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A convolution approach for parameters and delay systems identification

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Abstract: This paper deals with on-line parameters and delay estimation of systems involving retarded phenomena. Using a convolution based approach, it generalizes to arbitrary inputs existing estimation results based on structured ones, while keeping the generalized eigenvalue structure. The proposed estimation algorithms allow for updated estimations, avoiding possible singularities in the matric pencil. Theoretical results are supported by numerical simulation.

Keywords: Time delay systems, Identification

1. INTRODUCTION

The real time delay identification is one of the most crucial open problems in the field of delay systems (see, e.g., Richard (2003)). On the one hand, various powerful control techniques (predictors, flatness-based predictive control, finite spectrum assignments, observers, ...) may be applied if the dead-time is known. On the other hand, most existing identification techniques for time-delay systems (see, e.g., Orlov et al. (2006); Drakunov et al. (2006) for adaptive techniques or Ren et al. (2005) for a modified least squares technique) generally suffer from poor speed performance. The developments in Belkoura et al. (2006, 2009) have considered the on line identification of delay systems with particular (structured) inputs. This paper considers the identification problem from general input-output trajectories. Although the parameter estimation technique is still inspired from the fast identification techniques that were proposed Fliess and Sira-Ramirez (2003) for linear, finite-dimensional models, this paper considers a new approach to deal with the delay estimation. Let us recall that those techniques are not asymptotic, and do not need statistical knowledge of the noises corrupting the data (See, e.g., Fliess et al. (2007) for linear and nonlinear diagnosis, Fliess et al. (2003) for signal processing, and Beltran-Carvajal et al. (2005) for successful laboratory experiments).

The contribution of this paper is in the continuity of the works presented in Belkoura et al. (2009) and Belkoura et al. (2011), with an extension to arbitrary inputs and a unified formulation. Moreover, a new identification approach provides a linear formulation in terms of the unknown parameters, allowing for recursive estimation techniques.

The paper is organized as follows. Section 2 gives a brief summary of the existing approaches for estimation problems of systems with structured entries. Section 3 presents the convolution based approach allowing for arbitrary inputs, and the connection between these two techniques is provided on a simple example. Based on projection methods, Section 4 introduces the numerical resolution of the generalized eigenvalue problem with non square matrices. An algorithm allowing for a recursive estimation of parameters and delay is presented. Numerical simulations are provided in Section 5. Most of the obtained results are formulated in the distribution framework, and the next paragraph is devoted to a brief summary of the definitions and tools we shall need in the sequel. The reader may consult Schwartz (1966); Hirsch and Lacombe (1999) for more on distribution theory.

The distribution framework

Throughout this paper, functions will be considered through the distributions they define, i.e. as continuous linear functionals on the space $\mathcal{D}$ of C^∞-functions having compact support in $[0, \infty)$. This framework allows the definition of the Dirac distribution $u = \delta$ and its derivative $u = \dot{\delta}$ as $\langle u, \varphi\rangle = \varphi(0)$ and $\langle u, \varphi\rangle = -\dot{\varphi}(0)$, $\varphi \in \mathcal{D}$ respectively. More generally, every distribution is indefinitely differentiable, and if u is a continuous function except at a point a, its distributional derivative writes:

$$u^{(1)} = \frac{du}{dt} + \sigma_a \delta_a. \qquad (1)$$

where $\delta_a := \delta(t-a)$, $\sigma_a := u(t_a+) - u(t_a-)$, and $\frac{du}{dt}$ stands for the distribution stemming from the usual derivative (function) of u defined almost everywhere. Note that with $a = 0$, this result, and its extension to higher order derivation, is nothing but the analog part of the familiar Laplace transform $\mathcal{L}(\dot{y}) = sy(s) - y_0$.

We proceed this introductory section with some well-known definitions and results from the convolution products, and as usual, denote $\mathcal{D}'_+$ the space of distributions with support contained in $[0, \infty)$. It is an algebra with respect to convolution with identity δ. For $u, v \in \mathcal{D}'_+$, this product is defined as $\langle u * v, \varphi\rangle = \langle u(x).v(y), \varphi(x + y)\rangle$, and can be identified with the familiar convolution product $(u * v)(t) = \int_0^\infty u(\theta)v(t-\theta)d\theta$ in case of locally bounded functions u and v. Derivation, integration and translation can also be defined from the convolutions

$$\dot{u} = \dot{\delta} * u, \qquad \int u = H * u, \qquad u(t-\tau) = \delta_\tau * u,$$

where H is the familiar Heaviside step function. We also recall the following well known property:

$$u(t) * v(t-\tau) = u(t-\tau) * v(t) = \delta_\tau * u * v. \tag{2}$$

As for the supports, one has for $u, v \in \mathcal{D}'_+$:

$$\text{supp } u * v \subset \text{supp } u + \text{supp } v, \tag{3}$$

where the sum in the right hand side is defined by

$$X + Y = \{x + y\,;\, x \in X, y \in Y\} \tag{4}$$

In our subsequent developments, the specific need for the distributional framework also lies in its ability to cancel the singular terms simply by means of multiplication with some appropriate functions. Multiplication of two distributions (say α and u) always make sense when at least one of the two terms (say α) is a smooth function, and the cancelation procedure presented in this paper will be derived from the general Schwartz (1966) Theorem on multiplication.

Theorem 1. Schwartz (1966) If u has a compact support K and is of order r (necessarily finite), $\alpha\, u = 0$ whenever α and its derivatives of order $\leq r$ vanish on K.

Particularly, for any smooth function α, one has

$$\alpha\, \delta_\tau = \alpha(\tau)\, \delta_\tau, \tag{5}$$

and this result generalises to Dirac distributions of arbitrary order r as:

$$\alpha\, \delta_\tau^{(r)} = 0, \tag{6}$$

$$\forall \alpha \text{ s.t. } \alpha^{(k)}(\tau) = 0, \quad k = 0, \ldots, r. \tag{7}$$

Finally, with no danger of confusion, we shall sometimes denote $u(s)$, $s \in \mathbb{C}$, the Laplace transform of u.

2. SUMMARY OF THE IDENTIFICATION TECHNIQUES FOR STRUCTURED INPUTS

For the sake of completeness and comparison with the general approach presented in Section 3, a brief summary on identification results of systems with structured input is presented. Structured entries have been initially introduced (see e.g. Fliess and Sira-Ramirez (2003)) to refer to entities (mainly perturbations) that can be annihilated by means of simple multiplications and derivations. Piecewise polynomials, exponential and harmonic functions, impulsive functions and theirs derivatives (Dirac functions used to take into account initial conditions in the distribution framework) are typical examples of structured signals. Further developments on the material of this section can be found in Belkoura et al. (2009).

2.1 Single delay identification

Let us consider a first order system with a delayed input governed by:

$$\dot{y} + a\, y = b\, u(t-\tau), \tag{8}$$

where a, b, and τ are constant parameters, and the coefficient a is assumed to be known (for the moment). Consider also a step input $u = u_0 H$. A first order derivation yields

$$\ddot{y} + a\dot{y} = b\, u_0\, \delta_\tau. \tag{9}$$

By virtue of the annihilation procedure (7), the right hand side of equation (9) can be canceled by means of a multiplication with a function α such that $\alpha(\tau) = 0$, and the choice of the polynomial $\alpha(t) = t - \tau$ results in

$$t(\ddot{y} + a\dot{y}) = \tau\, (\ddot{y} + a\dot{y}). \tag{10}$$

As an equality of singular distributions, this relation doesn't make sense for any t (otherwise we would have $\tau = t$). However, $k \geq 2$ successive integrations (or convolution with regular function h_k with support $\in (0,\infty)$) result in functions equality from which the delay τ becomes available. More precisely, since supp $h_k * \delta_\tau \subset (\tau, \infty)$, we can easily show that all the obtained functions will vanish on $(0, \tau)$ and the delay is consequently not identifiable on this interval. Conversely, being nonzero for (almost) all $t > \tau$, the delay is (almost) everywhere identifiable on (τ, ∞) and given by the relation:

$$\tau = \frac{h_k * [t\ddot{y} + a\, t\dot{y})]}{h_k * [\ddot{y} + a\, \dot{y}]}, \qquad t > \tau, \tag{11}$$

The numerator and denominator of (11) are computed according to the pre-integration by parts:

$$t\dot{y} = (ty)^{(1)} - y, \tag{12}$$

$$t\ddot{y} = (ty)^{(2)} - 2\dot{y}, \tag{13}$$

avoiding any output derivatives. Figure 1 shows the realization scheme of the delay estimator, using the filter $h_3(s) = \frac{1}{s^3}$.

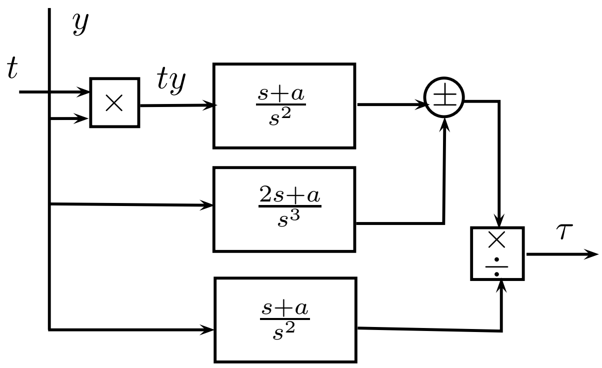

Fig. 1. Realization scheme of the delay estimator.

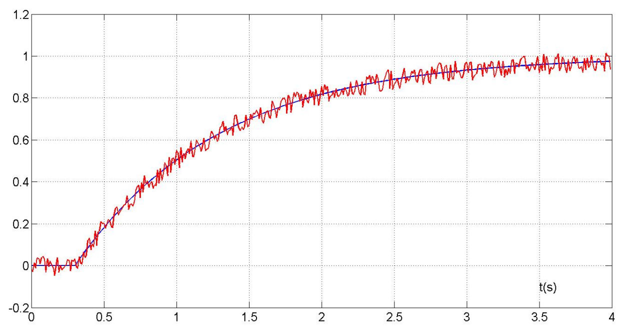

Fig. 2. Output trajectory of the delayed process (8).

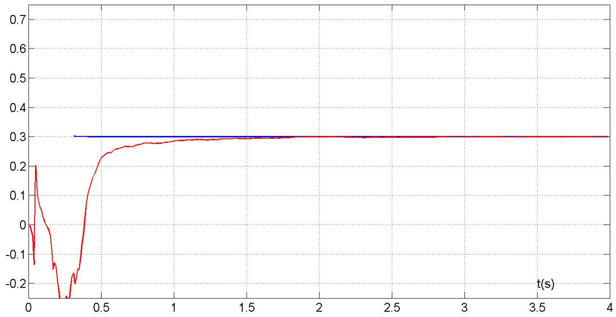

Fig. 3. Delay estimation of (8) in noise free (blue line) and noisy case (red line). (actual value $\tau = 0.3$)

2.2 Parameters and delay identification

Let us now consider a linear system with unknown parameters and delay described by:

$$G(s) = \frac{K\, e^{-\tau\, s}}{a_2 s^2 + a_1 s + 1}, \tag{14}$$

denoting $e = e^{-j\omega t}$, a derivation of the differential equation derived from (14), followed by the multiplication by $\alpha = (1 - \lambda e)$ result respectively in:

$$a_2 y^{(3)} + a_1 y^{(2)} + y^{(1)} = K\delta_\tau, \tag{15}$$

$$(1 - \lambda e)\ (a_2 y^{(3)} + a_1 y^{(2)} + y^{(1)}) = 0. \tag{16}$$

where the delay to be estimated is derived from $\lambda = e^{j\omega\tau} \in \mathbb{C}$, with tunable frequency ω. We shall focus on the identification of the coefficients $\{\lambda, a_2, a_1\}$, and provided a sufficiently large period $2\pi/\omega$, the delay is deduced from the unique argument $\tau = \arg(\lambda)/\omega$. Due to the terms λa_i, $i = 1, 2$, (16) is not linear w.r.t. the unknown coefficients, but may be written in the following form:

$$[(y^{(3)}, \cdots, y^{(1)}) - \lambda(ey^{(3)}, \cdots, ey^{(1)})] \begin{pmatrix} a_2 \\ a_1 \\ 1 \end{pmatrix} = 0. \tag{17}$$

As in the previous paragraph, successive integrations (or convolution with regular functions) transform the equality of singular distributions of (16) into one of continuous functions. Denoting $\Theta = (a_2, a_1, 1)^T$ the (normalized) vector of parameters, the specific structure of (17) leads to following generalized eigenvalue problem for possibly non square pencils:

$$(A_0 - \lambda\ A_1)\,\Theta = 0, \tag{18}$$

where, using a Matlab-like notation, the entries of the $m \times 3$ trajectory-dependent matrices A_0 and A_1 are given by

$$A_0(i,:) = h_i * (y^{(3)}, \cdots, y^{(1)}), \quad i = 1, \ldots, m,$$
$$A_1(i,:) = h_i * (ey^{(3)}, \cdots, ey^{(1)}) \quad i = 1, \ldots, m. \tag{19}$$

The use of successive filters $h_i(s)$ of relative degree > 2 allow one the one hand, to ensure causal relations from (17) and on the other hand, to obtain enough equations for a simultaneous estimation of both parameters and delay. The implementation of $A_0(i,j)$ and $A_1(i,j)$ is performed according to the integration by parts formulas. Therefore, the identification problem has been transformed into the eigenvalue problem (18) in which, at each t, the unknown delay $\tau = \arg(\lambda)/\omega$ is derived from one eigenvalue, while the parameters a_1, a_2 are obtained from the corresponding normalized eigenvector.

3. IDENTIFICATION APPROACH FOR THE UNSTRUCTURED CASE

When facing arbitrary inputs, the above annihilation procedure no longer applies, but algebraic estimation results can be still obtained by mean of an approach combining multiplication and cross convolution, as described below. Provided the system is initially at rest (null initial condition), state delay may also be identified.

3.1 The cross convolution approach

We first focus on a single delay identification regardless of any process dynamics. When considered on the whole real line, a delay between two functions $a(t)$ and $b(t)$ reads as in (20) and leads to (21) once multiplied by any deviated known function $\alpha(t - \tau)$.

$$a(t) = b(t - \tau), \tag{20}$$

$$\alpha(t - \tau)a(t) = (\alpha b)(t - \tau). \tag{21}$$

Using (2), a convolution product derived from these two relations results in equation (23) with no deviated argument in the original functions a and b.

$$[\alpha(t - \tau)a(t)] * b(t - \tau) = a(t) * (\alpha b)(t - \tau), \tag{22}$$

$$\Rightarrow [\alpha(t - \tau)a(t)] * b(t) = a(t) * (\alpha b)(t). \tag{23}$$

If the adopted function $\alpha(t - \tau)$ admits an expansion separating its arguments t and τ, i.e.:

$$\alpha(t - \tau) = \sum_{i \text{ finite}} \lambda_i(t)\mu_i(\tau), \tag{24}$$

for some known functions λ and μ, then an algebraic relation is obtained allowing for a non asymptotic and explicit delay formulation, as illustrated in the simple following examples:

$$\alpha(t) = t\ \Rightarrow \tau = \frac{ta * b - a * tb}{b * a} \tag{25}$$

$$\alpha(t) = e^{\gamma t}\ \Rightarrow e^{\gamma\tau} = \frac{b * e^{\gamma t}a}{a * e^{\gamma t}b} \tag{26}$$

Provided the involved convolution products are well defined, this delay formula holds for all nonzero values of their denominators. More precisely, if the signal b consists in measurements on $(0, \infty)$, then $\operatorname{supp} a \subset (\tau, \infty)$ and hence, by virtue of (3), both numerator and denominator of (25) and (26) have their support within (τ, ∞). Therefore, the delay is not identifiable for $t < \tau$. However, as in the finite dimensional case (see, e.g., Fliess and Sira-Ramirez (2007)), the input signal b being used in this algebraic approach does not necessarily exhibit the classical "persistency of excitation" requirement. Although a local loss of identifiability may occur due to the zero crossing of the denominator, only non trivial trajectories are required.

Once again, when facing derivatives, one of the nice features of multiplication by polynomial or exponential functions lies in the ability to use simple integration by parts formulas to avoid any derivation in the identification algorithm. The next paragraph illustrates the time lag identification for the simple first order case.

3.2 Application to a delay identification

Consider the following linear first order process with delayed input:

$$\dot{y} + y = k\,u(t-\tau), \tag{27}$$

which correspond to the formulation (20) with $a = \dot{y} + y$ and $b = ku$. In order to avoid multiplications by unbounded functions (polynomials), and hence the amplification of noise and neglected dynamics, a decaying exponential functions is considered and equation (26) reads:

$$\lambda = \frac{u * e^{\gamma t}(\dot{y} + y)}{(\dot{y} + y) * e^{\gamma t} u} \tag{28}$$

where we have denoted $\lambda = e^{-\gamma\tau}$ for some tunable positive parameter γ. Note that the static gain value k is not required nor identified. Denoting $e(t) = e^{-\gamma t}$ and taking into account the integration by parts formula, $\int e\dot{y} = ey + \gamma \int ey$, on gets:

$$\lambda = \frac{ey * u + (1+\gamma)\int(ey * u)}{eu * y + \int eu * y}, \tag{29}$$

while the delay is obtained from $\tau = \log(\lambda)/\gamma$. For this simple example, and since only a constant delay has to be identified, an additional step considering the integral of the square of equation (29) (*i.e.* $\int (29)^2$) avoids the possible singularities resulting from the zero crossing of the denominator $eu * y + \int eu * y$. This finally results in the delay estimation:

$$\lambda = \left[\frac{\int_0^t \left[u * ey + \gamma \int_0^\theta (u * ey)\right]^2 d\theta}{\int_0^t \left[eu * y + \int_0^\theta (eu * y)\right]^2 d\theta} \right]^{\frac{1}{2}}. \tag{30}$$

A simulation result with noisy data is depicted in Figure 4, for an input $u(t) = sin(t/2)sin(2t)(0.2 + sin(t/2))$, $\gamma = 0.2$, and a delay $\tau = 0.3\,s$. The simulation step size has been fixed to $0.05\,s$, and the integrals involved in the convolutions have been approximated by simple sums. The trajectory of the estimated delay based on equation (30) is depicted in Figure 5.

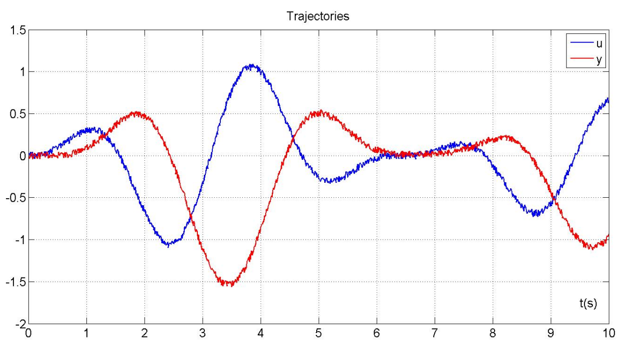

Fig. 4. Trajectories of the process in eq.(27).

3.3 Simultaneous parameters and delay identification

The aim of this paragraph is to provide a structure of the estimation problem when simultaneous parameters and delay identification is required. The above delay estimation procedure is considered in this section for signals a and b describing a linear system governed by:

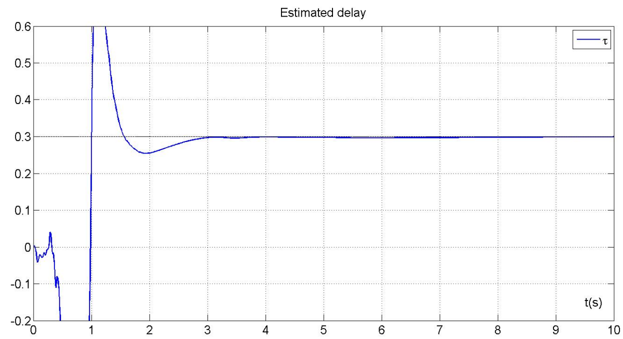

Fig. 5. Estimated delay of eq.(27).

$$\sum a_i\, y^{(i)}(t) = Ku(t-\tau), \tag{31}$$

whose left and right hand sides can be identified with the functions a and b of the previous section (equation 20) as:

$$a = \sum a_i\, y^{(i)}, \qquad b = Ku. \tag{32}$$

The exponential function $\alpha(t) = e^{-\gamma t}$, with $\gamma \in \mathbb{C}$ may be adopted for the multiplication step, leading to a reformulation of equation (23) as:

$$\lambda \sum a_i \left[u * (e^{-\gamma t} y^{(i)})\right] = \sum a_i \left[y^{(i)} * (e^{-\gamma t} u)\right], \tag{33}$$

where the unknown delay to be identified is contained in the unknown term $\lambda = e^{-\gamma\tau}$. Note that the process gain K has been removed by this procedure and will not be estimated. Depending on the regularity of the input u, equation (33) may be non causal and is non linear with respect to the delay and parameters a_i. As in the structured input case, successive filters $h_j(s)$ allow causal relations from and provide enough equations for a simultaneous estimation of both parameters and delay. Recalling $\Theta = (a_2, a_1, 1)^T$ the (normalized) vector of parameters, this results in the following estimation problem:

$$(A_0 - \lambda\, A_1)\,\Theta = 0, \tag{34}$$

$$A_0(i,j) = s^i h_j(s)[y * e^{-\gamma t} u](s), \tag{35}$$

$$A_1(i,j) = (s+\gamma)^i h_j(s)[e^{-\gamma t} y * u](s), \tag{36}$$

where the realization of each term of the filtering of equation (33), avoiding any measurement's derivative, is based on the familiar property of Laplace transform, $\mathcal{L}[e^{\gamma t} f(t)](s) = F(s-\gamma)$. In the available data $A_0(i,j)$ and $A_1(i,j)$, the notation $[a*b](s)$ emphasizes the measurement based convolution products subject to the filtering procedure. It is worth noticing that in case of a structured input u admitting a simple operational description (for instance a step u(s)=1/s), the realization of the entries of A_0 and A_1 reduces to filtering of y and $e^{-\gamma t} y$ as:

$$A_0(i,j) = s^i h_j(s) u(s+\gamma) y(s), \tag{37}$$

$$A_1(i,j) = (s+\gamma)^i h_j(s) u(s) y(s+\gamma) \tag{38}$$

3.4 Structured vs unstructured method

This section shows how the unstructured identification scheme generalizes the structured approach of Section 2. We illustrate with the simplest input case, i.e. a step input $b = 1/s$. Differentiation of (20) and (21) reads:

$$\dot{a} = \delta_\tau$$
$$\alpha(t-\tau) \times \dot{a} = (\alpha\delta) * \dot{a}.$$

In case of a polynomial choice $\alpha(t) = t$, $\alpha\delta = 0$, and this yields $(t-\tau) \times \dot{a} = 0$, leading after integration (or convolution with h) to

$$(h * \dot{a})\tau = [h * (t\dot{a})] \tag{39}$$

which correspond to (11) when expanding a as in (32). When considering cancellation with an exponential function $\alpha = e^{\gamma t}$, on has from (21) and the annihilation technique of the introductory Section:

$$\lambda(e\dot{a}) = (\alpha\delta) * \dot{a} = (\alpha(0)\delta) * \dot{a} = \dot{a} \tag{40}$$

and hence after integration:

$$h * \dot{a} = \lambda\, h * (e\dot{a}) \tag{41}$$

which correspond to the result obtained in (17) when expanding a as in (32). Therefore, the structured estimation technique can be viewed as a particular case of the the unstructured one in which, prior to the cross convolution operation, a differentiation has been used to reduce the right hand side of (20) to a singular distribution.

4. NUMERICAL RESOLUTION OF THE SIMULTANEOUS PARAMETERS AND DELAY IDENTIFICATION

When considering the identification problems (18) and (34), and for each value of t, the unknown delay consists in one eigenvalue while the parameters are derived from the associated normalized eigenvector. In order to uniquely estimate the desired eigenpair one has to consider additional integrations leading to a rectangular eigenvalue problem. However, as mentioned in Wright and Trefethen (2002), rectangular eigenvalue problems "have the awkward feature that most matrices have no eigenvalues at all, whilst for those that do, an infinitesimal perturbation will in general remove them". In Boutry et al. (2005), solving perturbed pencils is formulated in terms of the following a minimal perturbation approach: Given any $m \times n$ matrices A_0, A_1, find:

$$\min_{\bar{A}_0, \bar{A}_1 \{\lambda_k, v_k\}_{k=1}^n} \|A_0 - \bar{A}_0\|_F^2 + \|A_1 - \bar{A}_1\|_F^2 \tag{42}$$
$$\text{subject to} \quad \{(A_0 - \lambda A_1)v_k = 0, \quad \|v_k\|_2^2 = 1\}_{k=1}^n \tag{43}$$

where $\|()\|_F$ stands for the Frobenius norm. Except for the scalar ($n = 1$) case where analytical solution are obtained, the proposed algorithms are however asymptotic and cannot be considered online. To overcome this difficulty, this section will provide simple estimation algorithms based on projection techniques.

The developments are based on the QR decomposition of a matrix (Golub and Van Loan (2012)), and similar approaches using CS (Cosine Sine) or GSVD (Generalized singular value) decompositions have shown similar performances. We recall that any $m \times n$ matrix X can be factored as the product of an $m \times m$ unitary matrix Q ($Q^H Q = I$, where the symbol H stands for the conjugate transpose) and an $m \times n$ upper triangular matrix R. Further, this decomposition reads:

$$X = QR = [Q_x \ Q_y] \begin{bmatrix} R_x \\ 0 \end{bmatrix} = Q_x R_x \tag{44}$$

where R_x is $n \times n$ upper triangular matrix and Q_x has orthogonal columns. If X is full column rank and we require that the diagonal elements of R_x are positive, the decomposition $X = Q_x R_x$, also called thin factorization, is unique. Conversely, when X is rank deficient, a column permutation can be used such that the decomposition reads as follows where R_{11} is upper triangular and Π is a permutation matrix:

$$X\Pi^H = Q \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \tag{45}$$

4.1 A constrained eigenvalue problem

The material on constrained eigenvalue problems presented in this section can be found in Golub (1973). Consider a $n \times n$ constrained eigenvalue problem:

$$(B_0\lambda + B_1)\, x = 0,$$
$$\text{subject to} \quad C^H x = 0. \tag{46}$$

If $\operatorname{rank} C = q$, and using an orthogonal decomposition of the form:

$$C = Q^H \begin{pmatrix} R & S \\ 0 & 0 \end{pmatrix} \Pi \tag{47}$$

where R is upper triangular, S is $q \times (n-q)$, $Q^H Q = I$ and Π is permutation matrix, then the eigenvalues of the constrained problem are the eigenvalues of

$$(G_0 + \lambda G_1)\, z = 0, \tag{48}$$

for the matrices G_i, $i = 0, 1$ given by the $(n-q) \times (n-q)$ bottom right of the following matrix:

$$QB_iQ^H = \begin{pmatrix} \times & \times \\ \times & G_i \end{pmatrix} \tag{49}$$

The eigenvectors x_i of the original constrained problem are derived from those of the unconstrained one z_i using:

$$x_i = Q^H \begin{pmatrix} 0 \\ I_{n-q} \end{pmatrix} z_i, \quad i = 1, \ldots, s \leq q. \tag{50}$$

Note that in the symmetric problem considered in Golub (1973), $s = q$. Now let us consider our estimation problems given by (18) or (34), and denoted here:

$$(A_0\, \lambda + A_1)\Theta = 0, \tag{51}$$

and introduce the main assumption of this paper:

Assumption A1: The rectangular linear pencil (51) with $m \geq 2n$ admits, at each t, the unique eigenvector Θ with associated eigenvalue λ given by the delay.

Based on this assumption we shall be able to: (i) estimate the vector coefficient Θ trough the resolution of a linear system, (ii) estimate the eigenvalue λ an its associated delay trough a scalar equation. Consider the QR decomposition:

$$A_i = Q_i R_i, \quad i = 0, 1. \tag{52}$$

One may write

$$(A_0 + \lambda A_1) = (Q_0\, Q_1) \begin{pmatrix} \lambda R_0 \\ R_1 \end{pmatrix} \tag{53}$$

Multiplying by $(Q_0\, Q_1)^H$ yields:

$$\begin{pmatrix} I & Q_0^H Q_1 \\ Q_1^H Q_0 & I \end{pmatrix} \begin{pmatrix} \lambda R_0 \\ R_1 \end{pmatrix} \Theta = 0. \tag{54}$$

Now applying the (always defined) Schur Complement with respect to the top left term of the above matrix, and denoting:

$$P_0 = I - Q_0 Q_0^H, \tag{55}$$

the orthogonal projector onto the complement of range of A_0, our estimation problem (46) has been transformed into the constrained eigenvalue problem:

$$(R_0\lambda + Q_0^H Q_1 R_1)\Theta = 0, \tag{56}$$

$$Q_1^H P_0 Q_1 R_1 \Theta = 0. \tag{57}$$

Following Golub (1973), if the $(m-n)\times n$ matrix contraint $Q_1^H P_0 B_1$ is of rank r for some $r \leq n$, we can derive from these relations an eigenvalue problem of size $(n-r)\times(n-r)$. The eigenpair of (51) being assumed unique, $r = n-1$, leading to a scalar eigenvalue problem, and a matrix contraint of rank $n-1$ that uniquely determine the unknown vector parameter. Hence applying the decomposition (47) to the constraint (57):

$$Q_1^H P_0 Q_1 R_1 = Q_p^H \begin{pmatrix} R_p & S_p \\ 0 & 0 \end{pmatrix} \Pi_p, \tag{58}$$

we can state our estimation result as:

Proposition 2. Consider the estimation problem (51), the associated decomposition (52) and projector (55). Under assumption A1, the unknown (normalized) vector parameter Θ satisfies the linear equation.

$$Q_1^H P_0 A_1 \Theta = 0 \tag{59}$$

Moreover, and form the QR decomposition (58), the unknown delay is governed by the scalar equation:

$$(Q_p R_0 Q_p^H)_{nn}\ \ \lambda + (Q_p Q_0^H A_1 Q_p^H)_{nn} = 0 \tag{60}$$

Under our principal assumption, we have shown that the linear constraint in (57) contains enough information in order to determine the dynamic of the process by solving a linear system, and regardless of the delay. When considering the delay estimation, the resolution of equation (60) requires the orthogonal QR decomposition of $Q_1^H P_0 A_1$, which amounts to solving first the linear equation in the unknown parameters. Note also that in order for the considered projection approach to satisfy the rank condition, one must consider $m \times n$ matrices A_i $i = 0, 1$ with $m \geq 2n$.

By virtue of the projectors and rank properties, $P_0^2 = P_0$ and $\operatorname{rank} X^H X = \operatorname{rank} X$, a left multiplication by R_1^H of (59) and the uniqueness assumption also read:

$$P_0 A_1 \Theta = 0, \qquad \operatorname{rank} P_0 A_1 = n-1. \tag{61}$$

It should be stressed that whenever A_0 is not full column rank, the QR decomposition and hence the projector P_0 is not uniquely defined. In the next section we shall take advantage of the stationarity of Θ to provide a recursive estimation algorithm.

4.2 Vector parameter estimation

Without loss of generality, we may assume that eigenvalue $\lambda \neq 0$ for all t and hence consider the eigenvalue problem problem $(B_0 + \lambda^{-1} B_1)\Theta = 0$. Proceeding as in the previous section with the corresponding projector $P_1 = I - Q_1 Q_1^H$ and combining both linear constraints results in:

$$M\Theta = 0, \qquad M = \begin{bmatrix} A_1^H P_0 A_1 \\ A_0^H P_1 A_0 \end{bmatrix}. \tag{62}$$

Now since the vector parameter Θ is assumed here constant, one may apply any recursive method to update its estimation and deal with the noise effect. More precisely, following Kontoghiorghes et al. (1999), a block recursive least-squares (LS) estimation can be formulated as follows, where the subsequent matrices, including the matrix the matrix M in (62), will be denoted with a subscript t to indicate the running time. Let

$$Q_1^H M_1 = \hat{R}_1 = \begin{bmatrix} R_1 & u_1 \\ 0 & s_1 \end{bmatrix}, \tag{63}$$

be the QR decomposition of M_1, where R_1 is upper triangular, non singular matrix, and Q_1 has orthogonal columns. The least-squares estimation of Θ_1 is derived from the solution of

$$[R_1\, u_1]\Theta_1 = 0, \tag{64}$$

giving he initial estimate. Next, after computing the orthogonal factorization

$$Q_{t+1}^H \begin{bmatrix} \hat{R}_t \\ M_{t+1} \end{bmatrix} = \hat{R}_{t+1} = \begin{bmatrix} R_{t+1} & u_{t+1} \\ 0 & s_{t+1} \end{bmatrix}, \tag{65}$$

the updated LS estimator is obtained by solving

$$[R_{t+1}\, u_{t+1}]\Theta_{t+1} = 0. \tag{66}$$

It is worth noticing that from the QR decomposition of the time varying perturbed pencil matrices A_i, statistical distributions of the perturbations are generally not available. To the best of our knowledge, and unlike static decomposition where bound on the norms of the deviated factors Q and R can be estimated (see eg Stewart (1993)), there are no such bounds nor statistical distribution when considering time varying decomposition $Q(t)R(t)$. It is however clear that the sensitivity of our estimation problem highly depends on the time varying condition numbers $\kappa(A_i)$ of the involved matrices.

4.3 Delay estimation

The delay estimation can be obtained from the scalar equation (60) based on a QR decomposition of the matrix constraint in (57). Alternatively, one may use the updated vector parameters estimation of the above section, reducing the delay estimation problem to the following $(m \times 1)$ vector valued relation that can be solved in the least square sense:

$$a_0\, \lambda + a_1 = 0, \qquad a_i = A_i\Theta, \quad i = 0, 1. \tag{67}$$

As in the single delay case of Section 3.2, one can further apply the additional step consisting of integration an squaring to avoid possible local singularities resulting from the zero crossing. Note that in this scalar case ($n = 1$), the problem (42) admits an analytical solution (Boutry et al. (2005)), and the optimal solution for λ is given by the (+)-root of the quadratic equation:

$$\lambda^2 a_0^H a_1 + \lambda(a_1^H a_1 - a_0^H a_0) - a_1^H a_0 = 0, \tag{68}$$

the (+)-root being defined as $(-\beta + \sqrt{\beta^2 - 4\alpha\gamma})/(2\alpha)$ for a quadratic form $\alpha x^2 + \beta x + \gamma$. Finally, in case we adopt an harmonic multiplicative function

$$\alpha(t) = e^{j\omega t}, \quad \omega = 2\pi/\Delta, \qquad t \geq 0 \tag{69}$$

and using the Frobenius norm, the delay estimation problem (67) can be stated alternatively as:

$$\text{minimize} \quad \|a_0\lambda - a_1\|_F \tag{70}$$

$$\text{subject to} \quad \lambda^H \lambda = 1. \tag{71}$$

This correspond to the simplest (scalar) version of the orthogonal Procruste problem (Golub and Van Loan (2012)), considering how a_0 can be rotated into a_1. In this scalar case, the optimal solution is given by:

$$\lambda = e^{j\omega\tau} = a_0^H a_1/(\|a_0\|\|a_1\|) \tag{72}$$

The period Δ of the harmonic function is chosen sufficiently large to ensure a unique estimated delay from the argument of λ. While the two later methods provide an optimal solution at each t, a simple block recursive least square estimation can still be considered from (67) using the recursive approach of the previous Section where the $(m \times n)$ (respectively $(n \times 1)$) matrices M_t (resp. Θ_t) are replaced by the $(m \times 2)$ (resp. (2×1)) terms:

$$N_t = [a_0 \ a_1], \qquad \Lambda_t^T = [\lambda \ 1] \tag{73}$$

5. NUMERICAL IMPLEMENTATION AND SIMULATION EXAMPLE

The implementation of the estimation algorithms only requires an arbitrary smooth function $\alpha(t)$ for the multiplication, and $m \geq 2n$ linearly independent filters $h_i(t)$. The design of such optimal function and filters is a important and challenging problem, as the pencil matrices A_i in (51) should: a/ show favourable robustness properties with regard to noise, and b/span fairly independent spaces in order to provide, trough the orthogonal projections, well conditioned constraint relations in (57). This open problem is outside the scope of the paper, and we adopt here the harmonic multiplicative function in (69) and low pass filters:

$$h_i(s) = s^i/(1 + \zeta s)^{m+n}, \quad i = 1, \ldots, m. \tag{74}$$

The overall estimation algorithm can be described as follows: (i) form the matrix pencil entries A_0 and A_1 according to eq. (37). (ii) compute their QR factorizations and corresponding projectors P_i, $i = 0, 1$ (eq. (55)). (iii) apply the recursive algorithm described in eq. (63-66) for the vector parameter estimation. (iv): use the estimates $a_i = A_i\hat{\theta}$, $i = 0, 1$ to estimate the delay form eq. (72).

5.1 Simulation example

This example considers the of the second order delayed system described by: :

$$G(s) = \frac{1.2\,e^{-0.3s}}{0.6s^2 + s + 1}, \tag{75}$$

and subject to the input $u = \sin(2t) + .5\sin(t)$. The SNR is fixed to 45dB for both input and output signals, the discretization step is $2ms$, and the convolution products are performed using simple sums. The rectangular eigenvalue problem $(A_0 + A_1)\Theta = 0$ is formed by $m \times n$ matrices as described in (34), with $m = 2n = 4$. The pair Δ and ζ for the multiplicative function and filters is fixed to 4 and 1 respectively. Figure 6 shows the simulated trajectories. Figure 7 shows the time history of the estimated

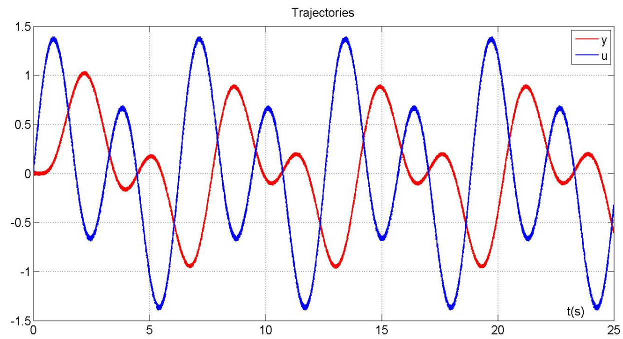

Fig. 6. Trajectories of eq.(75).

parameters $a_2 = 0.6$ and $a_1 = 1$ using a non recursive estimation (equation (62)), and in both free and noisy context. Note that as mentioned in Section 3, the gain process is not required nor estimated by this approach. Some fluctuations can be noticed in the noisy context, probably due to the vicinity of singularities of the matrix pencil (the pencil is singular when both matrices A_0 and A_1 are not column full rank). The benefits of using a

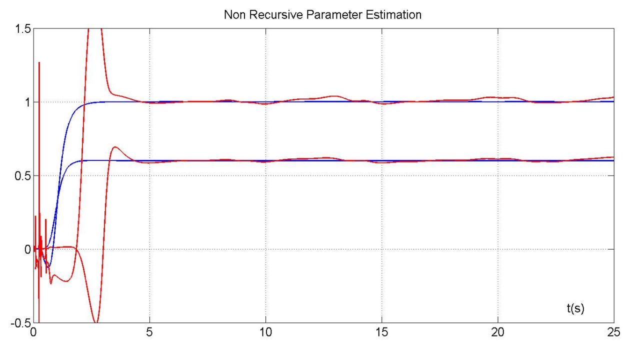

Fig. 7. Non recursive parameters estimation of eq.(75) in noise free (blue lines) and noisy context (red lines)

recursive parameter estimation is depicted in Figure 8, where the local singularities do not affect the estimation process. Finally, the estimated delay is shown in Figure 9, in noise free context(black line), noisy context using the minimal Frobenius norm approach described in (71) and (72) (blue line), and a noisy context with block recursive estimation using (73)(red line).

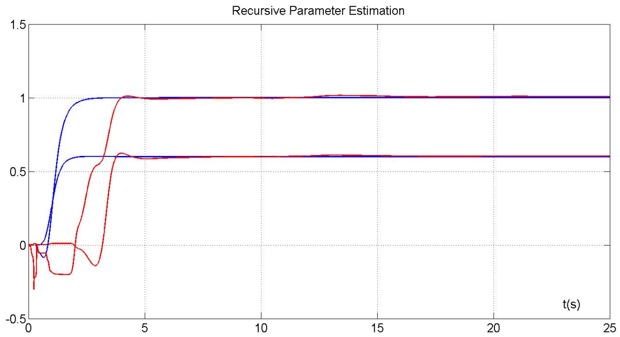

Fig. 8. Recursive parameters estimation of eq.(75) in noise free (blue lines) and noisy context (red lines)

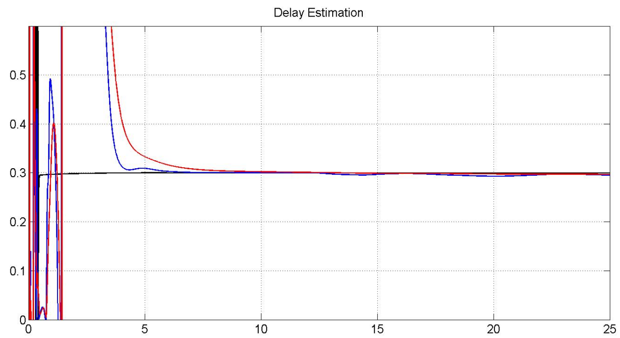

Fig. 9. Estimated delay (75) in noise free case (black line), noisy context with Procruste problem approach (eq. (71), blue line), and noisy context with updated estimation (eq. (73), red line)

6. CONCLUSION

The existing delay and parameters estimation results based on structured inputs have been extended to arbitrary ones, using a convolution approach. A connection between the two formalisms has been presented. Moreover, algorithms allowing for updating estimations and avoiding possible singularities are provided. Extensions to multivariable and multidelay cases, as well as the ability to tackle slowly varying parameters and delays, thanks to compact support filters, are under investigation. The ability to estimate parameters and delays using non asymptotic methods may provide new perspectives for real time control procedures.

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