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Three Lectures on Neutral Functional Differential Equations ⋆

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Abstract: The main idea of this cycle is that mixed initial boundary value problems for partial differential equations of hyperbolic type in two dimensions modeling lossless propagation are a valuable source of functional differential equations, in particular of neutral type. Starting from the simplest examples there are discussed such topics as basic theory (including various explanations for "what could actually define a neutral equation"), stability and forced oscillations. Rather than giving strictly rigorous proofs, the good motivations and final results are given priority. It is author's strong belief that well formulated applied problems are able to supply interesting, appealing while not always easy to solve problems.

Keywords: Neutral functional differential equations, Propagation systems, Qualitative theory

1. LECTURE TWO. OLD (FORGOTTEN) AND OPEN (POSSIBLY NEW) PROBLEMS

1.1 Basic theory

By basic theory it is understood the set of those results that validate the modeling by dynamical systems - from the point of view of the application expert (engineer, physicist, chemist, biologist) and make sense to the mathematical object as a non-void class of objects - from the point of view of the mathematician. We include here not only existence, uniqueness, continuation and continuity/smoothness with respect to parameters and initial conditions, but also existence of some invariant sets e.g. the positive orthant, which account of the inclusion in the model of some specific properties of the state variables which may be deduced directly from their physical significance.

The standard theorems on existence, uniqueness, continuation and continuity/smoothness are quite dependent on the state space chosen for system representation. Two are the aspects that have to be pointed out here: a) this state space is a function space hence it is infinite dimensional; consequently the possible topologies are no longer "equivalent" as it is in the case for finite dimensional vector spaces; b) the main concern in choosing a suitable function state space is compactness of the bounded sets; this is crucial in applying fixed point theorems or other tools for existence and uniqueness theorems; in finite dimensional spaces closed bounded sets are compact but this will be no longer the case in function spaces; it follows that compactness criteria become essential in choosing the state space. The best suited choice from these points of view has been the space $\mathscr{C}(-\tau,0;\mathbb{R}^n)$ of the continuous $\mathbb{R}^n$-valued defined on $(-\tau,0)$. For FDE of the delayed type we may refer to the classical references Krasovskii (1959); Halanay (1963); Hale and Lunel (1993).

The easiest way to cope with these problems is to consider a standard linear model of Răsvan (1998) where the construction of the solution by steps will be able to solve much of the problems. But we may deduce from all the models discussed a more general model that reads as follows

$$\begin{cases} \dot{x}^1(t) = f^1(t,x^1(t),x_t^2) \\ \mathscr{D}^2 x_t^2 = f^2(t,x^1(t),x_t^2) \end{cases} \tag{1}$$

with the usual notations of the field. This system is nonlinear and time-varying but the difference operator is time invariant (for simplicity). We shall just remember here how this system can be approached as a system of neutral type.

If the suggestion of Hale and Martinez Amores (1977) is to be applied then we may associate to the nonlinear system (1) the following system

$$\begin{cases} \dot{x}^1(t) = f^1(t,x^1(t),x_t^2) \\ \dfrac{d}{dt}[\mathscr{D}^2 x_t^2 - f^2(t,x^1(t),x_t^2)] = 0 \end{cases} \tag{2}$$

which is clearly of neutral type since in the second equation we differentiate a genuine nonlinear difference operator.It is also possible to use the suggestion of this author (Răsvan (1973),Răsvan (1975)). Denoting

$$z^2(t) = \int_{-\tau}^{t} x^2(\theta)d\theta \tag{3}$$

where τ is the maximal delay value then the initial system becomes

$$\begin{cases} \dot{x}^1(t) = f^1(t,x^1(t),\dot{z}_t^2) \\ \mathscr{D}^2 \dot{z}_t^2 = f^2(t,x^1(t),\dot{z}_t^2) \end{cases} \tag{4}$$

which clearly is of neutral type. Which of the above approaches is to be taken depends mainly on the basic theory that has to be constructed.

The problem of the state space has been already discussed in the previous lecture. A fruitful and far going idea of Hale

⋆ Presented at the Department of Automatic Control of the Research Institute for Advanced Studies CINVESTAV Mexico City NOVEMBER 2006 which has been reported firstly at the World Mathematics Congress in Moscow (1966) and then published jointly with K. R. Meyer Hale and Meyer (1967) allowed to "remain" in $\mathscr{C}$; this idea was that of the role of the difference operator:

instead of

$$\frac{d}{dt}x(t) = f(t,x_t), \tag{5}$$

a neutral system is given by

$$\frac{d}{dt}\mathscr{D}x(t) = f(t,x_t), \tag{6}$$

The papers and, further, the monograph Hale and Lunel (1993) account for the development based on this idea - basic theory, but also Liapunov stability, oscillations and other. System (1) as well as its linear version are somehow different since the component x^2 is, generally speaking, discontinuous. This problem may be coped with in several ways. The approach taken by Hale and Martinez Amores (1977) which reduces the analysis to system (2) allows a rather simple application of the standard results obtained within the theory on $\mathscr{C}$. The major drawback of this approach is that it leaves aside the discontinuous solutions. Consequently, if we keep in mind that the basic source for such systems are the IBVP for hyperbolic PDE and that we have a one-to-one correspondence between the solutions of the two mathematical objects, then it becomes clear that leaving aside the discontinuous solutions for (1) means leaving aside generalized solutions for the PDE. With respect to this, the introduction of the variable $z^2(t)$ by (3) is a step ahead: the integral is "smoothing" and $z^2(t)$ thus introduced is continuous. Moreover, the basic theory for the resulting system (2) is performed on the Sobolev space $\mathscr{W}_2^{(1)}(-\tau,0;\mathbb{R}^{n_2})$ which allows generalized solutions for the associated PDE. This basic theory relies on the results in Melvin (1973) which at their turn make use of some extensions of the fixed point theorems belonging to M. A. Krasnosel'skii.

We have to mention here also another line for the basic theory - due to the group of Graz (F. Kappel, K. Kunisch and W. Schappacher). The main feature of their approach is to consider $\mathbb{R}^n \times L^p(-\tau,0;\mathbb{R}^n)$ as state space for FDE; this is a useful extension to nonlinear systems of the $\mathscr{M}^2$ approach of Delfour and Mitter (but going back to Borisovič and Turbabin (1969)) and is also very adequate to deal with discontinuous solutions of (1). The most suitable for the basic theory of (1) is Kunisch (1979). The nonlinear neutral FDE considered there is

$$\frac{d}{dt}(x(t) - g(x_t)) = f(x(t),x_t) + h(t) \tag{7}$$

which incorporates e.g. (2); this means once again that the approach proposed in Hale and Martinez Amores (1977) is suitable; the difference made by Kunisch (1979) is the extension to L^p spaces. This extension is performed by introducing the averaging approximating equations

$$\begin{aligned} x(t) = x(0) - \frac{1}{\varepsilon}\int_0^{\varepsilon}(g(U_s\varphi) + g(U_s x_t))\mathrm{d}s + \\ + \int_0^t (f(x(s),x_s) + h(s))\mathrm{d}s \end{aligned} \tag{8}$$

where $U_s : \mathscr{L}^p(0,\infty;\mathbb{R}^n) \mapsto \mathscr{L}^p(0,\infty;\mathbb{R}^n)$ is defined by

$$(U_s\varphi)(\theta) = \begin{cases} \varphi(s+\theta)\ , & s+\theta < 0 \\ 0 & , s+\theta > 0 \end{cases} \tag{9}$$

The role of the averaging approximating equations is to allow a "good" definition of the solution of (7) i.e. independent of the choice of φ as a representative of the class $[\varphi] \in L^p$.

1.2 Invariant sets

We start from the idea that in some applications, *invariant sets* are very useful. Such invariant sets occur, for instance, when some physically significant variables have definite sign and this property is re-discovered as a property of the mathematical model thus validating it in some sense. The starting model will be the model of the combined heat/electricity generation where between the steam extraction of the turbine and the steam consumer there is a "long" pipe that involves propagation phenomena. Some features of the model (1) in the companion paper Răsvan (2009) will be recognized here. The model is taken from Răsvan (1981):

$$\begin{cases} T_a\dfrac{ds}{dt} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_g \\ T_1\dfrac{d\pi_1}{dt} = \mu_1(t) - \pi_1 \\ T_p\dfrac{d\pi_s}{dt} = \pi_1 - \beta_1\mu_2(t)\pi_s - \beta_2\xi_w(0,t) \\ T_2\dfrac{d\pi_2}{dt} = \mu_2(t)\pi_s - \pi_2. \\ T_c\dfrac{\partial\xi_p}{\partial t} + \dfrac{\partial\xi_w}{\partial\lambda} = 0 \\ \psi_c^2 T_c\dfrac{\partial\xi_w}{\partial t} + \dfrac{\partial\xi_p}{\partial\lambda} = 0 \\ \xi_w(0,t) = \alpha_p\left[\pi_s(t) - \xi_p(0,t)\right] \\ \xi_w(1,t) = \psi_s\xi_p(1,t). \end{cases} \tag{10}$$

In this model, the state variables π_1,π_s,π_2 and ξ_p represent pressures and therefore they have to be nonnegative. From the mathematical point of view, this means that the model should have an invariant set: if the initial conditions $\pi_1(0),\pi_s(0),\pi_2(0)$ and $\xi_p(\lambda,0)$ are nonnegative, then $\pi_1(t),\pi_s(t),\pi_2(t)$ and $\xi_p(\lambda,t)$ should be nonnegative for all $t > 0$.

Let us mention that in Răsvan et al (2006) the case of two steam extractions is considered and the properties are of the same type as here; also the proofs are likewise.

The existence of such an invariant set is an argument speaking about the *correctness* of the model. In proving rigorously the existence of this invariant set, we shall take into account the fact that control functions $\mu_i(t)$ satisfy the following inequalities:

$$0 \le \mu_1(t) \le 1,\ 0 \le \gamma_0 \le \mu_2(t) \le 1 \tag{11}$$

due to the per-unit representation of the physical variables. Since (11) are valid, we see from the representation formula for $\pi_1(t)$:

$$\pi_1(t) = e^{-\frac{t}{T_1}}\pi_1(0) + \frac{1}{T_1}\int_0^t e^{-\frac{t-\theta}{T_1}}\mu_1(\theta)d\theta$$

that $\pi_1(t) \geq 0$ for all $t \geq 0$ if $\pi_1(0) \geq 0$. The same property is not so obvious for $\pi_2(t)$ and $\xi_p(\lambda,t)$.

Taking the approach of Subsection 1.1 (Răsvan (2009)) we may associate the following system of functional differential equations:

$$\begin{cases} T_a\dfrac{ds}{dt} = \alpha\pi_1 + (1-\alpha)\pi_2 - v_g \\ T_1\dfrac{d\pi_1}{dt} = \mu_1(t) - \pi_1 \\ T_p\dfrac{d\pi_s}{dt} = \pi_1 + \left(-\beta_1\mu_2(t) + \dfrac{\beta_2\alpha_p}{1+\alpha_p\psi_c}\right)\pi_s + \\ \qquad +2\dfrac{\beta_2\alpha_p\psi_c}{1+\alpha_p\psi_c}\,\eta_2(t-\psi_cT_c)) \\ T_2\dfrac{d\pi_2}{dt} = \mu_2\pi_s - \pi_2. \\ \eta_1(t) = \dfrac{1-\alpha_p\psi_c}{1+\alpha_p\psi_c}\eta_2(t-\psi_cT_c) + \dfrac{\alpha_p}{1+\alpha_p\psi_c}\pi_s(t) \\ \eta_2(t) = \dfrac{1-\psi_s\psi_c}{1+\psi_s\psi_c}\eta_1(t-\psi_cT_c), \end{cases} \tag{12}$$

together with the representation formulae:

$$\begin{aligned} \xi_p(\lambda,t) &= \psi_c(\eta_1(t-\psi_cT_c\lambda) + \eta_2(t+\psi_cT_c(\lambda-1))) \\ \xi_w(\lambda,t) &= \eta_1(t-\psi_cT_c\lambda) - \eta_2(t+\psi_cT_c(\lambda-1)). \end{aligned} \tag{13}$$

We may prove the following result:

Theorem 1. Consider system (12) with $0<\alpha<1$, $v_g>0$, $\beta_1>0$, $\beta_2>0$, $\alpha_p>0$, $\psi_c>0$, $0<\alpha_p\psi_c<1$, $0<\psi_s\psi_c<1$ and $\mu_i(t)$ satisfying (11). If:

$$\pi_1(0)\geq 0,\ \pi_s(0)\geq 0,\ \pi_2(0)\geq 0,\ \eta_i^0(t)\geq 0,$$
$$-\psi_cT_c \leq t < 0, \quad i=\overline{1,2},$$

then:

$$\pi_1(t)\geq 0,\ \pi_s(t)\geq 0,\ \pi_2(t)\geq 0,\ \eta_i(t)\geq 0,\ i=\overline{1,2} \quad \forall t>0.$$

At its turn this theorem together with the representation formulae (13) allow to obtain the same property for the basic system (10).

Theorem 2. Consider the system (10) with $0<\alpha<1$, $v_g>0$, $\beta_1>0$, $\beta_2>0$, $\alpha_p>0$, $\psi_c>0$, $0<\alpha_p\psi_c<1$, $0<\psi_s\psi_c<1$ and $\mu_i(t)$ satisfying (11). If:

$$\pi_1(0)\geq 0, \quad \pi_s(0)\geq 0, \quad \pi_2(0)\geq 0,$$
$$\xi_p(\lambda,0)+\psi_c\xi_w(\lambda,0)\geq 0, \quad \xi_p(\lambda,0)-\psi_c\xi_w(\lambda,0)\geq 0,$$
$$0\leq\lambda\leq 1,$$

then:

$$\pi_1(t)\geq 0, \quad \pi_s(t)\geq 0, \quad \pi_2(t)\geq 0,$$
$$\xi_p(\lambda,t)+\psi_c\xi_w(\lambda,t)\geq 0,\ \xi_p(\lambda,t)-\psi_c\xi_w(\lambda,t)\geq 0,$$
$$0\leq\lambda\leq 1,\ \forall t>0.$$

1.3 Linear systems and exponential stability

A. Let us consider the system

$$\begin{aligned} \dot{x}_1 &= A_0x_1(t) + A_1x_2(t-\tau) \\ x_2(t) &= A_2x_1(t) + A_3x_2(t-\tau) \end{aligned} \tag{14}$$

Since it may be shown that its solutions define a semigroup of operators on e.g. $\mathbb{R}^{n_1}\times L^2(-\tau,0;\mathbb{R}^{n_2})$ it is but natural to obtain its exponential stability from the location in $\mathbb{C}^-$ of the roots of a certain characteristic equation. Its RHS is a quasi-polynomial displaying delayed and neutral chains of roots. As well known, the neutral chains may approach asymptotically the imaginary axis $j\mathbb{R}$ thus allowing non-damped oscillations on some frequencies. This phenomenon is avoided provided the matrix A_3 has its eigenvalues inside the unit disk $\mathbb{D}_1$: this is nothing more than the stability condition for the difference operator associated to the second equation of (14). Under this condition the roots of the characteristic equation located in $\mathbb{C}^-$ never approach $j\mathbb{R}$ asymptotically being in fact located in some half-plane $\{s\in\mathbb{C},\ \Re e(s)\leq -\alpha<0\}$. Consequently all solutions of (14) approach 0 exponentially (for $t\to+\infty$).

Due to R. Datko it is by now clear that stability of the difference operator is a necessary condition for the exponential stability of a linear system of neutral type. As an example we consider the characteristic equation occurring in Kabakov (1946) Denoting

$$\gamma_1 = \frac{1-\alpha_p\psi_c}{1+\alpha_p\psi_c}\ ,\ \gamma_2 = \frac{1-\psi_s\psi_c}{1+\psi_s\psi_c}\ ,\ a_0 = \frac{(1-\gamma_1)(1-\gamma_2)}{1-\gamma_1\gamma_2}\frac{\delta}{2\alpha_p}\ ,$$

$$b_0 = \frac{(1-\gamma_1)(1+\gamma_2)}{1-\gamma_1\gamma_2}\frac{\delta}{2\alpha_p}$$

we put the quasi-polynomial in Kabakov (1946) under the standard form for second order quasi-polynomials from Čebotarev and Meĭman (1949)

$$\begin{aligned} &\frac{1-\gamma_1\gamma_2}{1+\gamma_1\gamma_2}\left[\left(\frac{\psi_m}{\psi_cT_c}\right)^2 z^2 + \frac{\psi_D}{\psi_cT_c}\,z + 1 + a_0\right]\cosh z + \\ &+\left[\left(\frac{\psi_m}{\psi_cT_c}\right)^2 z^2 + \frac{\psi_D}{\psi_cT_c}\,z + 1 + b_0\right]\sinh z = 0 \end{aligned} \tag{15}$$

We use first the necessary conditions of Stodola type. Since $(\psi_m/\psi_cT_c)^2>0$ it is then necessary that $1-\gamma_1\gamma_2>0$ hence the difference operator has to be here exponentially stable. Fortunately this is the case since $0<\gamma_i<1$, $i=1,2$. Under the circumstances all coefficients are strictly positive and the Stodola conditions are automatically fulfilled. Moreover we are in the so called *Case I* of Čebotarev and Meĭman (1949) whose inequalities may be found there. Note that this is the only case which may provide stability for arbitrarily large delays.

This was a second degree quasi-polynomial and we have seen that the Routh-Hurwitz like conditions are not easy to be checked - some of the cases were even overlooked because of a mistake in the originary proof. Consequently the search for equivalent (or almost equivalent conditions) which are feasible from the computational point of view also from the point of view of the design engineer is very popular if we are to judge after the quantity of the publications. Let us mention here the *frequency domain approach* which corresponds in the case of systems described by ODE or by rational transfer functions to various applications of the Cauchy principle of the argument, also to root locus or to D-splitting. With respect to this framework it is worth mentioning the "crossing" problem - to identify the bifurcation values of the coefficients corresponding to the crossing of $j\mathbb{R}$ by a pair of roots when the coefficients are perturbed. Due to terms as $e^{-j\omega\tau}$ and to the fact that crossings take place for some values $(\omega\tau)_k$, if the perturbed parameter is the delay then to the same crossing value $(\omega\tau)_k$ may correspond several delays. The smallest will give the so-called delay-dependent stability estimate. To this problem it is associated another: the sense of the crossing - from $\mathbb{C}^-$ to $\mathbb{C}^+$ or vice-versa - which is analyzed by complex functions techniques.

B. Motivated by control problems, the interest in linear systems stability is enhanced by the more recent interest in robust stability. Within the framework of the robust stability an interesting approach is that based on stability radii. For system (14) this approach is illustrated by Halanay and Răsvan (1997), Răsvan (1995), Răsvan (2000). We state a standard theorem for completeness

Theorem 3. Consider system (14) and the system perturbed with structured perturbations

$$\begin{cases} \dot{x} = (A_0 + B_1\Delta C_1)x(t) + (A_1 + B_1\Delta C_2)y(t-\tau) \\ y(t) = (A_2 + B_2\Delta C_1)x(t) + (A_3 + B_2\Delta C_2)y(t-\tau) \end{cases} \tag{16}$$

Let

$$T(s) = \begin{pmatrix} C_1 & C_2e^{-s\tau} \end{pmatrix} (H_0(s))^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

where

$$detH_\Delta(\lambda) =$$

$$\begin{pmatrix} \lambda I - A_0 - B_1\Delta C_1 & -(A_1 + B_1\Delta C_2)e^{-\lambda\tau} \\ -A_2 - B_2\Delta C_1 & I - (A_3 + B_2\Delta C_2)e^{-\lambda\tau} \end{pmatrix} = 0$$

and $H_0(\lambda) = H_\Delta(\lambda)$ for $\Delta = 0$ be the RHS of the characteristic equation associated with (14). Then the complex stability radius associated to the structured perturbations defined by (16),

$$r_{\mathbb{C}} = r_{\mathbb{C}}(A_i;B_i,C_i) = \inf\{\Delta | \Delta \in \mathbb{C}^{n_1\times n_2},\ \Lambda(\Delta)\cap\mathbb{C}^+ = \emptyset\},$$

where $\Lambda(\Delta)$ is the set of the roots of $\det H_\Delta(s) = 0$ - the characteristic equation of (16) - is given by

$$r_{\mathbb{C}} = [max_{\omega\in\mathbb{R}}|T(j\omega)|]^{-1} \tag{17}$$

The above theorem has been communicated in Răsvan (1995); its proof is performed along the standard lines of stability radii theory. It is important to mention that the class of structured perturbations considered here is interesting in that it does not modify the delay structure of the system or the neutral character of the equations with deviated argument. For real stability radii and other associated problems for (14) the reader is sent to *(op.cit.)*.

1.4 Forced linear systems

We shall turn now to the forced system (14)

$$\begin{aligned} \dot{x}_1 &= A_0x_1(t) + A_1x_2(t-\tau) + f_1(t) \\ x_2(t) &= A_2x_1(t) + A_3x_2(t-\tau) + f_2(t) \end{aligned} \tag{18}$$

, a forced linear system for which a formula of the variations of constants may be obtained by using again the new variable (3) thus reducing the system, as already shown above, to a standard system of neutral type; another approach for such a reduction is that of Hale and Martinez Amores (1977). A more general framework of the problem is in Hale and Huang (1995). This more recent paper also considers the reduction to the neutral FDE and makes use of what is called *the true adjoint*. For our purpose we consider sufficient the formula of Răsvan (1973)

$$\begin{cases} x^1(t) = Z_{11}(t)x_0^1 + \int_{-\tau}^{0} Z_{12}(t-\theta)x_0^2(\theta)\mathrm{d}\theta + \\ \quad + \sum_1^2 \int_0^t Z_{1i}(t-\theta)f^i(\theta)\mathrm{d}\theta \\ x^2(t) = \dot{Z}_{21}(t)x_0^1 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\tau}^{0} Z_{22}(\theta)x_0^2(\theta)\mathrm{d}\theta + \\ \quad + \frac{\mathrm{d}}{\mathrm{d}t}\sum_1^2 \int_0^t Z_{2i}(t-\theta)f^i(\theta)\mathrm{d}\theta \end{cases} \tag{19}$$

where the matrices $Z_{ij}(t)$ $(i,j = 1,2)$ are solutions of the following equations

$$\begin{cases} \dot{Z}_{11} = Z_{11}A_0 + Z_{12}A_2 \\ Z_{12}(t) = Z_{12}(t-\tau)A_3 + Z_{11}(t-\tau)A_1 \end{cases} \tag{20}$$

and

$$\begin{cases} \dot{Z}_{21} = Z_{21}A_0 + Z_{22}A_2 \\ Z_{22}(t) = Z_{22}(t-\tau)A_3 + Z_{21}(t-\tau)A_1 + I \end{cases} \tag{21}$$

with the initial conditions

$$Z_{ij}(t) \equiv 0,\ t<0; Z_{ii}(0) = I_{n_i},\ Z_{ij}(0) = 0,\ i \neq j$$

Obviously the matrix solutions of these systems may be constructed by steps. They satisfy exponential *a priori* estimates and their asymptotic behavior may be deduced using Laplace transform arguments. It may be shown using the construction by steps that $Z_{2i}(t)$ are discontinuous at $t = k\tau$, $k \in \mathbb{N}$ and for this reason the second formula in (19) has that form. Worth mentioning that analogous formulae may be obtained for the case with several delays or with distributed delay, also for the case of the time varying coefficients. We would like to cite now some problems which are considered to belong to *linear oscillations in a broad sense*. We do not know if these problems have been solved for systems of neutral type or for systems like (18).

A. The first interesting application is the *Perron condition* for system (18). As in the ODE or the retarded FDE case, it is said that *system (18) satisfies the Perron condition if for any continuous $f^i(t)$, $i = 1,2$ which are bounded for $t > 0$, the solution of (18) with zero initial conditions $x_o^1 = 0$, $x_o^2(\theta) \equiv 0$, $-\tau \le \theta \le 0$ is bounded on* $\mathbb{R}_+$. A standard result would be as follows: if system (18) satisfies the Perron condition then system (14) is exponentially stable and, in particular, the difference operator associated to the equation is such.

The proof of this result should be rather standard (see Halanay (1963) for the ODE and retarded FDE); it is in fact based on the technique of R. Bellman which strongly relies on the Banach-Steinhaus lemma. The case on Banach spaces has been discussed first by Daleckii and Krein (1970). It is worth mentioning here also the pioneering results of P. Bohl which are earlier to those of Perron; for this reason the Perron condition is sometimes called the Bohl-Perron condition. Further generalizations of the Perron condition may be obtained using the approach of Massera and Schäffer (1966) : the most important are the results on exponential dichotomies and splittings.

B.The next problem to those connected to the Perron condition and to exponential dichotomies is that of *forced linear oscilla-*

tions i.e. with f^i in (18) being periodic or almost periodic. As pointed out in Kolmanovskii and Nosov (1981) these problems which are connected with the Fredholm alternative, are better solved when a formula of variations of constants is available; for this reason an adaptation of the adjoint theory due to Henry (1971) appears as necessary.

C. A special case of the problem of linear oscillations is the problem of Malkin (1956): if in (18) the coefficients are T-periodic and $f^i(t)$ have the form

$$f^i(t) = \sum_1^m e^{\imath\omega_k t} g_k^i(t) \tag{22}$$

where $\omega_k \in \mathbb{R}$ and g_k^i, $k = 1, \ldots, m$, $i = 1, 2$ are T-periodic, then system (18) has an almost periodic solutions of the form

$$x^i(t) = \sum_1^m e^{\imath\omega_k t} z_k^i(t) \tag{23}$$

where $z_k^i(t)$ are T-periodic. Due to the general theory if e.g. the linear free system (14) (nevertheless with periodic coefficients) is exponentially stable, then the solution (23) is exactly the unique bounded solution of (18) on $\mathbb{R}$ which is almost periodic and exponentially stable. Judging after the result for retarded FDE Halanay (1963) the formula of the variations of constants appears again as crucial.

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