (KF) is the optimal minimum variance state estimator (Kalman, 1960; Hongpeng et al., 2013; Welch, 2004). For uncertain linear stochastic systems, extended Kalman filter (EKF) is the most used approach to estimate the states as well as the parameters of the system (Marins et al., 2001; Wang and Papageorgiou, 2005; Hovland and Antoine, 2006; Bolognani et al., 2003). However, such an approach is suboptimal and in many practical applications it leads to unobservable and/or undetectable system and hence unstable estimator, especially if the time horizon is long enough. On the other hand, the application of Kalman filter necessitates the knowledge of the covariance matrices of the input and output noise vectors. In most of the situations, the covariance matrices of the input and output noise vectors are either unknown or approximately known. Several techniques are implemented to handle this problem and showed to give good performance (Bos et al., 2005). The adaptive robust extended Kalman filter (AREKF), is widely used in the case of unknown noise covariances, and leads to stable state estimator. In this approach the design of the estimator is based on the stability analysis, and determines whether the error covariance matrix should be reset according to the magnitude of the innovation (Xiong et al., 2009). Another problem may occur if the noise of the system is non-Gaussian (Saab,1995; Nsour et al., 2013). In this case Kalman filter can be applied to estimate the state, where the noise is considered to be white Gaussian noise and the first and the second moments are used to develop the algorithm of the Kalman filter, but the results are suboptimal rather than optimal (Gustafsson and Nordlund, 2005). Also, the particle filter (Hoteit et al., 2008; Chou and Nakajima, 2016; Won et al., 2010) can be used to deal with non-Gaussian problems. This filter is easy to implement and tune. However, its main drawback is that it is quite computer intensive since the computational complexity increases quickly with the increase of state vector dimensionality.

In (Zhang et al., 2010) a new estimation algorithm, based on Bayes's theorem, is developed where the filter is experimented by applying its observation data.

In this paper, we present a new state estimator for linear discrete-time dynamical systems. The design of the proposed filter is based on pole placement in which bounds are imposed on the output estimation error of the system. With this approach it is not needed to allocate the poles of the estimator far from those of the system to achieve fast response of the estimated states. The poles of the new filter are allocated very near from the original system's poles. Therefore, we avoid the existence of excessive overshoot during the transient period of the estimation process, especially if the output measurements are corrupted by noise signals. Such a technique can also be applied to stochastic systems as well. In this case the main advantages of developed filter are:

It does not depend on the knowledge of the covariance matrices of the corrupting noise signals.

It can handle estimation problems with Gaussian or non-Gaussian noise signals.

It leads to stable results with uncertain system parameters.

The only information needed is to have just a guess of the variances of the noise signals especially at the output. The convergence of the developed approach for both the deterministic and the stochastic cases are analyzed. Illustrative examples for both the deterministic and the stochastic cases are presented to show the effectiveness of the proposed filter. For the stochastic case, Monte Carlo simulation is used to compare the results with the well-known Kalman filter.

The rest of the paper is organized as follows. The problem formulation, the design of the developed estimator for discrete-time deterministic linear systems and the stability analysis of the new filter are given in section 2. In section 3 the case of stochastic discrete-time linear system is considered and the stability of the estimator is also analyzed. Illustrative examples for both the deterministic and stochastic systems are presented in section 4. The paper is concluded in section5.

2. THE DEVELOPED CONSTRAINTED ESTIMATOR

Consider the following uncertain dynamical discrete-time linear system:

$$\begin{aligned} x_{k+1} &= \left(A + \Delta A_k\right) x_k + w_k \\ y_{k+1} &= C x_{k+1} + v_{k+1} \end{aligned} \tag{1}$$

where $x_k \in R^n$ is the state vector, $y_k \in R^r$ is the measured output vector; $A \in R^{n\times n}$ is the system matrix with the nominal values of the parameters; $\Delta A_k \in R^{n\times n}$ includes the uncertainties of the parameters which are uncorrelated white Gaussian or non- Gaussian with zero means and standard deviation either known or unknown. The system (1) is assumed to be stable and the pair (A, C) is observable; $C \in R^{r\times n}$ is the output matrix; $w_k \in R^n$ and $v_k \in R^r$ are respectively, two zero mean independent input and output noise vectors with covariance matrices $Q_k = E\left\{w_k w_k^T\right\} \in R^{n\times n}$, $R_k = E\left\{v_k v_k^T\right\} \in R^{m\times m}$; and $k \in \{1, 2, ...\}$ is the discrete time. System (1) satisfies the following model properties:

$E\left\{x_k w_j^T\right\} = 0 \quad \forall\ j \ge k$; $E\left\{x_k v_j^T\right\} = 0 \quad \forall\ k, j$;

$E\left\{y_k v_j^T\right\} = 0 \quad \forall\ k < j$; $E\left\{y_k w_j^T\right\} = 0 \quad \forall\ j > k$;

$E\{\Delta A_k\} = 0$; the uncertain parameters of the system are uncorrelated with the input and output noise vectors; x_k is independent of $\Delta A_j \quad \forall\ j \ge k$; and y_k is independent of $\Delta A_j \quad \forall\ j > k$.

Knowing the estimate $\hat{x}_{k|k} \in R^n$ using the measurements $\{y_1, y_2 .....y_k\}$, then as we receive the measurements y_{k+1}, it is desired to estimate the state vector $\hat{x}_{k+1|k+1}$ such that:

$$\underline{\varepsilon} \le y_{k+1} - \hat{y}_{k+1|k+1} \le \overline{\varepsilon} \tag{2}$$

where $\hat{y}_{k+1|k+1} \in R^r$ is the estimated output; and $\underline{\varepsilon} \in R^r$, $\overline{\varepsilon} \in R^r$ are respectively the lower and upper bounds of the estimation error element by element.

For the deterministic case, since the best estimation is achieved if $y_{k+1} = \hat{y}_{k+1|k+1}$, the vectors $\underline{\varepsilon}$, $\overline{\varepsilon}$ can be given a very small numbers. However, for the stochastic case y_{k+1} is corrupted by the measurement noise. Therefore, the best estimate y_{k+1} is the expected value of $\hat{y}_{k+1|k+1}$. Since this is unknown, a reasonable value of the i^{th} element of $\left|\underline{\varepsilon}_i\right|, \left|\overline{\varepsilon}_i\right|$ is in the range $\left[\sigma_i, 2\sigma_i\right]$ where σ_i is the standard deviation of the corresponding output noise signal. The proposed estimator is realized through the following three steps:

Step 1: Prediction

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} \\ \hat{y}_{k+1|k} &= C\hat{x}_{k+1|k} = CA\hat{x}_{k|k} \end{aligned} \tag{3}$$

where $\hat{x}_{k+1/k} \in R^n$ is the predicted estimate of the state vector; and $\hat{y}_{k+1|k} \in R^r$ is the predicted estimate of the output given the set of measurements $\{y_1, y_2 .....y_k\}$.

Step 2: Filtering

Once the measurement vector y_{k+1} is received, the filtered estimate of the state vector $\hat{x}_{k+1|k+1}$ and the corresponding output vector $\hat{y}_{k+1|k+1}$ are given in by:

$$\begin{aligned} \hat{x}_{k+1|k+1} &= A\hat{x}_{k|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right] \\ \hat{y}_{k+1|k+1} &= C\hat{x}_{k+1|k+1} \end{aligned} \tag{4}$$

where $L \in R^{n \times r}$ is the observer gain.

Using (1) and (4), the estimation error defined by $\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$ is such that:

$$\begin{aligned} \hat{x}_{k+1|k+1} &= A_c\tilde{x}_{k|k} + (I - LC)\Delta A_k x_{k|k} \\ &+ (I - LC)w_k - Lv_{k+1} \end{aligned} \tag{5}$$

where $A_c = (A - LCA)$ (6)

The observer gain L is designed such that the estimation error $\tilde{x}_{k+1|k+1} \to 0$ as $k \to \infty$. However, as the poles of A_c are allocated closer to the origin of the z-plane, the settling time of the estimation error is shorter but the estimator shows worst relative stability (large overshoot) which may be undesirable if the estimated states are used to generate closed loop control policies. To overcome this drawback, the gain matrix L is chosen such that the poles of A_c matrix are near to those of the A matrix of the system. However, to achieve faster response of the estimation error we go to the third step of the proposed estimation technique.

Step 3: The Linear Update Estimator

Since the estimator is not orthogonal to the innovation space as in the ideal case of Kalman filter in which the system is certain and the input and output noise vectors are uncorrelated white Gaussian, then it is possible to update the estimator resulting from step 2 above.

This is the main idea of the new developed filter to get a very fast response of the estimated states while avoiding the undesirable large overshoot during the transient period of the estimator. This idea is achieved by satisfying the imposed set of inequality constraints (2). Such an objective is fulfilled in this step.

Let us define a new estimated state vector $\hat{\hat{x}}^{i}_{k+1|k+1}$ for which:

$$\hat{\hat{x}}^{0}_{k+1|k+1} = \hat{x}_{k+1|k+1} \tag{7}$$

The estimation error is then defined by:

$$\tilde{x}^{0}_{k+1|k+1} = x_{k+1} - \hat{\hat{x}}^{0}_{k+1|k+1} \tag{8}$$

Now, we have one of the following two cases which may arise:

Case I: The estimated outputs satisfy the imposed set of constraints (2). In this case, we let:

$$\hat{x}_{k+1|k+1} = \hat{\hat{x}}^{0}_{k+1|k+1} \tag{9}$$

and we proceed to the next sampling instant of time.

Case II: If a subset of the estimated outputs violates the imposed constraints, then this subset is identified and has to be saturated to the lower or the upper bounds which have been violated. Let us define the subset of the violated outputs by $z_{k+1} \in R^{l}$ where l is the number of the violated constraints and the vector z_{k+1} is given by:

$$z_{k+1} = Dx_{k+1} + v'_{k+1} \tag{10}$$

where the matrix $D \in R^{l \times n}$ contains the rows of the matrix C corresponding to the set of the violated outputs; and v'_{k+1} is a noise vector containing the elements of the v_{k+1} corresponding to the set of the violated outputs. For this subset of violated constraints, it is desired to update the estimator (4) to satisfy the equality constraints given by:

$$z_{k+1} - \hat{\hat{z}}^{i}_{k+1|k+1} = \mu \tag{11}$$

where $\hat{\hat{z}}^{i}_{k+1|k+1} = D\hat{\hat{x}}^{i}_{k+1|k+1}$; and $\mu \in R^{l}$ contains the elements of $\underline{\varepsilon}$ or $\overline{\varepsilon}$ corresponding to the violated outputs. Equation (10) will be assumed as a new received subset of measurements through which the estimator will be corrected. Such a procedure is valid, since as mentioned before, equation (4) will not lead to orthogonal projection on the innovation space.

The update estimator is proposed to take the form:

$$\begin{aligned} \hat{\hat{x}}^{i}_{k+1|k+1} &= \hat{\hat{x}}^{i-1}_{k+1|k+1} + H[z_{k+1} - \hat{\hat{z}}^{i-1}_{k+1|k+1}] \\ \hat{\hat{z}}^{i-1}_{k+1|k+1} &= D\hat{\hat{x}}^{i-1}_{k+1|k+1} \end{aligned} \tag{12}$$

where $H \in R^{n \times l}$ is a gain matrix to be defined later.

This process has to be repeated iteratively. As $z_{k+1} - \hat{\hat{z}}^{i}_{k+1|k+1} = \mu$ element by element, we go to the next sampling instant of time.

After the first iteration, the updated error is given by:

$$\tilde{x}^{1}_{k+1|k+1} = x_{k+1} - \hat{\hat{x}}^{1}_{k+1|k+1} \tag{13}$$

Substituting (11) into (13) with the iteration number i=1, we get:

$$\tilde{x}^{1}_{k+1|k+1} = x_{k+1} - \left\{\hat{\hat{x}}^{0}_{k+1|k+1} + H(z_{k+1} - \hat{\hat{z}}^{0}_{k+1|k+1})\right\} \tag{14}$$

Using (12) and (10) into (14), we have:

$$\begin{aligned} \tilde{x}^{1}_{k+1|k+1} &= (I - HD)\left[x_{k+1} - \hat{\hat{x}}^{0}_{k+1|k+1}\right] - Hv'_{k+1} \\ &= (I - HD)\tilde{x}^{0}_{k+1|k+1} - Hv'_{k+1} \end{aligned} \tag{15}$$

Using the same procedure it can be shown that the updated estimation error at $(i+1)^{th}$ iteration is given by:

$$\begin{aligned} &\tilde{x}^{i+1}_{k+1|k+1} = (I - HD)\tilde{x}^{i}_{k+1|k+1} - Hv'_{k+1} \\ &\text{or}: \tilde{x}^{i+1}_{k+1|k+1} = (I - HD)^{i+1}\tilde{x}^{0}_{k+1|k+1} - \sum_{j=0}^{i}(I - HD)^{i-j}Hv'_{k+1} \end{aligned} \tag{16}$$

Using (5), equation (16) can be written in the following form:

$$\tilde{x}^{i+1}_{k+1|k+1} = (I - HD)^{i+1} A_c \tilde{x}_{k|k} + (I - HD)^{i+1}(I - LC)\Delta A_k x_{k|k} + (I - HD)^{i+1}(I - LC)w_k - (I - HD)^{i+1} L v_{k+1} - \sum_{j=0}^{i} (I - HD)^{i-j} H v'_{j+1} \tag{17}$$

To guarantee the stability of $\tilde{x}^{i+1}_{k+1|k+1}$, the gain matrix H is designed such that:

$$\left|\lambda\{(I - HD)A_c\}\right| < 1 \tag{18}$$

The existence of the matrix H is guaranteed if the pair (A_c, D) is observable. It is worth mentioning that the computational requirement of this observer is very moderate since: 1) the update phase is applied only at the points violating the constraints which are usually very few, 2) the gain matrix H is calculated once in this phase, 3) the procedure to be applied in the update step requires a very few number of iterations to converge to the desired results. Validation of these facts will be verified later on in our simulation.

3. STABILITY OF THE LINEAR UPDATE ESTIMATOR

In this section we demonstrate firstly that the linear update estimator is equivalent to the least square quadratic estimator (LSQ) with special weighting matrices to allocate the poles at the desired values. Then, we show that such an estimator is a Gauss-Newton's method which belongs to the class of contraction mapping algorithms.

Since the estimator is not orthogonal to the innovation space, the update procedure is applied only at the sampling instants at which some of the constraints are violated, and all the information are known, then the linear update estimator is equivalent to a static estimation problem in which we correct the current estimated states iteratively using the same set of measurements (Aljuwaiser, 2017).

3.1 Least Square Quadratic Estimator

Given the assumed new received vector of measurements z_{k+1}, we define the following LSQ estimator problem to be solved in order to update the estimator $\hat{x}^1_{k+1|k+1}$:

$$\min_{\hat{x}^1_{k+1|k+1}} J = \frac{1}{2}\left\|\hat{x}^1_{k+1|k+1} - \hat{x}^0_{k+1|k+1}\right\|^2_{W^{-1}} + \frac{1}{2}\left\|z_{k+1} - \hat{z}^1_{k+1|k+1}\right\|^2_{M^{-1}} \tag{19}$$

where $W \in R^{n\times n}$ and $M \in R^{l\times l}$ are positive definite weighting matrices.

The necessary condition of optimality is such that:

$$\frac{\partial J}{\partial \hat{x}^1_{k+1|k+1}} = W^{-1}[\hat{x}^1_{k+1|k+1} - \hat{x}^0_{k+1|k+1}] - D^T M^{-1}[z_{k+1} - \hat{z}^1_{k+1|k+1}] = 0 \tag{20}$$

Let $\hat{x}^1_{k+1|k+1} = \hat{x}^0_{k+1|k+1} + \Delta\hat{x}^0_{k+1|k+1}$ (21)

Then $\hat{z}^1_{k+1|k+1} = D\hat{x}^1_{k+1|k+1}$ (22)

or ; $\hat{z}^1_{k+1|k+1} = D\hat{x}^0_{k+1|k+1} + D\Delta\hat{x}^0_{k+1|k+1}$ (23)

Now using (21), (22) and (23) into (20) while using (10) and after simple algebraic manipulation, we get:

$$\begin{aligned}\hat{x}^1_{k+1|k+1} &= \hat{x}^0_{k+1|k+1} \\ &+ (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D x_{k+1} \\ &- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D \hat{x}^0_{k+1|k+1} \\ &+ (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}\end{aligned} \tag{24}$$

The error between the actual and the estimated state, as defined by, $\tilde{x}^1_{k+1|k+1} = x_{k+1} - \hat{x}^1_{k+1|k+1}$, is given by:

$$\begin{aligned}\tilde{x}^1_{k+1|k+1} &= [I - (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D]\tilde{x}^0_{k+1|k+1} \\ &- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}\end{aligned} \tag{25}$$

let $H = [W^{-1} + D^T M^{-1} D]^{-1} D^T M^{-1}$

Therefore, $\tilde{x}^1_{k+1|k+1}$ is such that:

$$\begin{aligned}\tilde{x}^1_{k+1|k+1} &= [I - (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D]\tilde{x}^0_{k+1|k+1} \\ &- (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}\end{aligned}$$

$$\tilde{x}^1_{k+1|k+1} = [I - HD]\tilde{x}^0_{k+1|k+1} - Hv'_{k+1} \tag{26}$$

Equation (26) is equivalent to equation (16) if the weighting matrices W and M as chosen such that the poles are allocated as desired by (18).

3.2 Gauss- Newton's Method

In this section we show that the linear update estimator is a Gauss- Newton's method. To fulfill this objective, consider the following nonlinear least square problem:

$$\min.\ f(\alpha) = \frac{1}{2} r(\alpha) r^T(\alpha) \tag{27}$$

where:

$$r(\alpha) = \Gamma \begin{bmatrix} \hat{z}^1_{k+1|k+1} - z_{k+1} \\ \hat{x}^1_{k+1|k+1} - \hat{x}^0_{k+1|k+1} \end{bmatrix} \qquad \Gamma^T \Gamma = \begin{bmatrix} M^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \tag{28}$$

let: $\Delta\hat{z}^0_{k+1|k+1} = Dx_{k+1} - D\hat{x}^0_{k+1|k+1} + v'_{k+1}$ (29)

Then by using (21), (22) and (29), $r(\alpha)$ can be written as:

$$r(\alpha) = \Gamma \begin{bmatrix} -\Delta\hat{z}^0_{k+1|k+1} + D\Delta\hat{x}^0_{k+1|k+1} \\ \Delta\hat{x}^0_{k+1|k+1} \end{bmatrix} \tag{30}$$

The necessary condition of optimality is such that:

$$\frac{\partial f}{\partial \Delta \hat{\hat{x}}^0_{k+1|k+1}} = D^T M^{-1}(-\Delta \hat{\hat{z}}^0_{k+1|k+1} + D\Delta \hat{\hat{x}}^0_{k+1|k+1})$$
$$+ W^{-1}\Delta \hat{\hat{x}}^0_{k+1|k+1} = 0 \tag{31}$$
$$= -D^T M^{-1}\Delta \hat{\hat{z}}^0_{k+1|k+1}$$
$$+ (W^{-1} + D^T M^{-1} D)\Delta \hat{\hat{x}}^0_{k+1|k+1} = 0$$

from which one gets:

$$\Delta \hat{\hat{x}}^0_{k+1|k+1} = (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} \Delta \hat{\hat{z}}^0_{k+1|k+1} \tag{32}$$

Using (29) into (32) we have:

$$\begin{aligned}\Delta \hat{\hat{x}}^0_{k+1|k+1} &= (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D x_{k+1} \\ &\quad - (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D \hat{\hat{x}}^0_{k+1\backslash k+1}\end{aligned} \tag{33}$$

Therefore $\hat{\hat{x}}^1_{k+1|k+1}$ is given by:

$$\begin{aligned}\hat{\hat{x}}^1_{k+1|k+1} &= \hat{\hat{x}}^0_{k+1|k+1} + \Delta \hat{\hat{x}}^0_{k+1|k+1} \\ &= \hat{\hat{x}}^0_{k+1\backslash k+1} + (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D x_{k+1} \\ &\quad - (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} D \hat{\hat{x}}^0_{k+1|k+1} \\ &\quad + (W^{-1} + D^T M^{-1} D)^{-1} D^T M^{-1} v'_{k+1}\end{aligned} \tag{34}$$

Equation (34) is equivalent to equation (24) which proves that the linear update estimator is a Gauss- Newton's method.

To complete our analysis, the convergence of the mean value of the output error vector $\tilde{z}^i_{k+1|k+1} = z_{k+1} - \hat{\hat{z}}^i_{k+1|k+1}$ is demonstrated. Using (5), (11) with i=1, the error vector $\tilde{z}^1_{k+1|k+1}$ is given by:

$$\tilde{z}^1_{k+1|k+1} = D\tilde{x}^0_{k+1|k+1} + v'_{k+1} - DHD\tilde{x}^0_{k+1|k+1} - DHv'_{k+1}$$
$$\tilde{z}^1_{k+1|k+1} = \left(I - DH\right) D\tilde{z}^0_{k+1|k+1} \tag{35}$$

In general, at the i^{th} iteration we get:

$$\tilde{z}^i_{k+1|k+1} = (I - DH)^i \tilde{z}^0_{k+1|k+1}$$

For which the norm is given by:

$$\left\|\tilde{z}^i_{k+1|k+1}\right\| = \left\|(I - DH)^i \tilde{z}^0_{k+1|k+1}\right\| \le \left\|(I - DH)^i\right\| \left\|\tilde{z}^0_{k+1|k+1}\right\| \tag{36}$$

If after i^{th} iteration, the norm of matrix $(I - DH)^i$ is less than one, then as $i \to N$ $\left\|\bar{\tilde{z}}^i_{k+1|k+1}\right\| \to \|\mu\|$ and the mean value of the output of the system converges to the mean value of the desired output which satisfies the imposed set of constraints.

4. STABILITY OF THE LINEAR STATE ESTIMATOR

In this section, we analyze the stability of the proposed estimator (Aljuwaiser 2017). Using the results of steps (1), (2) and (3) given in section 3 while using (3) and (4) then, for the first iteration, we get:

$$\hat{\hat{x}}^1_{k+1|k+1} = \hat{\hat{x}}^0_{k+1\backslash k+1} + H\left[z_{k+1} - \hat{\hat{z}}^0_{k+1|k+1}\right]$$

or $\hat{\hat{x}}^1_{k+1|k+1} = \hat{x}_{k+1|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right] + H\left[z_{k+1} - \hat{\hat{z}}^0_{k+1|k+1}\right]$

For the 2[nd] iteration, we have:

$$\begin{aligned}\hat{\hat{x}}^2_{k+1|k+1} &= \hat{\hat{x}}^1_{k+1\backslash k+1} + H\left[z_{k+1} - \hat{\hat{z}}^1_{k+1|k+1}\right] \\ &= \hat{x}_{k+1|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right] + H\left[z_{k+1} - \hat{\hat{z}}^0_{k+1|k+1}\right] \\ &\quad + H\left[z_{k+1} - \hat{\hat{z}}^1_{k+1|k+1}\right]\end{aligned}$$

Since the linear update estimator is a Gauss-Newton's method which requires N number of iterations to converge to the required bounds as specified by the imposed set of constraints, then the filtered estimate at the $(k+1)^{th}$ sampling instant is given by:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + L\left[y_{k+1} - \hat{y}_{k+1|k}\right] + \sum_{j=1}^{N} H\left[z_{k+1} - \hat{\hat{z}}^{j-1}_{k+1|k+1}\right] \tag{37}$$

At the end of the convergence, the last term in $\hat{x}_{k+1|k+1}$ has a finite value, and can be written in the form:

$$\alpha_{k+1} = \sum_{j=1}^{N} H\left[z_{k+1} - \hat{\hat{z}}^{j-1}_{k+1|k+1}\right] \tag{38}$$

or $\alpha_{k+1} = K_{k+1}\left[y_{k+1} - \hat{y}_{k+1|k}\right]$ (39)

where K_{k+1} is a gain matrix of adjustable parameters such that:

$$K_{k+1}\left[y_{k+1} - \hat{y}_{k+1|k+1}\right] = \sum_{j=1}^{N} H\left[z_{k+1} - \hat{\hat{z}}^{j-1}_{k+1|k+1}\right] \tag{40}$$

Using (1), (37) and (40) it is easy to shown that:

$$\begin{aligned}\tilde{x}_{k+1|k+1} &= A_c{}^{k+1}\tilde{x}_0 + \sum_{j=0}^{k} A_c{}^{k-j}\Big\{-K_{j+1} CA\ \tilde{x}_{j|j} \\ &\quad + \left(I - LC - K_{j+1}C\right)\Delta A_j x_j \\ &\quad + \left(I - LC - K_{j+1}C\right) w_j - \left(L + K_{j+1}\right) v_{j+1}\Big\}\end{aligned}$$

let $L'_k = L + K_j$

then $\tilde{x}_{k+1|k+1}$ can be written as:

$$\begin{aligned}\tilde{x}_{k+1|k+1} &= A_c{}^{k+1}\tilde{x}_0 + \sum_{j=0}^{k} A_c{}^{k-j}\Big\{-K_{j+1} CA\ \tilde{x}_{j|j} \\ &\quad + \left(I - L'_{j+1}C\right)\Delta A_j x_j + \left(I - L'_{j+1}C\right) w_j - L'_{j+1} v_{j+1}\Big\}\end{aligned} \tag{41}$$

With assumptions that the equilibrium point is stable, observable or detectable, and the set or a subset of the imposed constraints (2) is violated only during the transient period of the estimator, then $\lim_{k\to\infty}\left(A_c - K_k CA\right) = A_c$.

Therefore, we conduct the following theorem:

Theorem 1:

1- If the mean value of all the solutions of the homogenous equation:

$$\tilde{x}_{k+1|k+1} = A_c \tilde{x}_{k|k} \quad \text{for } k \geq 0 \tag{42}$$

remain bounded as $k \to \infty$ then:

a-The same is true for the mean value of all the solutions of the non-homogenous system:

$$\begin{aligned} \tilde{x}_{k+1|k+1} &= \left(A_c - K_{k+1}CA\right)\tilde{x}_{k|k} \\ &+ \left(I - L'_{k+1}C\right)\Delta A_k x_k \qquad \text{for } k \geq 0 \\ &+ \left(I - L'_{k+1}C\right)w_k - \left(L'_{k+1}\right)v_{k+1} \end{aligned} \tag{43}$$

provided that $\sum_{k=0}^{\infty} \lambda_k < \infty$ for $k \geq 0$ (44)

where $\left\| E\left\{K_{k+1}CA\, \tilde{x}_{k|k}\right\}\right\| = \lambda_k \left\|\bar{\tilde{x}}_{k|k}\right\|$ (45)

and $E\left\{\tilde{x}_{k|k}\right\} = \bar{\tilde{x}}_{k|k}$

b- If $\lim_{k\to\infty}\left(A_c - K_{k+1}CA\right) = A_c$ (46)

Then $\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \to 0$ as $k \to \infty$ and hence the mean value of both the homogenous and the non-homogenous systems are exponentially stable.

2-If the expected value of the norm of all the solutions of the homogenous equation (42) remains bounded as $k \to \infty$, then the same is also true for the expected value of the norm of all the solutions of the non- homogenous system (43):

provided that $\sum_{k=0}^{\infty}\left\{\beta_k + \rho_k\right\} < \infty$ for $k \geq 0$ (47)

where:

$$\begin{aligned} &\left\|\left(I - L'_{k+1}C\right)\right\| \sum_{r=1}^{n} \sigma_{wr} + \left\|\left(L'_{k+1}\right)\right\| \; E\left\{\left\|\tilde{x}_{k|k}\right\|\right\} \sum_{r=1}^{m} \sigma_{vr} \\ &E\left\{\left\|\left(I - L'_{k+1}C\right)\Delta A_k x_k\right\|\right\} = \rho_k E\left\{\left\|\tilde{x}_{k|k}\right\|\right\} \end{aligned} \tag{48}$$

$$E\left\{\left\|K_{k+1}CA\right\|\right\} = \beta_k E\left\{\left\|\tilde{x}_{k|k}\right\|\right\} \tag{49}$$

and $\sum_{r=1}^{n}\sigma_{wr}^2$, $\sum_{r=1}^{m}\sigma_{vr}^2$ are the summations of the squares of the standard deviations of the input and output noise vectors respectively.

Proof:

1a- Equation (42) can be written as:

$$\tilde{x}_{k+1|k+1} = A_c^{\,k+1}\tilde{x}_0 \quad \text{for } k \geq 0 \tag{50}$$

Since the mean value of all the solutions of (50) are bounded, then there exists c_1 such that for $k \geq 0$

$$\left\|A_c^{\,k+1}\right\| \leq c_1 \tag{51}$$

The expected value of (41) is given by:

$$\bar{\tilde{x}}_{k+1|k+1} = A_c^{\,k+1}\bar{\tilde{x}}_0 + \sum_{j=0}^{k} A_c^{\,k-j} E\left\{-K_{j+1}CA\tilde{x}_{j|j}\right\} \tag{52}$$

Taking the norm of equation (52) while using (51) we have:

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k}\left\|A_c^{\,k-j}\right\|\left\|E\left\{-K_{j+1}CA\tilde{x}_{j|j}\right\}\right\|$$

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq c_1\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} c_1\lambda_j\left\|\bar{\tilde{x}}_{j|j}\right\| \tag{53}$$

Let $g_j = c_1\lambda_j$ (54)

Using (54) into (53), we have:

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq c_1\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} g_j\left\|\bar{\tilde{x}}_{j|j}\right\| \tag{55}$$

From (44) there exists c_2 such that:

$$\sum_{k=0}^{\infty}\lambda_k < c_2$$

Using Gronwall lemma (Walsh, Ye & Bushnell, 2002), equation (55) can be written as follows:

$$\begin{aligned} \left\|\bar{\tilde{x}}_{k+1|k+1}\right\| &\leq c_1\left\|\bar{\tilde{x}}_0\right\|\prod_{j=0}^{k}(1+g_j) \leq c_1\left\|\bar{\tilde{x}}_0\right\|\exp\sum_{j=0}^{k}g_j \\ &\leq c_1\left\|\bar{\tilde{x}}_0\right\|\exp(c_1c_2) \end{aligned} \tag{56}$$

This shows that the mean value of the non- homogenous system is bounded.

1b- Since the equilibrium is observable and the observer gain matrix L is designed such that the estimator is asymptotically stable, then there exists 0≤δ<1 such that:

$$\left\|A_c^{k+1}\right\| < \delta^k \quad \text{for } k \geq K$$

Therefore, for the homogeneous and the non- homogeneous systems we have:

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k}\left\|A_c^{\,k-j}\right\|\lambda_j\left\|\bar{\tilde{x}}_{j|j}\right\|$$

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} c_1\lambda_j\left\|\bar{\tilde{x}}_{j|j}\right\|$$

$$\left\|\bar{\tilde{x}}_{k+1|k+1}\right\| \leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\| + \sum_{j=0}^{k} g_j\left\|\bar{\tilde{x}}_0\right\| \tag{57}$$

Using Gronwall lemma, equation (57) takes the form:

$$\begin{aligned} \left\|\bar{\tilde{x}}_{k+1|k+1}\right\| &\leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\|\prod_{j=0}^{k}(1+g_j) \\ \left\|\bar{\tilde{x}}_{k+1|k+1}\right\| &\leq \left\|A_c^{\,k+1}\right\|\left\|\bar{\tilde{x}}_0\right\|\exp\sum_{j=0}^{k}g_j \leq \delta^{k+1}\left\|\bar{\tilde{x}}_0\right\|\exp(c_1c_2) \end{aligned} \tag{58}$$

Then, as $k \to \infty$, $\delta^{k+1} \to 0$. This proves that the mean values of the homogenous and the non-homogenous systems are exponentially stable.

2-From (41), the norm of the error $\tilde{\boldsymbol{x}}_{\boldsymbol{k}+1|\boldsymbol{k}+1}$ is given by:

$$\left\|\tilde{x}_{k+1|k+1}\right\| \le \left\|A_c{}^{k+1}\right\|\left\|\tilde{x}_0\right\| + \sum_{j=0}^{k}\left\|A_c{}^{k-j}\right\|\left\{\left\|-K_{j+1}CA\tilde{x}_{j|j}\right\|\right.$$

$$\left. + \left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\| + \left\|\left(I - L'_{j+1}C\right)w_j\right\| + \left\|\left(L'_{j+1}\right)v_{j+1}\right\|\right\}$$

$$\left\|\tilde{x}_{k+1|k+1}\right\| \le c_1\left\|\tilde{x}_0\right\| + \left\{\sum_{j=0}^{k} c_1\left\|-K_{j+1}CA\tilde{x}_{j|j}\right\| + \left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\|\right.$$

$$\left. + \left\|\left(I - L'_{j+1}C\right)w_j\right\| + \left\|\left(L'_{j+1}\right)v_{j+1}\right\|\right\} \tag{59}$$

The expected value of equation (59) is such that:

$$E\left\{\left\|\tilde{x}_{k+1|k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} + \sum_{j=0}^{k} c_1\left\{E\left\{\left\|-K_{j+1}CA\tilde{x}_{j|j}\right\|\right\}\right.$$

$$+ E\left\{\left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\|\right\}$$

$$\left. + \left\|\left(I - L'_{j+1}C\right)\right\| E\left\{\left\|w_j\right\|\right\} + \left\|\left(L'_{j+1}\right)\right\| E\left\{\left\|v_{j+1}\right\|\right\}\right\}$$

$$E\left\{\left\|\tilde{x}_{k+1|k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} + \sum_{j=0}^{k} c_1\left\{E\left\{\left\|-K_{j+1}CA\tilde{x}_{j|j}\right\|\right\}\right.$$

$$+ E\left\{\left\|\left(I - L'_{j+1}C\right)\Delta A_j x_j\right\|\right\}$$

$$\left. + \left\|\left(I - L'_{j+1}C\right)\right\| \sum_{r=1}^{n} \sigma_{wr} + \left\|\left(L'_{j+1}\right)\right\| \sum_{r=1}^{m} \sigma_{vr}\right\}$$

Let $g'_j = c_1[\beta_j + \rho_j]$ (60)

$$E\left\{\left\|\tilde{x}_{k+1|k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} + \sum_{j=0}^{k} g'_j E\left\{\left\|\tilde{x}_{j|j}\right\|\right\} \tag{61}$$

Again, using Gronwall lemma, we have:

$$E\left\{\left\|\tilde{x}_{k+1|k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} \prod_{j=0}^{k}(1+g'_j) \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} \exp\sum_{j=0}^{k} g'_j \tag{62}$$

From (47) there exists c_3 such that $\sum_{k=0}^{\infty}\{\beta_k + \rho_k\} < c_3$.

Hence (62) is give by:

$$E\left\{\left\|\tilde{x}_{k+1|k+1}\right\|\right\} \le c_1 E\left\{\left\|\tilde{x}_0\right\|\right\} \exp(c_1 c_3) \tag{63}$$

This proves that the mean value of the norm of non-homogeneous system is bounded.

However, one has to notice that the mean value of the norm of the solution of the non-homogenous system depends on the standard deviations of the corrupting noise signals as well as the uncertain parameters. As the levels of the noise signals and the uncertainties increase, especially the output noise signals, the algorithm will have the tendency to diverge. This fact is well known and it is now justified mathematically.

5. SIMULATION AND RESULTS

In this section, the developed algorithm is applied to both deterministic and stochastic discrete-time linear estimation problems. For the stochastic case, since uncertain discrete-time linear stochastic systems are considered, and for the purpose of comparison with other estimators, Monte Carlo simulation (Eckhardt, 1987) is used as a vehicle to fulfill this objective. Two indicators are applied in our analysis. They are defined as follows:

The Root Mean Square Index (RMSI)**:**

This is a constant indicator to be calculated for each state. It is defined as:

$$RMSI_i = \sqrt{\frac{\sum_{j=1}^{NOMI}\sum_{k_0}^{k_f}\left[x_{i_j}(k) - \hat{x}_{i_j}(k\,|\,k)\right]^2}{NOMI * \left(k_f - k_0 + 1\right)}} \tag{64}$$

where NOMI is the number of Monte Carlo iterations, k_0 is the initial time and k_f is the final time.

The Root Mean Square Estimation Error (RMS(k))***:***

This is a time varying indicator to be estimated for each state and is given by:

$$RMS_i(k) = \sqrt{\frac{\sum_{j=1}^{NOMI}\left[x_{i_j}(k) - \hat{x}_{i_j}(k\,|\,k)\right]^2}{NOMI}} \tag{65}$$

Example 1:

Consider the following crane model (Solihin et al., 2010; Aljuwaiser, 2017):

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_{33} & 0 \\ 0 & a_{42} & a_{43} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ b_{31} \\ b_{41} \end{bmatrix} u(t) + w(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} x(t) + v(t) \tag{66}$$

where $a_{33} = -1/\tau$; $a_{42} = -g/l$; $a_{43} = 1/l\tau$; $b_{31} = k_a\tau$; $b_{41} = -k_a/\tau l$; $l = 0.4m$ is Payload cable length; $\tau = 0.09s$ is identified time constant; $g = 9.81m/s^2$ is gravitational acceleration and $k_a = 0.26m/s$ is identified speed constant. The state $x_1 = h$ is the horizontal trolley position; $x_2 = \theta$ is the angular swing; $x_3 = \dot{h}$ is the change in the horizontal trolley position; and $x_4 = \dot{\theta}$ is the change in the angular swing. The controller $u(t)$ is considered to be zero and the system is discretized with sampling period $\Delta T = 0.01\,s$. The initial condition of the state $x(0)$ and the estimated state $\hat{x}_{0|0}$ are given by $x(0) = [0.1 \;\; 1 \;\; 1 \;\; 1]^T$, $\hat{x}_{0|0} = [0 \;\; 0 \;\; 0 \;\; 0]^T$; the covariance of the input noise vector $w(k)$ is $Q = E\left\{w(k)w^T(k)\right\} = \text{diag}[q \;\; q \;\; q \;\; q]$ where q is given different values as shown in Table 1; and finally the covariance of the output noise vector $v(k)$ is $R = E\{v(k)v^T(k)\} = I$.

Case 1: The deterministic case

The estimated states using the developed filter are denoted by x_{ik}^{df}. The results are compared with the estimated states using Luenberger observer to be denoted by x_{ik}^{lo}. The gain matrix of Luenberger observer is adjusted to achieve almost the same settling time. The results are shown in Figs. 1-6. From these results it is clear that the Luenberger observer leads to a very large overshoot of the estimated states except for the first (measured) state, while the developed filter leads to satisfactory results for all of the state variables.

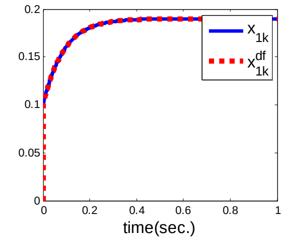

Fig. 1. x_{1k} and $\hat{x}_{1k}$ for the developed filter.

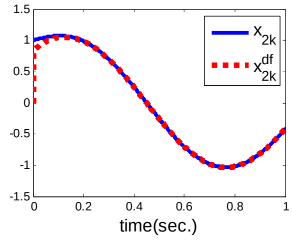

Fig. 2. x_{2k} and $\hat{x}_{2k}$ for the developed filter.

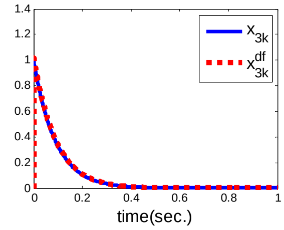

Fig. 3. x_{3k} and $\hat{x}_{3k}$ for the developed filter.

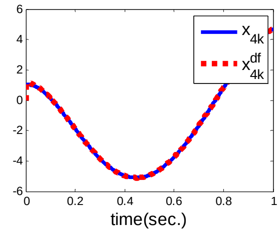

Fig. 4. x_{4k} and $\hat{x}_{4k}$ for the developed filter.

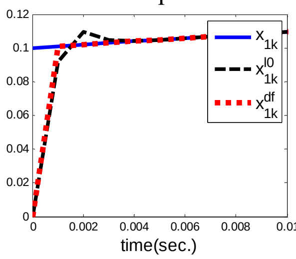

Fig. 5. x_{1k} and $\hat{x}_{1k}$ for luenberger observer and developed filter.

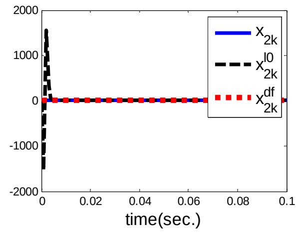

Fig. 6. x_{2k} and $\hat{x}_{2k}$ for luenberger observer and developed filter.

Case 2: Stochastic case with certain parameters and white Gaussian noise vectors with known covariances

In this case study, the results of the developed filter, to be denoted by DF, are compared with the results using Kalman filter, to be denoted by KF. It is worth mentioning that, for all the stochastic cases 100 Monte Carlo simulation runs are performed and the two indicators, defined by (64) and (65),are used to compare the performance of each filter. Simulation results are presented in Figs. 7, 8 for the case in which the covariance matrix of the input noise vector is given by $Q = \mathrm{diag}[0.1\ 0.1\ 0.1\ 0.1]$. As it is expected, for such an ideal case, the results of Kalman filter are the best, and hence leads to the least RMS(k). The RMSI is also calculated for each state variable using different values of the covariances of the input noise signals. The achieved results are presented in Table 1. Again, KF leads to the best results in such case studies. Moreover, Fig. 13 shows the average estimation error for the developed filter.

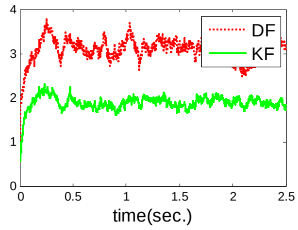

Fig. 7. RMS_3 for x_{3k} (Case2).

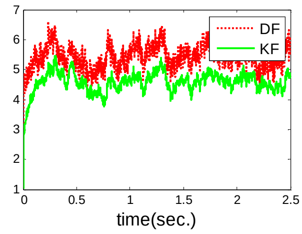

Fig. 8. RMS_4 for x_{4k} (Case2).

Case 3: Stochastic with certain parameters, non-Gaussian noise vectors with known covariances

In this case study, the input and the outputs corrupting noise signals are assumed to be white Non-Gaussian with uniform distribution and known covariance matrices. Simulation results are shown in Figs. 9,10 for $Q = \mathrm{diag}[0.1\ 0.1\ 0.1\ 0.1]$ as well as in Table 1 for the different values of the input covariance matrix Q. From these results it is clear that the performance of the developed filter is better than KF.

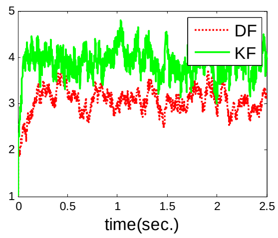

Fig. 9. RMS_3 for x_{3k} (Case3).

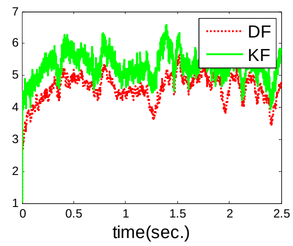

Fig. 10. RMS_4 for x_{4k} (Case3).

Case 4: Stochastic with uncertain parameters, white Gaussian noise vectors with unknown covariances

In this case the input and the output corrupting noise signals are assumed to be white Gaussian with unknown covariance matrices and the system parameters are considered uncertain. First the covariance matrices of the input and the output noise signals are estimated using the method developed in (Xiong et al., 2009) and applied when needed. Firstly, the developed filter is compared with the Kalman filter while using the nominal values of the system parameters. The results are denoted by KF. On the other hand, to estimate the states and the system parameters, Extended Kalman filter is used and the results are denoted by EKF. Also the results are compared with the iterated constrained state estimator (Hassan, 2012) to be denoted by ICE. Simulation results are shown in Figs. 11, 12 for $Q = \mathrm{diag}[0.1\ 0.1\ 0.1\ 0.1]$. From these figures as well as the results demonstrated in Table 2, it is clear that the least values of the RMS(k) and the RMSI indicators are achieved while using the developed filter.

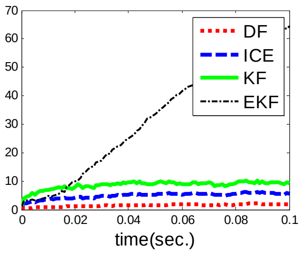

Fig. 11. RMS_3 for x_{3k} (Case4).

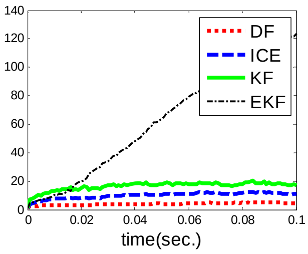

Fig. 12. RMS_4 for x_{4k} (Case4).

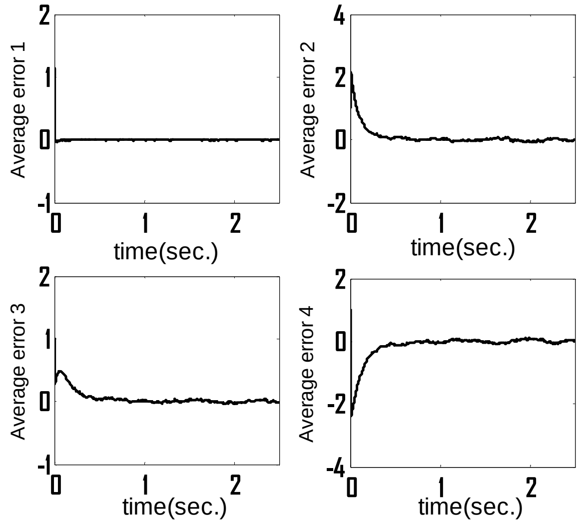

Fig. 13. The average error between the states and the estimated states for the developed filter.

Example 2:

Consider the following servo motor model (Ramli et al., 2007; Aljuwaiser, 2017):

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ 0 & -\dfrac{1}{J_m}\left(D_m+\dfrac{k_t k_b}{R_a}\right)\end{bmatrix}x(t)+\begin{bmatrix}0\\ \dfrac{k_t}{R_a J_m}\end{bmatrix}u(t)+w(t) \qquad (67)$$
$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}x(t)+v(t)$$

where J_m is motor inertia $=30\times10^{-6}\,\text{kgm}^2$; k_b is back-emf constant= $60\times10^{-3}\,\text{Vrad}^{-1}$; k_t is motor torque constant $=17\times10^{-3}\,\text{NmA}^{-1}$; R_a is armature resistance = 3.2 Ω; D_m =equivalent viscous density by the motor = small (cannot be quoted). The state $x_1=\theta$ is the angular displacement of the motor shaft and $x_2=\dot{\theta}$ is the angular velocity of the motor shaft. The controller $u(t)$ is considered to be zero and the system is discretized with sampling period $\Delta T=0.01\,s$. Again, the covariance of the input noise vector $w(k)$ is $Q=E\left\{w(k)w^T(k)\right\}=\text{diag}[q\;\; q\,]$ where the value of q=0.1 and the covariance of the output noise vector $v(k)$ is $R=E\left\{v(k)v^T(k)\right\}=I.$

Table 1. Comparison between DF, KF for case 2&3.

			Case2		Case3	
Q	R	State	DF	KF	DF	KF
0.01	1	x_1	0.32	0.27	0.36	0.37
		x_2	1.46	1.25	1.48	2.62
		x_3	2.77	0.65	2.68	3.10
		x_4	3.98	2.74	3.82	4.74
0.1	1	x_1	0.71	0.49	0.94	1.18
		x_2	3.49	3.36	3.5	4.81
		x_3	3.06	1.81	3.18	4.16
		x_4	5.34	4.49	4.62	5.37
1	1	x_1	1.56	0.72	1.56	1.84
		x_2	5.21	4.97	5.21	7.12
		x_3	4.98	2.10	4.98	6.57
		x_4	6.10	5.91	6.10	7.88

Table 2. Comparison between DF, KF and ICE for case4.

			Case4		
Q	R	State	DF	KF	ICE
0.01	1	x_1	0.41	0.59	0.45
		x_2	2.30	6.27	3.02
		x_3	3.11	4.65	3.89
		x_4	4.28	7.81	5.95
0.1	1	x_1	0.81	1.03	0.86
		x_2	2.31	6.86	5.19
		x_3	2.16	7.72	5.47
		x_4	7.24	15.62	10.65
1	1	x_1	2.01	2.97	2.51
		x_2	7.44	13.62	10.48
		x_3	5.92	11.47	6.56
		x_4	9.09	18.91	12.19

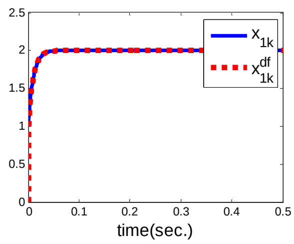

Fig. 14. x_{1k} and $\hat{x}_{1k}$ forthe developed filter.

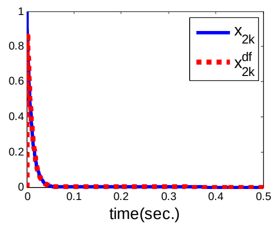

Fig. 15. x_{2k} and $\hat{x}_{2k}$ for the developed filter.

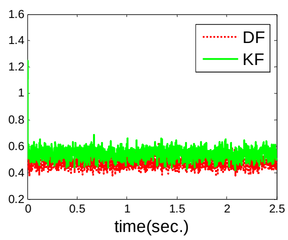

Fig. 16. RMS_1 for x_{1k} (Case3).

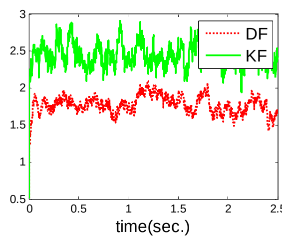

Fig. 17. RMS_2 for x_{2k} (case3).

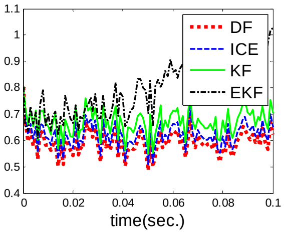

Fig. 18. RMS_1
for x_{1k} (case4).

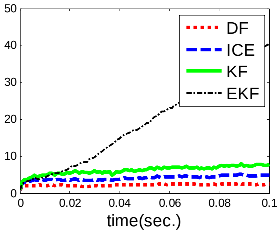

Fig. 19. RMS_2
for x_{2k} (case4).

The present estimation problem is solved for the case studies 1,3,4 as in the first example. For case 1, Figs. 14, 15 show the results of the actual and the estimated states for the deterministic case. A 100 Monte Carlo simulations are executed for the other two cases with $Q = \mathrm{diag}[0.1\ 0.1]$. Figs. 16,17 show the results of case 3, while the results of case 4 are demonstrated in Figs 18,19. Again, from these results it is clear that the developed filter is also the best for case 3 and 4.

Discussion:

Based on our simulation results, one can conclude the following:

From Tables 1, 2 it is clear that the least *RMSI* values are achieved with the application of the developed estimator in case 3 and 4.

The estimated states using the developed approach are stable either for case 4 in which the EKF is unstable (Figs. 11,12).

In the case study 1, Luenberger observer is designed such that the settling time of the estimation errors are more or less the same as the developed filter. In this case the results of Luenberger observer showed large overshoot at the start of the estimation process except for the measured states.

The average estimation error of the developed filter is almost stable at least for the worked examples.

In noisy cases, in order to insure that the estimated outputs are bounded and not tracking the actual measurements, the upper and lower bounds are chosen within the ranges as stated in section 2.

6. CONCLUSIONS

In this paper a new state estimator is developed for either deterministic or uncertain stochastic discrete-time linear systems. The proposed observer is based on pole placement technique with an added update phase to handle a set of imposed inequality constraints on the output estimation error. For the stochastic case, the proposed estimator can deal with either certain or uncertain systems, Gaussian or non-Gaussian noise signals with known or unknown statistical data. The state estimation errors of the proposed technique reach the desired zero steady state values very shortly without showing any undesired overshoots. The stability of the developed approach is rigorously investigated. Illustrative examples are presented to show the effectiveness of the developed technique. For the deterministic case, simulation results show that the developed estimator leads to superior results than those achieved while using Luenberger observer. On the other hand, for the stochastic case, it leads to much better results when compared with the results of the well-known Kalman filter except for the ideal case in which the system is certain and the input and output noise signals are white Gaussian with known covariances.

REFERENCES

Aljuwaiser, E. (2017). New on-line observer based controller for uncertain constrained stochastic nonlinear systems, PhD. Thesis, *Cairo University*.

Åström, K. J. (2012). Introduction to stochastic control theory. *Courier Corporation*.

Bolognani, S., Tubiana, L. and Zigliotto, M. (2003). Extended kalman filter tuning in sensorless pmsm drives. *IEEE Transactions on Industry Applications*, 39, 1741–1747.

Bos, R., Bombois, X. and Van den Hof, P. M. (2005). Designing a kalman filter when no noise covariance information is available. *Proceedings of the 16th IFAC World Congress*, 16, 212–212.

Brittain, K., Corner, L., Robinson, L. and Bond, J. (2010). Ageing in place and technologies of place: the lived experience of people with dementia in changing social,physical and technological environments. *Sociology of health & illness*, 32, 272–287.

Chou, Y. C. & Nakajima, M. (2016). Particle filter planar target tracking with a monocular camera for mobile robots. *Intelligent Automation & Soft Computing*, 1-9.

Eckhardt, R. (1987). Stan ulam, john von neumann, and the monte carlo method. *Los Alamos Science*, 15, 131–136.

Goodwin, G. C., Graebe, S. F., & Salgado, M. E. (2001). Control system design (Vol. 240). *New Jersey: Prentice Hall.*

Hassan, M. F. (2012). Iterated constrained state estimator for nonlinear discrete-time systems with uncertain parameters. *Int. J. Innov. Comput. Inf. Control*, 8, 6141–6160.

Hongpeng, Y., Chao, P., Yi, C., & Qu, F. (2013). A Robust Object Tracking Algorithm Based on Surf and Kalman Filter. *Intelligent Automation & Soft Computing*, *19*, 567-579.

Hoteit, I., Pham, D.-T., Triantafyllou, G. and Korres, G. (2008). A new approximate solution of the optimal nonlinear filter for data assimilation in meteorology and oceanography. *Monthly Weather Review*, 136, 317–334.

Hovland, G., & Antoine, M. (2006). Scheduling of gas turbine compressor washing. *Intelligent Automation & Soft Computing, 12*, 63-73.

Kalman, R. E. (1960). A new approach to linear filtering and prediction problems. *Journal of Fluids Engineering*, 82, 35–45.

Maggiore, M., Ord´o˜nez, R., Passino, K. M. and Adibhatla, S. (2003). Estimator design in jet engine applications. *Engineering Applications of Artificial Intelligence*, 16, 579–593.

Mahmoud, M. S., Hassan, M. F. and Darwish, M. G. (1985). Large-scale control systems: theories and techniques. *M. Dekker*.

Marins, J. L., Yun, X., Bachmann, E. R., McGhee, R. B. and Zyda, M. J. (2001). An extended kalman filter for quaternion-based orientation estimation using marg sensors. *Proceedings of 2001 IEEE/RSJ International Conference on Intelligent Robots and Systems*. 4, 2003–2011.

Nsour, A., Abdallah, A.-S. and Zohdy, M. (2013). An investigation into using kalman filtering for phase estimation in bluetooth receivers for gaussian and non-gaussian noise. *IEEE International Conference on Electro/Information Technology* (EIT), 1–5.

Phillips, C. L. and Nagle, H. T. (2007). Digital control system analysis and design. *Prentice Hall Press*.

Ramli, M., Rahmat, M. and Najib, M. (2007). Design and modeling of integral control state-feedback controller for implementation on servomotor control. *6th WSEAS International Conference on Circuits, Systems, Electronics, Control & Signal Processing*, Cairo, Egypt, Citeseer.

Saab, S. S. (1995). Discrete-time kalman filter under incorrect noise covariances. *Proceedings of the American Control Conference*, 2, 1152–1156.

Schön, T., Gustafsson, F. and Nordlund, P.-J. (2005). Marginalized particle filters for mixed linear/nonlinear state-space models. *IEEE Transactions on Signal Processing*, 53, 2279–2289.

Solihin, M. I., Akmeliawati, R., Muhida, R. and Legowo, A. (2010). Guaranteed robust state feedback controller via constrained optimization using differential evolution,). *6th International Colloquium on Signal Processing and Its Applications* , 1–6.

Walsh, G. C., Ye, H., Bushnell, L. G., Stability analysis of networked control systems. *IEEE Transactions on Control Systems Technology*, 10, 438-446

Wang, W. and Gao, Z. (2003). A comparison study of advanced state observer design techniques. *Proceedings of the American Control Conference*, 6, 4754–4759.

Wang, Y. and Papageorgiou, M. (2005). Real-time freeway traffic state estimation based on extended kalman filter: a general approach, *Transportation Research Part B: Methodological*, 39, 141–167.

Welch, G. (2004). Gray bishop, an introduction to the kalman filter. unc-chapel hill, *Technical report*, TR 95-014.

Williams, P. (2010). Stochastic physics and climate modelling. *T. Palmer* (Ed.), Cambridge University Press.

Won, S.-h. P., Melek, W. W. and Golnaraghi, F. (2010). A kalman/particle filter-based position and orientation estimation method using a position sensor/inertial measurement unit hybrid system. *IEEE Transactions on Industrial Electronics*, 57, 1787–1798.

Xiong, K., Zhang, H. and Liu, L. (2009). Adaptive robust extended kalman filter. *INTECH Open Access Publisher*.

Zhang, H., Mehr, A. S. and Shi, Y. (2010). Improved robust energy-to-peak filtering for uncertain linear systems. *Signal Processing* 90, 2667–2675.

Zhong, W. (2004). Duality system in applied mechanics and optimal control. *Springer Science & Business Media*.