Data-driven $\mathcal{H}_\infty$ Controller/Detector Design for a Quadruple Tank Process ⋆

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Abstract: This paper solves data-driven simultaneous fault detection and control (SFDC) problem for a multi-variable laboratory process named quadruple tank process which has an adjustable zero that can be located in both left and right half-plane. The knowledge of mathematical model of the process is not needed and only input/output time domain data (I/O) are used for solving a data-based $\mathcal{H}_\infty$ optimization problem. Solving this problem yields a data-driven $\mathcal{H}_\infty$ controller/detector unit of modest complexity that is able to achieve some control and fault detection objectives. The tradeoff between these objectives is established by tuning a scalar parameter and some weighting matrices. The effectiveness of the proposed methodology is demonstrated for both minimum phase and non-minimum phase plants in the presence of both incipient and abrupt faults.

Keywords: simultaneous fault detection and control, data-driven control, subspace predictor, $\mathcal{H}_\infty$ control.

1. INTRODUCTION

Occurring faults in control systems can yield performance degradation or even instability. Therefore, in order to satisfy the needs for safety, reliability, and performance of industrial processes, the controller is not sufficient and fault detection unit is necessary. It is highly desirable to investigate the fault detection and control problems simultaneously. Motivation behind that is drawn from the fact that in simultaneous fault detection and control(SFDC) problem, the controller and detector units coalesce into one unit which results in less overall complexity in comparison with designing two separate units and it is a reasonable approach since there is a fundamental trade-offs governing the design of controller and detector units, thus the design of each unit should take the other into consideration. The solution to SFDC problem leads to a controller/detector unit that produces two signals, one of them for fault detection and the other for satisfying some pre-defined control objectives.

Historically, model-based techniques have been used for solving the SFDC problem, for example, see Nett et al. (1988), Tyler & Morari (1994), Khosrowjerdi et al. (2004), Davoodi et al. (2013) and Weijie et al. (2014). In this way, see Ding (2009) for an extensive bibliography and review of the literature for motivations and the model-based approaches to the SFDC problem. In the recent years, data-driven methods have also provided an alternative solution for control and fault detection problems using process available data, see Hou & Wang (2013) and Ding (2014) for an extensive bibliography and review of literature. By this method, the modeling step can be omitted, therefore it can be useful when the mathematical model is difficult to establish or unavailable or when it is inaccurate and involves uncertainties. In data-driven methods, the system identification becomes a part of the controller/detector design procedure and the computations for identification a mathematical model are omitted. These reasons motivate us to introduce a novel data-driven approach to SFDC problem.

Here is some of the data-driven approaches for fault detection and control methods in the review of literature. Palanthandalam et al. (2009) have developed a subspace identification algorithm for input reconstruction from output measurements and known inputs. The proposed method can be applied for fault estimation. Wang et al. (2015) have designed a parity space-based fault detection and isolation system. In Ding et al. (2011) input/output data are used to identify a parity space based primary residual generator. The primary residual generator is implemented as a closed-loop diagnostic observer with just few additional steps. Yin et al. (2014) have proposed a data-driven fault detection scheme with robust residual generators directly constructed from available process data. Based on the method proposed by Ding et al. (2009), the parity space is first identified directly from the measured data. Then, the optimal parity vectors under a given performance index is selected as well as an optimization criterion. Dong et al. (2012) have introduced a data-driven system-inversion-based fault estimation filter for both linear time invariant(LTI) and linear time-varying (LTV) systems. Ding (2014) has reviewed the different data-driven subspace-based fault detection methods. The data-driven control has been first introduced by Favoreel et al. (1999) where a subspace method applied for linear

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quadratic Gaussian (LQG) controller design. Data-driven control has been also applied to design subspace-based model predictive control, for example, see Azizi-Mardi (2010), Huang & Kadali (2008), Dong et al. (2008)). There are some attempts to design the controller/detector unit using input/output data, see for example, Ding et al. (2012), Yin et al. (2014) and Zhang et al. (2014)), but these design methods cannot be considered as solutions to the data-driven SFDC problem.

In this paper, we have proposed a developed approach presented in Woodley et al. (2001). A novel data-driven $\mathcal{H}_\infty$ approach is proposed for SFDC problem in the presence of disturbances and faults, which leads to a controller/detector unit of less overall complexity. A multivariable laboratory process named quadruple tank process which has an adjustable zero has selected to test the proposed method. For designing the controller/detector unit only the input/output data is necessary and the knowledge of the mathematical model is not needed. The $\mathcal{H}_\infty$ controller/detector unit produces two signals for satisfying some control and fault detection objectives. The control objectives are tracking references and rejecting disturbances and the fault detection objective is estimating the filtered version of unknown faults. The tradeoff between these objectives is established by tuning a scalar parameter and some weighting matrices. An easily implementable design algorithm summarizes the methodology presented in the paper. The effectiveness of the proposed methodology is demonstrated for both minimum phase and nonminimum phase plants in the presence of both incipient and abrupt faults. The results show the effectiveness of our proposed method in all cases.

This paper is organized as follows. In Section 2, a brief review about the quadruple tank process which is selected to test the proposed method is presented. In Section 3, the data-driven SFDC problem is formulated as a time domain data-driven $\mathcal{H}_\infty$ optimization problem. In Section 4, a brief review of subspace predictor design which is essential for solving the data-driven SFDC problem defined in Section 3 is presented. In Section 5, an easily implementable design algorithm summarizes the proposed methodology for data-driven $\mathcal{H}_\infty$ controller/detector. In Section 6, this algorithm is applied to a linearized model of quadruple tank process and simulation results are presented. Concluding remarks are given in Section 7.

The notations used in this paper are fairly standard. For a given matrix A, A^T denotes its transpose. I denotes unity matrix with appropriate dimension. If $A = A^T$ then A is a symmetric matrix. If A and B are symmetric matrices, $A \geq B$ (respectively, $A > B$) denotes $A - B$ positive semi definite (respectively, positive definite). The space of real rational, stable and proper transfer matrices is denoted by $\mathcal{RH}_\infty$.

2. THE QUADRUPLE TANK PROCESS

The quadruple tank process consists of four interconnected water tanks and two pumps. The voltages to the pumps are inputs, see Johnsson, K. H. (2000) for more details. Figure 1 shows the schematic of four-tank process.

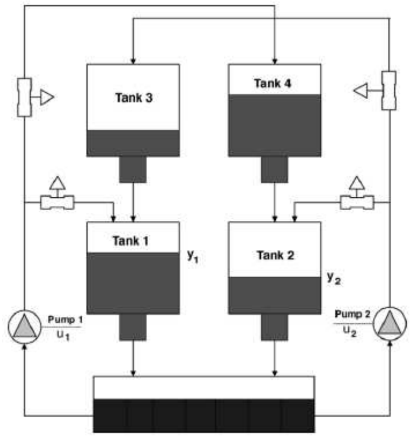

Fig. 1. Schematic of the four-tank process.

The nonlinear equations of the quadruple tank process are described as follows

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\alpha_1 k_1}{A_1}v_1 \tag{1}$$

$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\alpha_2 k_2}{A_2}v_2 \tag{2}$$

$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\alpha_2)k_2}{A_3}v_2 \tag{3}$$

$$\frac{dh_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\alpha_1)k_1}{A_4}v_1, \tag{4}$$

where, A_i is the cross-section of tank i, a_i is cross-section of the outlet hole of tank i, k_i is the gain of pump i, v_i is the voltage of pump i, h_i is water level of tank i. The parameters $\alpha_1, \alpha_2 \in (0, 1)$ are determined from how the valves are set prior to an experiment.

The linearized model of this process is described as follows

$$\dot{x} = \begin{pmatrix} \frac{-1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & \frac{-1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & \frac{1}{T_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{T_4} \end{pmatrix} x + \tag{5}$$

$$\begin{pmatrix} \frac{\alpha_1 k_1}{A_1} & 0 \\ 0 & \frac{\alpha_2 k_2}{A_3} \\ 0 & \frac{(1-\alpha_2)k_2}{A_3} \\ \frac{(1-\alpha_2)k_2}{A_4} & 0 \end{pmatrix} (u + f) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{-k_{d_1}}{A_3} & 0 \\ 0 & \frac{-k_{d_2}}{A_4} \end{pmatrix} d$$

$$y = \begin{pmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{pmatrix} x$$

where $x_i := h_i - {h_i}^0$, $u_i := v_i - {v_i}^0$, and ${h_i}^0, {v_i}^0$ are the values of h_i, v_i in the operating points, respectively. k_c is the sensor gain, and $T_i = \frac{A_i}{a_i}\sqrt{\frac{2{h_i}^0}{g}}$ is the time constant, f is the actuator fault associated with pumps 1 and 2 and d is the disturbance representing flow out of Tanks 3 and 4. The process has an adjustable zero that can be located in both left and right half-plane by changing the values of α_1, α_2. It follows that the system is non-minimum phase for $0 < \alpha_1 + \alpha_2 < 1$ and minimum phase for $1 < \alpha_1 + \alpha_2 < 2$. Usually this process is studied in

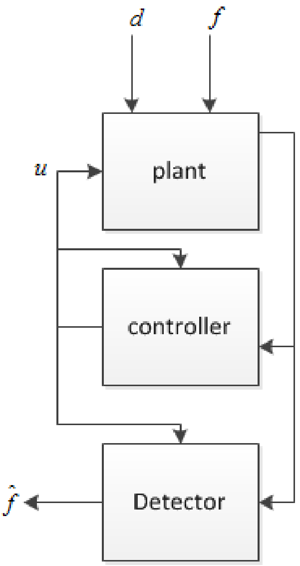

Fig. 2. Data-driven controller and detector design

two operating points: one of them for minimum phase and the other for non-minimum phase characteristics. Table 1 shows the values for these two operating points.

Table 1. parameters associated with minimum phase and non-minimum phase plants

parameters[unit]	minimum phase plant	non-minimum phase
$T_1[s]$	62	63
$T_2[s]$	90	91
$T_3[s]$	23	39
$T_4[s]$	30	56
$k_1[\frac{cm^3}{Vs}]$	3.33	3.14
$k_2[\frac{cm^3}{Vs}]$	3.35	3.29
$\alpha_1[-]$	0.7	0.43
$\alpha_2[-]$	0.6	0.34
$A_1[cm^2]$	28	28
$A_2[cm^2]$	32	32
$A_3[cm^2]$	28	28
$A_4[cm^2]$	32	32
$k_c[\frac{V}{cm}]$	0.5	0.5
$k_{d_1}[\frac{cm^3}{Vs}]$	1	1
$k_{d_2}[\frac{cm^3}{Vs}]$	1	1

In recent years the quadruple tank process has been used as a benchmark multi-variable system for studying performance of fault detection and control design, for example, Khosrowjerdi et al. (2005) have proposed a model-based approach to actuator fault detection problem for the quadruple tank process in a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ setting, Li et al. (2003) have developed the subspace algorithms in the continuous-time domain to identify the residual models from sampled data without separate identification of the system matrices for fault detection and isolation in the quadruple tank process. Kirubakarana et al. (2014) have applied model predictive control for reference tracking and disturbance rejection of a quadruple tank process. Biswas et al. (2009) have designed an sliding mode control for set point tracking of a quadruple tank process.

In this paper a new algorithm has developed that uses only the input/output data (I/O) for solving SFDC problem for the quadruple tank process. The traditional data-driven

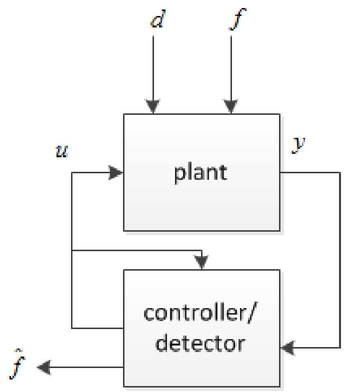

Fig. 3. Data-driven SFDC design

approach to design controller and detector is based on a separation principle as shown in Figure 2. In contrast of this approach, in this paper the controller and detector blocks, as shown in Figure 3, are unified that leads to less overall complexity that is able to achieve some control and fault detection objectives. The tradeoff between these objectives is established by tuning a scalar parameter and some weighting matrices. The control objective is to regulate the level of tanks 1, 2 to a pre-defined setting point. The fault detection objective is to detect the actuator faults and isolate them. The mathematical model of the process is assumed to be unknown. The mathematical model is only used to collect I/O data. Both incipient and abrupt faults are considered for minimum-phase and non-minimum phase cases. To the best of authors' knowledge, data-driven SFDC problem for quadruple tank process has not been solved in the literature. In this paper, an easily and constructive implementable algorithm is proposed for solving this problem that leads to a data-driven $\mathcal{H}_\infty$ controller/detector.

3. PROBLEM FORMULATION

Consider the plant G in Figure 3 whose dynamic can be described by linear system time-invariant (LTI) system

$$G : \begin{cases} x_{k+1} = A\,x_k + B\,u_k + B_d\,d_k + B_f\,f_k + e_k, \\ y_k = C\,x_k + D\,u_k + D_d\,d_k + D_f\,f_k + \nu_k \end{cases} \tag{6}$$

where $x_k \in R^n$, $u_k \in R^m$, $y_k \in R^l$and $d_k \in R^{k_d}$ are the state, the known input, the measured output and the disturbance, respectively. The unknown input $f \in R^{k_f}$ models a possible actuator, sensor, and/or component fault. With f_k set to zero, system (6) describes the fault-free system. Also, $e_k \in R^n$ and $\nu_k \in R^l$ denote process and measurement noise sequences that are normally distributed, white and statistically independent of the control input u_k and the initial condition x_0. A, B, B_d, B_f, C, D, D_d and D_f are assumed to be unknown constant matrices of appropriate dimensions. The controller/detector in Figure 3 must internally stabilize G in (6) and achieve specified control and fault detection objectives. In order to solve the SFDC problem some weighting matrices are defined, i.e. $W_j, j = f, d, r, u, \hat{f}$. The weights are defined below. To achieve control objectives such as minimizing tracking error with reasonable control effort, a typical control performance measure z_c can be defined by

$$z_c = \begin{pmatrix} z_r \\ z_u \end{pmatrix} = \begin{pmatrix} W_r(r-y) \\ W_u u \end{pmatrix} \tag{7}$$

where W_r and W_u are appropriate proper stable weighting matrices. In this paper the control objective is regulation, however the z_r is defined in general form, then in simulation the reference signal r set to zero. To achieve specified detection objectives, the controller/detector unit in Figure 3 must generate a signal $\hat{f} \in R^{k_f}$ for reconstructing a filtered version of the fault f which can give us some indication about the fault itself. This objective can be described in terms of a detection performance measure z_d is given by

$$z_d = W_{f1}(W_{f2}f - \hat{f}) = W \begin{pmatrix} f \\ \hat{f} \end{pmatrix} \tag{8}$$

where $W = \begin{pmatrix} W_{f1}W_{f2} & -W_{f1} \end{pmatrix}$ is an appropriate weighting transfer matrix in $\mathcal{R}H_\infty$ used for fault detection objectives (see Khosrowjerdi et al. (2004)). Without loss of generality, we assume that W_{f1} is an identity matrix. The fault detection objective can then be defined as follows

$$z_d = W_f f - \hat{f}, \tag{9}$$

where W_f is a appropriate proper stable weighting matrix. The weight W_f describes the relative importance and/or the expected or known frequency content of the fault f_k and is normally set as diagonal used for fault estimation purposes, for more details see Patton & Chen (1999). We can now define a combined control/detection performance measure z as follows

$$z = \begin{pmatrix} z_c \\ z_d \end{pmatrix}. \tag{10}$$

Define

$$w = \begin{pmatrix} r \\ d \\ f \end{pmatrix}, \quad v = \begin{pmatrix} u \\ \hat{f} \end{pmatrix} \tag{11}$$

Here $w \in R^{l+k_d+k_f}$ and $v \in R^{m+k_f}$ are the exogenous input and the combined control input, where $r \in R^l$ and $\hat{f} \in R^{k_f}$ are the reference input and the fault estimate, respectively. In order to express the signals in Figure 3 in terms of w and v, the following matrices are defined

$$r = K_1 w, \quad K_1 = \begin{pmatrix} I_l & 0_{l\times(k_d+k_f)} \end{pmatrix} \tag{12}$$
$$d = K_2 w, \quad K_2 = \begin{pmatrix} 0_{k_d\times l} & I_{k_d} & 0_{k_d\times k_f} \end{pmatrix} \tag{13}$$
$$f = K_3 w, \quad K_3 = \begin{pmatrix} 0_{k_f\times(k_d+l)} & I_{k_f} \end{pmatrix} \tag{14}$$
$$u = K_4 v, \quad K_4 = \begin{pmatrix} I_m & 0_{m\times(m+k_f)} \end{pmatrix} \tag{15}$$
$$\hat{f} = K_5 v, \quad K_5 = \begin{pmatrix} 0_{k_f\times m} & I_{k_f} \end{pmatrix} \tag{16}$$

Suppose the I/O data from a disturbance-free healthy system, i.e. the system (6) when $f = 0$ and $d = 0$ are now available. Given the experimental I/O data from a healthy system, the data-driven $\mathcal{H}_\infty$ controller/detector design problem is to choose v such that the finite horizon $\mathcal{H}_\infty$ gain from w to z has at most magnitude γ. The control signal u and the fault estimation signal $\hat{f}$ can be then derived directly from v.

The SFDC problem can be formulated as the min-max data-driven $\mathcal{H}_\infty$ optimization problem: *given the I/O data of the system (6) and $\gamma > 0$, determine the controller/detector unit in Figure 3 which generates v such that*

$$\min_v \sup_w J(\gamma) \le 0, \tag{17}$$

where J is a cost function which defined by

$$J(\gamma) = \sum_{t=0}^{i-1} ({z_t}^T z_t - \gamma^2 {w_t}^T w_t) \tag{18}$$

The length of the horizon (i) can be chosen arbitrarily. In the next section, a solution is proposed to the minimax optimization problem (17) that yields a data-driven $\mathcal{H}_\infty$ controller/detector.

4. SUBSPACE PREDICTOR ALGORITHM

In order to describe our solution to the problem (17), we briefly review the subspace predictor design methodology; for more details see Overschee & Moor (1999) and Woodley (2001). The formation of subspace predictor serves two purposes: i) it simultaneously reduces the effect of noise in the measured data, and ii) it establishes a method of extrapolating future plant input-output behavior from past I/O data. Suppose that the input/output data of length n from a disturbance-free healthy system is available with m inputs ($u_k \in R^m$), l outputs ($y_k \in R^l$).

$$\left(\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix} \right) \tag{19}$$

1. Choose the prediction horizon i which should be larger than the expected order of plant and set $j = N - 2i + 1$.
2. Define the following Hankel matrices

$$U_p = \begin{pmatrix} u_0 & u_1 & \cdots & u_{j-1} \\ u_1 & u_2 & \cdots & u_j \\ \vdots & \vdots & \ldots & \vdots \\ u_{i-1} & u_i & \cdots & u_{i+j-2} \end{pmatrix} \in R^{im\times j}$$

$$Y_p = \begin{pmatrix} y_0 & y_1 & \cdots & y_{j-1} \\ y_1 & y_2 & \cdots & y_j \\ \vdots & \vdots & \ldots & \vdots \\ y_{i-1} & y_i & \cdots & y_{i+j-2} \end{pmatrix} \in R^{il\times j}$$

$$U_f = \begin{pmatrix} u_i & u_{i+1} & \cdots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & \cdots & u_{i+j} \\ \vdots & \vdots & \ldots & \vdots \\ u_{2i-1} & u_{2i} & \cdots & u_{2i+j-2} \end{pmatrix} \in R^{im\times j}$$

$$Y_f = \begin{pmatrix} y_i & y_{i+1} & \cdots & y_{i+j-1} \\ y_{i+1} & y_{i+2} & \cdots & y_{i+j} \\ \vdots & \vdots & \ldots & \vdots \\ y_{2i-1} & y_{2i} & \cdots & y_{2i+j-2} \end{pmatrix} \in R^{il\times j},$$

where the subscript p and f represent past and future time, respectively.

3. Define $W_p = \begin{pmatrix} U_p \\ Y_p \end{pmatrix}$, W_p represent the past data.
4. Solve the following Frobenius norm optimization problem.

$$\min \| Y_f - \begin{pmatrix} L_w & L_u \end{pmatrix} \begin{pmatrix} W_p \\ U_f \end{pmatrix} \|_F^2 \tag{20}$$

The solution can be found by implementing QR decomposition as following

$$\begin{pmatrix} W_p \\ U_f \\ Y_f \end{pmatrix} = R^T Q^T = \begin{pmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \tag{21}$$

$$L = \begin{pmatrix} L_w & L_u \end{pmatrix} = \begin{pmatrix} R_{31} & R_{32} \end{pmatrix} \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}^{\dagger} \tag{22}$$

L_w and L_u can be used to form an estimate of outputs, namely

$$\begin{pmatrix} \hat{y}_k \\ \cdots \\ \hat{y}_{k+i-1} \end{pmatrix} = L_w (w_p)_k + L_u \begin{pmatrix} \hat{u}_k \\ \cdots \\ \hat{u}_{k+i-1} \end{pmatrix} \tag{23}$$

where $(w_p)_k$ is a vector made up past plant inputs and outputs

$$(w_p)_k = \begin{pmatrix} u_{k-i} \\ \vdots \\ u_{k-1} \\ y_{k-i} \\ \vdots \\ y_{k-i} \end{pmatrix} \tag{24}$$

It is worth mentioning that the subspace predictor algorithm introduced in this reference, uses I/O data which is collected from a healthy disturbance free system, i.e. system 6 when $f = 0$ and $d = 0$, while in this paper the I/O data is gathered from a healthy system and the limiting assumption $d = 0$ is not considered. Therefore the future output's estimation can be presented as following equation:

$$\hat{Y}_f = L_w W_p + L_u U_f + L_d D_{fd} \tag{25}$$

where D_{fd} is the Hankel matrix formed from disturbance signal as following

$$D_{fd} = \begin{pmatrix} d_0 & d_1 & \cdots & d_{j-1} \\ d_1 & d_2 & \cdots & d_j \\ \vdots & \vdots & \cdots & \vdots \\ d_{i-1} & d_i & \cdots & d_{i+j-2} \end{pmatrix} \in R^{ik_d \times j} \tag{26}$$

The nature of L_d is not important and it can be estimated by tuning the weighting matrix W_d which is explained in the next section.

5. A DATA-DRIVEN $\mathcal{H}_\infty$ CONTROLLER/DETECTOR

Motivated by the development in Overschee & Moor (1999) and Woodley et al. (2001), a solution is proposed to the problem (17) that leads a data-driven $\mathcal{H}_\infty$ controller/detector unit. Figure 4 shows a setup for data-driven $\mathcal{H}_\infty$ controller/detector design. A subspace predictor as proposed in Overschee & Moor (1999) is applied for

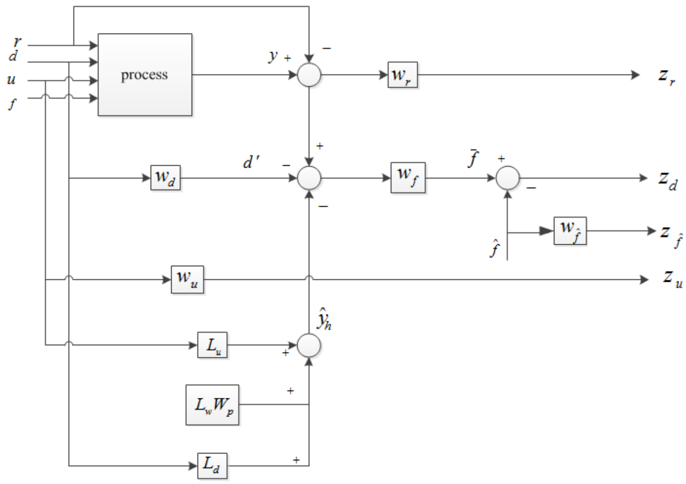

Fig. 4. A setup for data-driven $\mathcal{H}_\infty$ controller/detector design

the estimation of future plant output $\hat{y}$; see Section 4 for a review of subspace predictor. The subspace predictor is coupled to the weighting matrices W_r and W_u to represent the control objectives. W_r is usually chosen to be large at low frequency, and small at high frequency, while W_u is often chosen to be small at low frequency and large at high frequency. The weight W_d describes the relative importance and/or the expected or known frequency content of the disturbance. It is chosen to reduce the effect of disturbances on the fault estimation and on the output . As mentioned before, W_f is a proper stable weighting transfer matrix which is selected in this setup for reconstructing the filtered version of the fault f as given by $\bar{f} = W_f f$. Since the signal $\hat{f}$ is a virtual control signal, its amplitude should be limited. The matrix $W_{\hat{f}}$ is chosen to limit the amplitude of signal $\hat{f}$.

Assume the weighting transfer matrices W_r, W_d, W_f, W_u and $W_{\hat{f}}$ have the following minimal state space realizations

$$\mathbf{W_r} : \begin{cases} (x_{w_r})_{k+1} = A_{w_r} (x_{w_r})_k + B_{w_r} (r_k - y_k) \\ (z_r)_k = C_{w_r} (x_{w_r})_k + D_{w_r} (r_k - y_k) \end{cases} \tag{27}$$

$$\mathbf{W_d} : \begin{cases} (x_{w_d})_{k+1} = A_{w_d} (x_{w_d})_k + B_{w_d} d_k \\ d'_k = C_{w_d} (x_{w_d})_k + D_{w_d} d_k \end{cases} \tag{28}$$

$$\mathbf{W_f} : \begin{cases} (x_{w_f})_{k+1} = A_{w_f} (x_{w_f})_k + B_{w_f} (y_k - d'_k - (\hat{y}_h)_k) \\ \bar{f}_k = C_{w_f} (x_{w_f})_k + D_{w_f} (y_k - d'_k - (\hat{y}_h)_k) \\ (z_d)_k = \bar{f}_k - \hat{f}_k \end{cases} \tag{29}$$

$$\mathbf{W_u} : \begin{cases} (x_{w_u})_{k+1} = A_{w_u} (x_{w_u})_k + B_{w_u} u_k \\ (z_u)_k = C_{w_u} (x_{w_u})_k + D_{w_u} u_k \end{cases} \tag{30}$$

$$\mathbf{W_{\hat{f}}} : \begin{cases} (x_{w_{\hat{f}}})_{k+1} = A_{w_{\hat{f}}} (x_{w_{\hat{f}}})_k + B_{w_{\hat{f}}} \hat{f}_k \\ (z_{\hat{f}})_k = C_{w_{\hat{f}}} (x_{w_{\hat{f}}})_k + D_{w_{\hat{f}}} \hat{f}_k \end{cases} \tag{31}$$

Define

$$w = \begin{pmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+i-1} \end{pmatrix}, \; v = \begin{pmatrix} v_k \\ v_{k+1} \\ \vdots \\ v_{k+i-1} \end{pmatrix}, \; u = \begin{pmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+i-1} \end{pmatrix},$$

$$y = \begin{pmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+i-1} \end{pmatrix}, \; d' = \begin{pmatrix} d'_k \\ d'_{k+1} \\ \vdots \\ d'_{k+i-1} \end{pmatrix}, \; \hat{y}_h = \begin{pmatrix} (\hat{y}_h)_k \\ (\hat{y}_h)_{k+1} \\ \vdots \\ (\hat{y}_h)_{k+i-1} \end{pmatrix}$$

$$z_r = \begin{pmatrix} (z_r)_k \\ (z_r)_{k+1} \\ \vdots \\ (z_r)_{k+i-1} \end{pmatrix}, \; z_u = \begin{pmatrix} (z_u)_k \\ (z_u)_{k+1} \\ \vdots \\ (z_u)_{k+i-1} \end{pmatrix},$$

$$z_d = \begin{pmatrix} (z_d)_k \\ (z_d)_{k+1} \\ \vdots \\ (z_d)_{k+i-1} \end{pmatrix}, \; z_{\hat{f}} = \begin{pmatrix} (z_{\hat{f}})_k \\ (z_{\hat{f}})_{k+1} \\ \vdots \\ (z_{\hat{f}})_{k+i-1} \end{pmatrix}.$$

where k is the current time instant. By collecting data from (27)- (31), the following results are obtained

$$z_r = \Gamma_r (x_{w_r})_k + H_r K_{ref}\, w - H_r\, y \tag{32}$$

$$d' = \Gamma_d\, (x_{w_d})_k + H_d\, w \tag{33}$$

$$z_d = \Gamma_f (x_{w_f})_k + H_f (y - d' - \hat{y}_h), \tag{34}$$

$$z_u = \Gamma_u (x_{w_u})_k + H_u v \tag{35}$$

$$z_{\hat{f}} = \Gamma_{\hat{f}}\, (x_{w_{\hat{f}}})_k + H_{\hat{f}}\, v \tag{36}$$

where

$$K_{ref} = \begin{pmatrix} k_1 & 0 & 0 & \cdots & 0 \\ 0 & k_1 & 0 & \cdots & 0 \\ 0 & 0 & k_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & k_1 \end{pmatrix} \tag{37}$$

(38)

Γ_j's are the extended observability matrices formed from the impulse response of the weighting transfer matrices W_j's for $j = r, d, u, f, \hat{f}$ and given by

$$\Gamma_j = \begin{pmatrix} C_{w_j} \\ C_{w_j} A_{w_j} \\ \cdots \\ C_{w_j} A_{w_j}{}^{i-1} \end{pmatrix}, \quad j = r, d, \cdots, \hat{f}. \tag{39}$$

Here H_r, H_d, H_u, H_f and $H_{\hat{f}}$ are lower triangular toeplitz matrices formed from impulse response of the weighting transfer matrices W_j for $j = r, d, f, u, \hat{f}$ and given by

$$H_r = \begin{pmatrix} a_r & 0 & 0 & \cdots & 0 \\ b_r & a_r & 0 & \cdots & 0 \\ c_r & b_r & a_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_r & e_r & \cdots & b_r & a_r \end{pmatrix} \tag{40}$$

where $a_r = D_{w_r}$,$b_r = C_{w_r} B_{w_r}$, $c_r = C_{w_r} A_{w_r} B_{w_r}$, $d_r = C_{w_r} A_{w_r}{}^{i-2} B_{w_r}$, $e_r = C_{w_r} A_{w_r}{}^{i-3} B_{w_r}$, and

$$H_d = \begin{pmatrix} a_d & 0 & 0 & \cdots & 0 \\ b_d & a_d & 0 & \cdots & 0 \\ c_d & b_d & a_d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_d & e_d & \cdots & b_d & a_d \end{pmatrix} \tag{41}$$

where

$a_d = D_{w_d} K_2$, $b_d = C_{w_d} B_{w_d} K_2$, $c_d = C_{w_d} A_{w_d} B_{w_d} K_2$, $d_d = C_{w_d} A_{w_d}{}^{i-2} B_{w_d} K_2$, $e_d = C_{w_d} A_{w_d}{}^{i-3} B_{w_d} K_2$, and

$$H_f = \begin{pmatrix} a_f & 0 & 0 & \cdots & 0 \\ b_f & a_f & 0 & \cdots & 0 \\ c_f & b_f & a_f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_f & e_f & \cdots & b_f & a_f \end{pmatrix} \tag{42}$$

where $a_f = D_{w_f}$, $b_f = C_{w_f} B_{w_f}$, $c_f = C_{w_f} A_{w_f} B_{w_f}$, $d_f = C_{w_f} A_{w_f}{}^{i-2} B_{w_f}$, $e_f = C_{w_f} A_{w_f}{}^{i-3} B_{w_f}$, and

$$H_u = \begin{pmatrix} a_h & 0 & 0 & \cdots & 0 \\ b_h & a_h & 0 & \cdots & 0 \\ c_h & b_h & a_h & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_h & e_h & \cdots & b_h & a_h \end{pmatrix} \tag{43}$$

$a_h = D_{w_u} K_4$, $b_h = C_{w_u} B_{w_u} k_4$, $c_h = C_{w_u} A_{w_u} B_{w_u} K_4$, $d_h = C_{w_u} A_{w_u}{}^{i-2} B_{w_u} K_4$, $e_h = C_{w_u} A_{w_u}{}^{i-3} B_{w_u} K_4$, and

$$H_{\hat{f}} = \begin{pmatrix} a_{\hat{f}} & 0 & 0 & \cdots & 0 \\ b_{\hat{f}} & a_{\hat{f}} & 0 & \cdots & 0 \\ c_{\hat{f}} & b_{\hat{f}} & a_{\hat{f}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{\hat{f}} & e_{\hat{f}} & \cdots & b_{\hat{f}} & a_{\hat{f}} \end{pmatrix} \tag{44}$$

where $a_{\hat{f}} = D_{w_{\hat{f}}} K_5$, $b_{\hat{f}} = C_{w_{\hat{f}}} B_{w_{\hat{f}}} K_5$, $c_{\hat{f}} = C_{w_{\hat{f}}} A_{w_{\hat{f}}} B_{w_{\hat{f}}} K_5$, $d_{\hat{f}} = C_{w_{\hat{f}}} A_{w_{\hat{f}}}{}^{i-2} B_{w_{\hat{f}}} K_5$, $e_{\hat{f}} = C_{w_{\hat{f}}} A_{w_{\hat{f}}}{}^{i-3} B_{w_{\hat{f}}} K_5$.

We now present the following theorem which its proof can be found in Appendix A.

Theorem 1. If the measurements of plant input u, plant output y, and references r are available for times $k - i, \cdots, k-2, k-1$, then the strictly causal, finite horizon, model free subspace based level-γ, $\mathcal{H}_\infty$ based controller/detector for times $k, \cdots, k+i-1$ is

$$v_{opt} = (Q_u + K_v^T\, K_v + Q_{\hat{f}} - A\, H_d^T\, Q_f\, H_f - \gamma^2 I)^{-1} \times$$
$$\begin{pmatrix} -AK_{ref}^T\, Q_r + H_d^T\, Q_f + K_v^T\, H_f \\ A\, H_d^T\, Q_f\, L_w - K_v^T\, H_f\, L_w \\ -AH_r^T\, \Gamma_r \\ -H_u^T\, \Gamma_u \\ -AH_d^T\, H_f^T\, \Gamma_f + K_v^T\, \Gamma_f \\ A\, H_d^T\, Q_f\, \Gamma_d - K_v^T\, H_f\, \Gamma_d \\ -H_{\hat{f}}^T\, \Gamma_{\hat{f}} \end{pmatrix} \begin{pmatrix} y \\ w_p \\ x_{w_r} \\ x_{w_u} \\ x_{w_f} \\ x_{w_d} \\ x_{w_{\hat{f}}} \end{pmatrix}_k, \tag{45}$$

where

$$A = K_v^T\, H_f\, H_d (K_{ref}^T Q_r\, K_{ref} + H_d^T\, Q_f\, H_d - \gamma^2 I)^{-1}$$

$$K_v = H_f\, L_u\, K_u + K_{\hat{f}}$$

$$K_{\hat{f}} = \begin{pmatrix} k_5 & 0 & 0 & \cdots & 0 \\ 0 & k_5 & 0 & \cdots & 0 \\ 0 & 0 & k_5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & k_5 \end{pmatrix}$$

$$K_u = \begin{pmatrix} k_4 & 0 & 0 & \cdots & 0 \\ 0 & k_4 & 0 & \cdots & 0 \\ 0 & 0 & k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & k_4 \end{pmatrix}$$

$$Q_j = {H_j}^T H_j, \quad j = r, f, u, \hat{f}$$

Provided that

$$K_{ref}^T \, Q_r \, K_{ref} + H_d^T \, Q_f \, H_d - H_d^T \, H_f^T \, K_v \times (Q_u + K_v^T \, K_v + Q_{\hat{f}})^{-1} \times \, K_v^T \, H_f \, H_d < \gamma^2 I \quad (46)$$

v_{opt} is the vector of optimal future plant inputs and estimated fault $(\hat{f})$ at times $k, \cdots, k+i-1$. In fact, the first mi rows of v_{opt} is u_{opt} and the last $k_f i$ rows of v_{opt} is $\hat{f}$ where

$$u_{opt} = \begin{pmatrix} u_k \\ \vdots \\ u_{k+i-1} \end{pmatrix}, \hat{f} = \begin{pmatrix} \hat{f}_k \\ \vdots \\ \hat{f}_{k+i-1} \end{pmatrix} \quad (47)$$

The following algorithm summarizes the proposed procedure for data-driven $\mathcal{H}_\infty$ controller/detector design. *Data-driven $\mathcal{H}_\infty$ Controller/Detector Design*

(1) Collect the I/O data.
(2) Form Hankel matrices U_p, Y_p, U_f, Y_f, and W_p using (20).
(3) Calculate L_w and L_u according to (21).
(4) Choose W_j for $j = r, d, u, f, \hat{f}$.
(5) Initialize $(x_{w_j})_k$ for $j = r, d, u, f, \hat{f}$.
(6) Calculate the γ_{min} according to (46) and choose $\gamma > \gamma_{min}$.
(7) Form $(W_p)_k$ according to (24).
(8) Calculate v_{opt} for current time instant k.
(9) Calculate $\hat{f}_k, u_k$ from $(v_{opt})_k$.
(10) Take measurement y_k(from the real system) and r_k.
(11) Up to date $(x_{w_j})_k$ for $j = r, d, u, f, \hat{f}$.
(12) $k = k+1$, go to Step 7.

Do steps (7-12) i times, where i is the prediction horizon. This algorithm is constructive and can be implemented using standard scientific softwares such as Matlab.

6. SIMULATION

To illustrate the application of the results obtained in the paper, we apply the Data-driven $\mathcal{H}_\infty$ Controller/Detector Design algorithm which is described in the previous section to the linearized model of a quadruple tank process.

In this paper both incipient and abrupt faults are applied for minimum-phase and non-minimum phase plant. Incipient faults with maximum amplitude 1 occur between samples 600-1000 and samples 550-800, respectively for the first and second actuators. Abrupt faults are pulses with the amplitude 1, 1.3 and occur between 600-800, 550-800, respectively for the first and second actuators. Disturbances are pulses with the amplitude 1, 1.3 and occur between samples 750-830, 700-780, respectively for the first and second states. The reference input r is set to zero. Weighting transfer matrices are chosen in continuous-time domain and then discretized during performing the

	Abrupt Fault	Incipient Fault
W_r	$\begin{bmatrix} \frac{0.05(s+10)}{(s+0.5)} & 0 \\ 0 & \frac{0.03(s+10)}{(s+0.25)} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{(150s+1)} & 0 \\ 0 & \frac{1}{(150s+1)} \end{bmatrix}$
W_d	$\begin{bmatrix} \frac{1}{(s+5)} & 0 \\ 0 & \frac{1}{(s+5)} \end{bmatrix}$	$\begin{bmatrix} \frac{0.01}{(s+5)} & 0 \\ 0 & \frac{0.02}{(s+5)} \end{bmatrix}$
W_u	$\begin{bmatrix} \frac{2(s+2)}{(s+9)} & 0 \\ 0 & \frac{0.8(s+2)}{(s+5)} \end{bmatrix}$	$\begin{bmatrix} \frac{10(5s+1)}{(s+50)} & 0 \\ 0 & \frac{10(5s+1)}{(s+50)} \end{bmatrix}$
W_f	$\begin{bmatrix} \frac{5}{s+10} & 0 \\ 0 & \frac{5}{s+10} \end{bmatrix}$	$\begin{bmatrix} \frac{0.5}{0.01s+1} & 0 \\ 0 & \frac{0.5}{0.01s+1} \end{bmatrix}$
$W_{\hat{f}}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}$
N	519	1019
i	10	10

Fig. 5. weighting matrices and parameters associated with minimum phase plant

	Abrupt Fault	Incipient Fault
W_r	$\begin{bmatrix} \frac{0.1(s+5)}{(s+0.5)} & 0 \\ 0 & \frac{0.2(s+5)}{(s+0.25)} \end{bmatrix}$	$\begin{bmatrix} \frac{0.05(s+5)}{(s+1)} & 0 \\ 0 & \frac{0.05(s+6)}{(s+1)} \end{bmatrix}$
W_d	$\begin{bmatrix} \frac{0.1}{(s+1)} & 0 \\ 0 & \frac{0.2}{(s+15)} \end{bmatrix}$	$\begin{bmatrix} \frac{0.5}{s+10} & 0 \\ 0 & \frac{0.5}{s+10} \end{bmatrix}$
W_u	$\begin{bmatrix} \frac{5(s+0.6)}{(8s+50)} & 0 \\ 0 & \frac{5(s+0.6)}{(8s+55)} \end{bmatrix}$	$\begin{bmatrix} \frac{10(s+0.6)}{(8s+45)} & 0 \\ 0 & \frac{10(s+0.05)}{(8s+45)} \end{bmatrix}$
W_f	$\begin{bmatrix} \frac{0.25}{0.9s+0.5} & 0 \\ 0 & \frac{0.4}{0.9s+0.6} \end{bmatrix}$	$\begin{bmatrix} \frac{0.25}{(0.9s+0.5)} & 0 \\ 0 & \frac{0.2}{(0.9s+0.4)} \end{bmatrix}$
$W_{\hat{f}}$	$\begin{bmatrix} 0.35 & 0 \\ 0 & 0.15 \end{bmatrix}$	$\begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}$
N	1019	1019
i	10	10

Fig. 6. weighting matrices and parameters associated with non-minimum phase plant

Data-driven $\mathcal{H}_\infty$ Controller/Detector Design algorithm. The weighting transfer matrices are selected according to explanation in Section 5 and are tuned in order to establish the trades-off between control and fault detection objectives, i.e. to have reasonable control effort, good tracking and good fault estimation, simultaneously. The mentioned algorithm is then applied to the system (5).

Different weighting matrices and other parameters associated for minimum phase and non-minimum phase plant are mentioned in Figure 5 and 6.

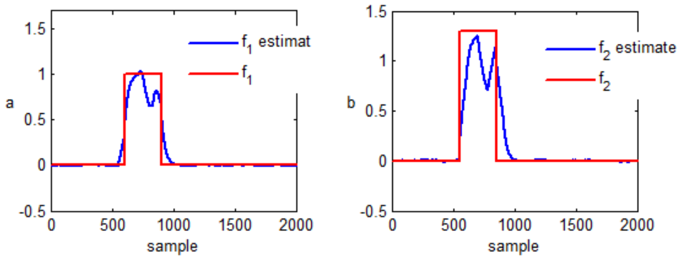

Fig. 7. The abrupt fault estimation for minimum phase plant, a:actuator 1, b: actuator2

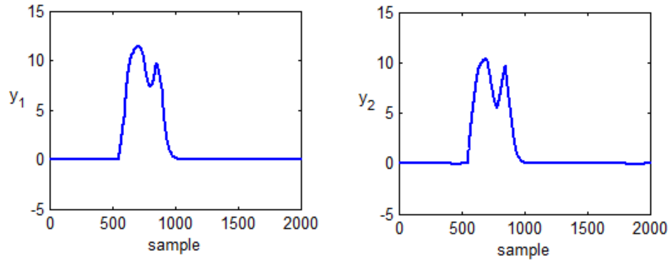

Fig. 8. The output y for minimum phase plant (abrupt fault case)

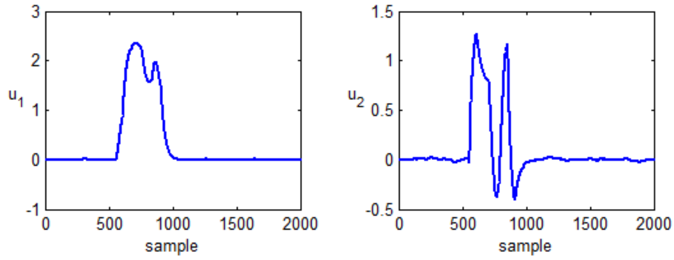

Fig. 9. The control input u for minimum phase plant(abrupt fault case)

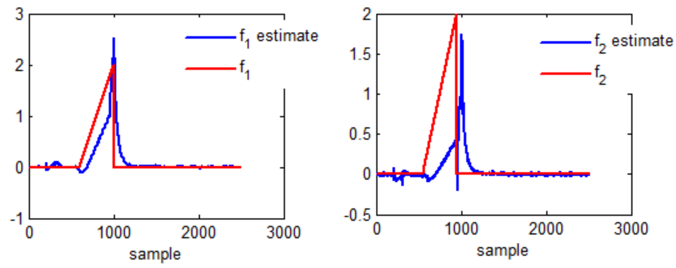

Fig. 10. The incipient fault estimation for minimum phase plant, a:actuator 1, b: actuator2

Figure 13-18 show the simulation results for the best value of γ for each case. As shown in these figures in all cases good fault estimation and good tracking are achieved, further more in all cases control efforts are reasonable. These simulations shows the advantages and efficiency of our proposed methodology.

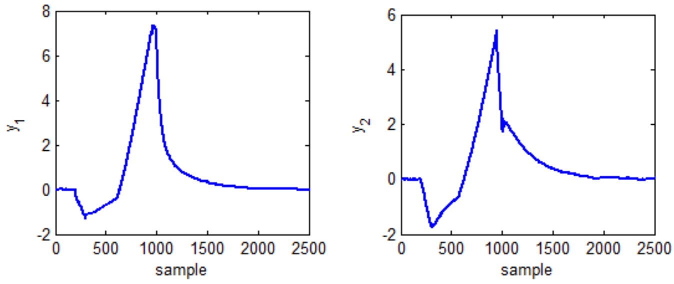

Fig. 11. The output y for minimum phase plant (incipient fault case)

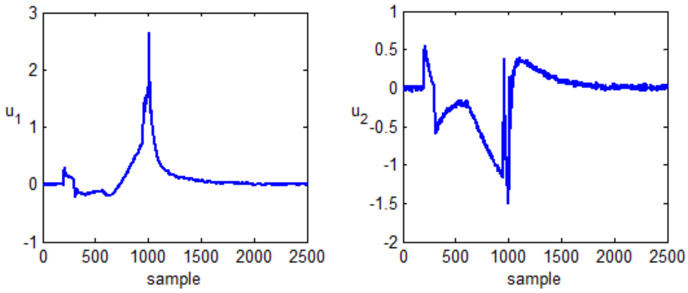

Fig. 12. The control input u for minimum phase plant (incipient fault case)

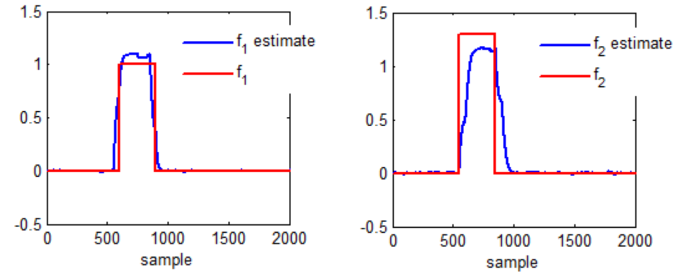

Fig. 13. The abrupt fault estimation for non-minimum phase plant, a:actuator 1, b: actuator2

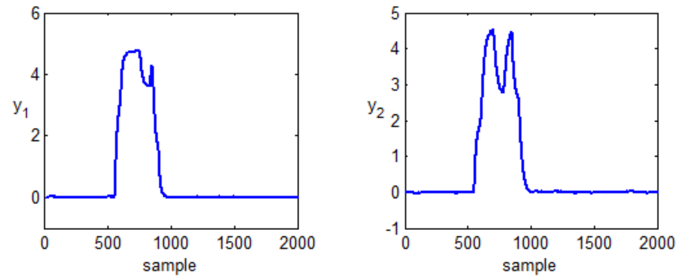

Fig. 14. The output y for non-minimum phase plant (abrupt fault case)

7. CONCLUSION

In this paper, motivated by recent development in data-driven control, a novel solution to SFDC problem for the multi-variable quadruple tank process with an adjustable zero has been developed. This problem is formulated as a data-based $\mathcal{H}_\infty$ optimization problem and its solution yields a data-driven $\mathcal{H}_\infty$ controller/detector unit of modest

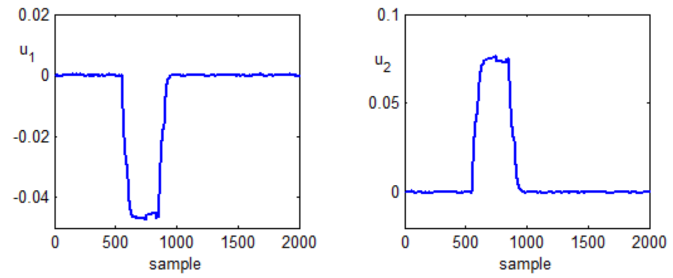

Fig. 15. The control input u for non-minimum phase plant (abrupt fault case)

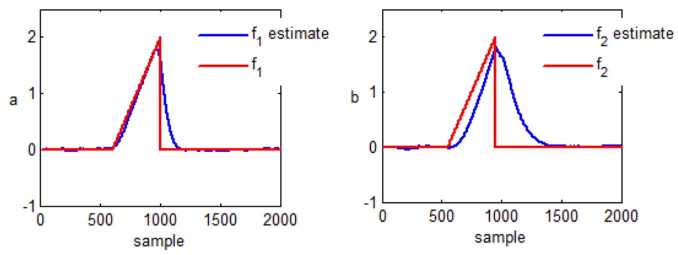

Fig. 16. The incipient fault estimation for non-minimum phase plant, a:actuator 1, b: actuator2

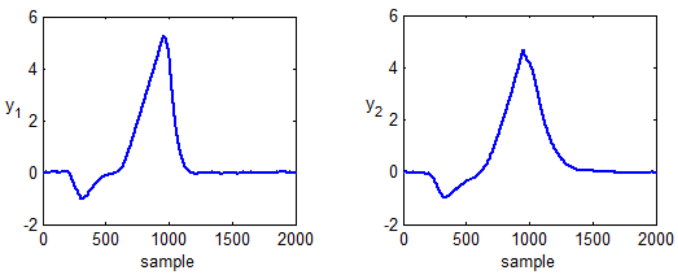

Fig. 17. The output y for non-minimum phase plant (incipient fault case)

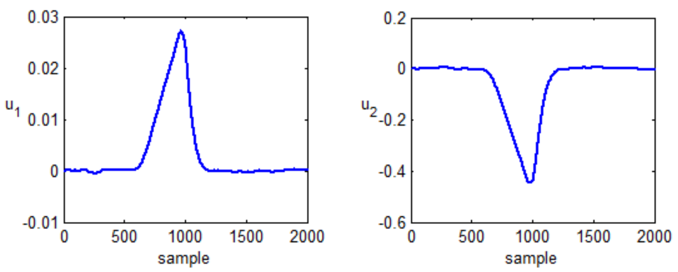

Fig. 18. The control input u for non-minimum phase plant (incipient fault case)

complexity. The design procedure is independent of the mathematical model and only I/O data is used for designing. Using this solution to SFDC problem, some control and fault detection objectives can be satisfied. Control objectives are regulation with minimizing the control effort and disturbance attenuation. The fault detection objective is also fault estimation. Using a scalar parameter and some appropriate weighting transfer matrices, the trade-off between control and fault detection objectives can be effectively established. The methodology presented in this paper is constructive and can be easily implemented using standard software tools. The simulation results show the advantages and efficiency of our algorithm. Further research work includes two aspects. The first one is assuming adaptive weights for designing the data driven controller/detector. The second one is that the proposed data-driven SFDC problem could be extended to a large class of uncertain nonlinear systems in the presence actuator and sensor faults.

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Appendix A. THE PROOF OF THEOREM 1

The following results are used in the proof of Theorem 1.

Lemma 2. (Schur Decomposition): If M is a symmetric matrix, then it can be expressed as

$$M = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix} = \tag{A.1}$$

$$\begin{pmatrix} I & A_2A_3^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 - A_2A_3^{-1}A_2^T & 0 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} I & A_2A_3^{-1} \\ 0 & I \end{pmatrix}^T$$

Lemma 3. (Ding (2009)): If A_1^{-1}, A_3^{-1}, and $\begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}^{-1}$ exist, then

$$\begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{pmatrix} \tag{A.2}$$

where

$$\begin{aligned} \bar{A}_1 &= (A_1 - A_2A_3^{-1}A_2^T)^{-1} \\ \bar{A}_2 &= -(A_1 - A_2A_3^{-1}A_2^T)^{-1}A_2A_3^{-1} \\ \bar{A}_3 &= -(A_3 - A_2^TA_1^{-1}A_2)^{-1}A_2^TA_1^{-1} \\ \bar{A}_4 &= (A_3 - A_2^TA_1^{-1}A_2)^{-1}. \end{aligned}$$

Suppose that the input/output data is collected from the unknown linear healthy system (the system (6) when $f = 0$). Calculate $\hat{y}$ according to Section 4. Substituting (33) and (23) into (34) yields

$$z = \begin{pmatrix} z_r \\ z_u \\ z_d \end{pmatrix} = \begin{pmatrix} \Gamma_r(x_{w_r})_k + H_rK_{ref}w - H_ry \\ \Gamma_u(x_{w_u})_k + H_uv \\ a \end{pmatrix} \tag{A.3}$$

where

$$\begin{aligned} a = {} & \Gamma_f(x_{w_f})_k + H_fy - H_f\Gamma_d(x_{w_d})_k - H_fH_dw \\ & -H_fL_wW_p - K_vv, \end{aligned}$$

and

$$K_{ref} = \begin{pmatrix} k_1 & 0 & 0 & \cdots & 0 \\ 0 & k_1 & 0 & \cdots & 0 \\ 0 & 0 & k_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & k_1 \end{pmatrix} \tag{A.4}$$

$$K_v = H_f\, L_u\, K_u + K_{\hat{f}} \tag{A.5}$$

$$K_{\hat{f}} = \begin{pmatrix} k_5 & 0 & 0 & \cdots & 0 \\ 0 & k_5 & 0 & \cdots & 0 \\ 0 & 0 & k_5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & k_5 \end{pmatrix} \tag{A.6}$$

$$K_u = \begin{pmatrix} k_4 & 0 & 0 & \cdots & 0 \\ 0 & k_4 & 0 & \cdots & 0 \\ 0 & 0 & k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & k_4 \end{pmatrix} \tag{A.7}$$

Substituting (A.3) into (18) results in

$$J = X^T \,(\, M_1 \quad M_2 \,)\, X \tag{A.8}$$

where

$$X = \begin{pmatrix} w \\ v \\ y \\ (w_p)_k \\ (x_{w_r})_k \\ (x_{w_u})_k \\ (x_{w_f})_k \\ (x_{w_d})_k \\ (x_{w_{\hat{f}}})_k \end{pmatrix}$$

and

$$M_1 = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & 0 \\ m_{51} & 0 & m_{53} & 0 & m_{55} \\ 0 & m_{62} & 0 & 0 & 0 \\ m_{71} & m_{72} & m_{73} & m_{74} & 0 \\ m_{81} & m_{82} & m_{83} & m_{84} & 0 \\ 0 & m_{92} & 0 & 0 & 0 \end{pmatrix} \tag{A.9}$$

$$M_2 = \begin{pmatrix} 0 & \bar{m}_{12} & \bar{m}_{13} & 0 \\ \bar{m}_{21} & \bar{m}_{22} & \bar{m}_{23} & \bar{m}_{24} \\ 0 & \bar{m}_{32} & \bar{m}_{33} & 0 \\ 0 & \bar{m}_{42} & \bar{m}_{43} & 0 \\ 0 & 0 & 0 & 0 \\ \bar{m}_{61} & 0 & 0 & 0 \\ 0 & \bar{m}_{72} & \bar{m}_{73} & 0 \\ 0 & \bar{m}_{82} & \bar{m}_{83} & 0 \\ 0 & 0 & 0 & \bar{m}_{94} \end{pmatrix} \tag{A.10}$$

$$Q_j = H_j{}^T H_j, \quad j = r, f, u, \hat{f}, \tag{A.11}$$

where

$$\begin{aligned}
&m_{11} = K_{ref}^T\, Q_r\, K_{ref} + H_d^T\, Q_f\, H_d - \gamma^2 I,\\
&m_{12} = H_d^T\, H_f^T\, K_v, m_{13} = -K_{ref}^T\, Q_r - H_d^T\, Q_f,\\
&m_{14} = H_d^T\, Q_f\, L_w, m_{15} = K_{ref}^T\, H_r^T\, \Gamma_r,\\
&m_{21} = K_v^T\, H_f\, H_d, m_{22} = Q_u + K_v^T\, K_v + Q_{\hat{f}}\\
&m_{23} = -K_v^T\, H_f, m_{24} = K_v^T\, H_f\, L_w,\\
&m_{31} = -Q_r\, K_{ref} - Q_f\, H_d, m_{32} = -H_f^T\, K_v,\\
&m_{33} = Q_r + Q_f, m_{34} = -Q_f\, L_w, m_{35} = -H_r^T\, \Gamma_r,\\
&m_{41} = L_w^T\, Q_f\, H_d, m_{42} = L_w^T\, H_f^T\, K_v,\\
&m_{43} = -L_w^T\, Q_f, m_{44} = L_w^T\, Q_f, Ł_w,\\
&m_{51} = \Gamma_r^T\, H_r\, K_{ref}, m_{53} = -\Gamma_r^T\, H_r,\\
&m_{55} = \Gamma_r^T\, \Gamma_r, m_{62} = \Gamma_u^T\, H_u,\\
&m_{71} = -\Gamma_f^T\, H_f\, H_d\,, m_{72} = -\Gamma_f^T\, K_v,\\
&m_{73} = \Gamma_f^T\, H_f, m_{74} = -\Gamma_f^T\, H_f\, L_w,\\
&m_{81} = \Gamma_d^T\, Q_f\, H_d, m_{82} = \Gamma_d^T\, H_f^T\, K_v,\\
&m_{83} = -\Gamma_d^T\, Q_f, m_{84} = \Gamma_d^T\, Q_f\, L_w,\\
&m_{92} = \Gamma_{\hat{f}}^T\, H_{\hat{f}}
\end{aligned}$$

and

$$\begin{aligned}
&\bar{m}_{12} = -H_d^T\, H_f^T\, \Gamma_f, \bar{m}_{13} = H_d^T\, Q_f\, \Gamma_d,\\
&\bar{m}_{21} = H_u^T\, \Gamma_u, \bar{m}_{22} = -K_v^T\, \Gamma_f,\\
&\bar{m}_{23} = K_v^T\, H_f\, \Gamma_d,\ \bar{m}_{24} = H_{\hat{f}}^T\, \Gamma_{\hat{f}},\\
&\bar{m}_{32} = H_f^T\, \Gamma_f, \bar{m}_{33} = -Q_f\, \Gamma_d,\\
&\bar{m}_{42} = -L_w^T\, H_f^T\, \Gamma_f, \bar{m}_{43} = L_w^T\, Q_f\, \Gamma_d,\\
&\bar{m}_{61} = \Gamma_u^T\, \Gamma_u, \bar{m}_{72} = \Gamma_f^T\, \Gamma_f,\\
&\bar{m}_{73} = -\Gamma_f^T\, H_f\, \Gamma_d, \bar{m}_{82} = -\Gamma_d^T H_f^T \Gamma_f,\\
&\bar{m}_{83} = \Gamma_d^T\, Q_f\, \Gamma_d, \bar{m}_{94} = \Gamma_{\hat{f}}^T\, \Gamma_{\hat{f}}\,.
\end{aligned}$$

then the problem (17) can be written as

$$\min_{v} \sup_{w} X^T \,(\, M_1 \quad M_2 \,)\, X \leq 0 \tag{A.12}$$

Thus, the optimal solution for v and the worst case w in the problem (17) can be derived by solving the following equation

$$\frac{\partial J}{\partial \begin{pmatrix} w \\ v \end{pmatrix}} = 0 \tag{A.13}$$

which yields

$$\begin{pmatrix} w \\ v_{opt} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}^{-1} \times \tag{A.14}$$
$$\begin{pmatrix} -m_{13} & -m_{14} & -m_{15} & 0 & -\bar{m}_{12} & -\bar{m}_{13} & 0 \\ -m_{23} & -m_{24} & 0 & -\bar{m}_{21} & -\bar{m}_{22} & -\bar{m}_{23} & -\bar{m}_{24} \end{pmatrix} X$$

Using (A.2) in Lemma 3 the second row of (A.14) can be written as (1). It is obvious that u and $\hat{f}$ can be calculated directly by v_{opt}. The sufficient condition for optimization is satisfied when the Hessian of the left hand side of (A.12) has $(k_f + k_d + l)i$ positive and $(k_f + m)i$ negative eigenvalues.

$$Hess = \frac{\partial^2 J}{\partial^2 \begin{pmatrix} w \\ v \end{pmatrix}} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{A.15}$$

It can be easily seen that

$$Hess = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix} \qquad (A.16)$$

It is worth mentioning that left multiplying (A.2) in Lemma 3 by $\begin{pmatrix} 0 & I \end{pmatrix}$ produces the relation

$$\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}^{-1} = (A_3 - A_2^T A_1^{-1} A_2)^{-1} \begin{pmatrix} -A_2^T A_1^{-1} & I \end{pmatrix}$$

Using *Schur decomposition* in Lemma 2, the Hessian can be written as follows

$$Hess = \Gamma^T \begin{pmatrix} A_1 - A_2 {A_3}^{-1} {A_2}^T & 0 \\ 0 & A_3 \end{pmatrix} \Gamma, \qquad (A.17)$$

where

$$\Gamma = \begin{pmatrix} I & 0 \\ {A_3}^{-1} {A_2}^T & I \end{pmatrix} .$$

In the Γ coordinates, the eigenvalues of the following matrix can be investigated

$$\begin{pmatrix} A_1 - A_2 {A_3}^{-1} {A_2}^T & 0 \\ 0 & A_3 \end{pmatrix} \qquad (A.18)$$

Since $A_3 > 0$ then the sufficient condition is satisfied when $A_1 - A_2 {A_3}^{-1} {A_2}^T < 0$. This yields to

$$\begin{aligned} &K_{ref}^T \, Q_r \, K_{ref} + H_d^T \, Q_f \, H_d - H_d^T \, H_f^T \, K_v \qquad (A.19) \\ &\times (Q_u + K_v^T \, K_v + Q_{\hat{f}})^{-1} \times \, K_v^T \, H_f \, H_d < \gamma^2 I \end{aligned}$$

This completes the proof.