The Fokker-Planck equation for a Stochastic Single Machine-Infinite Bus System

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Abstract: In power systems, the single machine-infinite bus system is formalized as a second-order differential equation and the equation is called a machine swing equation. After accounting for the noise influence, the Single Machine-Infinite Bus (SMIB) system assumes the structure of stochastic differential equations. This paper analyses a stochastically influenced SMIB system using the Fokker-Planck equation. The Fokker-Planck equation has a central importance in the theory of Markov stochastic processes. That has found its applications in stability, estimation and control of stochastic systems as well.
In this paper, first we develop the Fokker-Planck model for the SMIB system. Then, we achieve the noise analysis of the stochastic single machine infinite bus system by deriving conditional moments. It is imperative to know 'conditional moments' before designing a dynamic controller for the power system involving single machine infinite bus system. This paper bridges a 'gap between stochastic differential equations, partial differential equations and power systems dynamics'as well.

Keywords: Kolmogorov-Fokker-Planck equation, Single Machine-Infinite Bus (SMIB), Itô stochastic differential equation, partial differential equations

1. INTRODUCTION

In power systems, the Single Machine Infinite Bus system is well studied. The system accounts for the generator excitation, transmission line parameters and terminal voltage. The stability of equilibrium point of the machine can be examined using swing equation in combination with the notion of the derivative of the Lyapunov function. A second-order non-linear differential equation describes the machine swing equation. After accounting for correction terms stemming from the random perturbations associated with the single machine-infinite bus system, stochastic differential system formalism arises. The random forcing correction terms are attributed to renewable energy generations, random loads, the stochastic perturbations of rotor speed, rotor vibrations due to electrical harmonics, mechanical asymmetry and aging, damping coefficient, exogenous voltage flicker in the power system etc. The resulting stochastic system is described by a vector Stochastic Differential Equation (SDE). Generally, random perturbations are modelled as Gaussian white noise processes, generalized stochastic processes. Mumford (2000) argues stochastic differential equation descriptions in his philosophical paper in lieu of the deterministic setting for dynamical systems. The SDE description will refine stochastic stability conditions, estimation algorithms and control laws for dynamical systems. The white noise-driven stochastic SMIB system and effect of the noise intensity on power systems are nicely explained in (Wei and Luo, 2009), see (Wang and Crow, 2013) as well. In their paper, first the Fokker-Planck equation for a stochastic SMIB system was developed and then numerical studies of the SMIB were demonstrated. The numerical experimentation is about the conditional probability density trajectory of the SMIB system, which utilizes a system of parameters and initial data. (Odun-ayo and Crow, 2013) analysed power system stability using stochastic energy function. (Qin and Li, 2014) added randomness in damping coefficient and explained a chaotic behaviour of power systems by choosing a set of noise intensities. To understand noise influence on the erosion of safe basins in power systems, (Wei et al., 2010) will be a good source, see (Zhang et al., 2012) as well. The voltage analysis of the stochastic SMIB system is available in (Ghanavati et al., 2013; Cotilla-Sanchez et al., 2012), which are relatively quite scarce.

The intent of this paper is to develop and analyse single machine-infinite bus system using a formal setting of stochastic interpretations in lieu of the white noise setting. After accounting for random perturbations in deterministic dynamics of the SMIB system, we arrive at a randomly perturbed SMIB system. Then, the Fokker-Planck equation, a celebrated stochastic method, is utilized to accomplish the noise analysis of the stochastic problem considered here. The Fokker-Planck equation allows to study Markovian stochastic differential systems in the sense of conditional probability density evolution. After combining the Fokker-Planck equation and scalar stochastic function, we arrive at conditional moment evolution equations. Notably, we exploit the notion of conditional probability density as well as conditional moment evolution equations for the stochastic system considered here in lieu of conditional probability density (Wang and Crow, 2000). This paper demonstrates an appealing application of partial differential equations, stochastic differential equations to power system dynamics.

This paper is organized as follows: section 2 derives a theory of a stochastic single machine-infinite bus system. Section 3 discusses the conditional mean and variance equations, derived using the Kolmogorov-Fokker-Planck equation. Section 4 is about numerical simulations. Concluding remarks are given in section 5.

2. STOCHASTIC SINGLE MACHINE-INFINITE BUS (SMIB) SYSTEM

In deterministic setting, the swing equation of a Single Machine-Infinite Bus (SMIB) system is given by the following second-order non-linear differential equation (Kundur, 1994; Chen et al., 2005):

$$M\ddot{\delta}+D\dot{\delta}+\frac{VE'_a}{X}\sin\delta=P_m. \tag{1}$$

Note that the term V denotes the voltage magnitude of the infinite bus, M and D are the combined inertia constant and the damping coefficient of the generator and turbine respectively. The power system parameters E'_a and δ are the transient emf and the rotor angle of the generator respectively. The reactance X is the sum of the generator transient reactance X'_d and the line reactance X_l. The term P_m is the input mechanical power and $P_e=\frac{VE'_a}{X}\sin\delta$ is the electrical power of generator. The schematic diagram of equation (1) is illustrated in figure (1) of the paper.

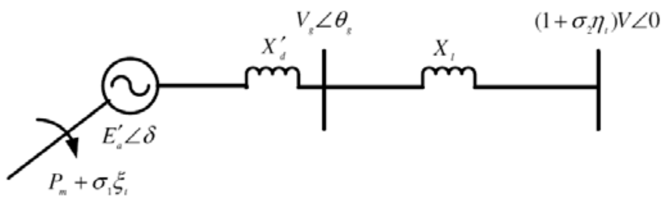

Fig. 1. Stochastic single machine-infinite bus system.

Here, we revisit the swing equation that accounts for noise in voltage magnitude of the infinite bus, $V(1+\sigma_2\eta_t)$, and the input mechanical power with additive noise $P_m+\sigma_1\xi_t$. Thus, we get

$$M\ddot{\delta}+D\dot{\delta}+\frac{(1+\sigma_2\eta_t)VE'_a}{X}\sin\delta=P_m+\sigma_1\xi_t, \tag{2}$$

where η_t and ξ_t are the independent white Gaussian noise processes. That are added to the voltage magnitude of the infinite bus and mechanical power respectively. The parameters σ_1,σ_2 are the noise intensities of the white noise processes. The mechanical power P_m assumed to be constant.

$$\ddot{\delta}=-\frac{D}{M}\dot{\delta}-\frac{VE'_a}{M\ X}\sin\delta+\frac{P_m}{M}+\frac{\sigma_1}{M}\dot{B}_1(t)-\frac{\sigma_2VE'_a}{M\ X}\sin\delta\ \dot{B}_2(t).$$

For the notational clarity and consistence, we replace the terms ξ_t and η_t with $\dot{B}_1(t)$ and $\dot{B}_2(t)$ respectively. The terms $B_1(t)$ and $B_2(t)$ are two independent Brownian motion processes. In phase space formulations, we consider the state vector $y_t=(y_1,y_2)^T=(\delta,\omega)^T$, where $\omega=\dot{\delta}$ is the angular velocity of the rotor. Thus,

$$dy_1=y_2dt,$$

$$dy_2=(-\frac{D}{M}y_2-\frac{VE'_a}{M\ X}\sin y_1+\frac{P_m}{M})dt$$
$$+\frac{\sigma_1}{M}dB_1(t)-\frac{\sigma_2VE'_a}{M\ X}\sin y_1dB_2(t).$$

Furthermore, the above system can be re-stated as

$$dy_t=f(y_t,t)dt+G(y_t,t)dB_t, \tag{3}$$

where,

$$f(y_t,t)=\begin{pmatrix} y_2 \\ -\frac{D}{M}y_2-\frac{VE'}{M\ X}\sin y_1+\frac{P_m}{M}\end{pmatrix},$$

$$G(y_t,t)=\begin{pmatrix} 0 & 0 \\ \frac{\sigma_1}{M} & -\frac{\sigma_2VE'_a}{M\ X}\sin y_1\end{pmatrix},$$

$$B_t=\left(B_1(t)\quad B_2(t)\right)^T.$$

Here, the term $f(y_t,t)$ is the system non-linearity and the $G(y_t,t)$ is the process noise coefficient matrix. We denote a vector Brownian motion using the notation B_t.

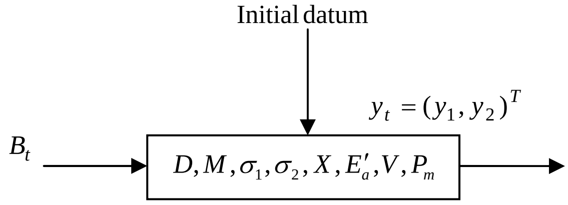

Fig. 2. An SMIB system SDE diagram.

Figure 2 shows the SMIB system SDE diagram. The parameters $D,M,\sigma_1,\sigma_2,X,E'_a,V,P_m$ are the system parameters of the stochastic system of the paper.

3. A FOKKER-PLANCK EQUATION FOR A STOCHASTIC SMIB SYSTEM

The Fokker-Planck equation is a parabolic linear homogeneous equation of order two in partial differentiation for the transition probability density. For Markov processes, the conditional probability density becomes the transition probability density. The Fokker-Planck equation accounts for a time derivative of the conditional proability density, a first-order spatial derivative and a second-order spatial derivative. In literature, the Fokker-Planck equation has found applications in theoretical studies of stochastic processes as well as found its striking applications in practical problem, e.g. the two body problem (Sharma and Parthasarathy, 2007) Duffing-van der Pol system (Sharma, 2008), the wind turbine-generator system (Hirpara and Sharma, 2014), phase noise analysis (Chow and Zhou, 2007) as well as underwater vehicles dynamic (Sharma and Hirpara, 2014). After applying ‘the Kolmogorov-Fokker-Planck equation for Markov processes’ (Karatzas and Shreve, 1988) to the randomly perturbed system of the paper, we get

$$dp = \left(-\sum_i \frac{\partial f_i(y\ ,\ t)p}{\partial y_i} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 (GG^T)_{ij}(y,t)p}{\partial y_i\,\partial y_j}\right)dt$$

$$= \left(-\frac{\partial y_2 p}{\partial y_1} + \frac{\partial}{\partial y_2}\left(\frac{D}{M}y_2\right)p + \frac{\partial}{\partial y_2}\left(\frac{VE'_a}{M\ X}\right)\sin y_1 p + \frac{1}{2}\frac{\sigma_1^2}{M^2}\frac{\partial^2 p}{\partial y_2^2}\right.$$

$$\left.+\frac{1}{2}\frac{\sigma_2^2 V^2 E_a'^2}{M^2X^2}\sin^2 y_1\frac{\partial^2 p}{\partial y_2^2}\right)dt, \tag{4}$$

where the term p denotes the conditional probability density $p = p(y,t|y_{t_0},t_0)$. In the Kolmogorov-Fokker-Planck operator $L(.)$ setting, equation (4) is given by $dp = L(p)dt$. The Kolmogorov backward operator $L'(.)$ for the Itô SDE is the adjoint of the forward operator $L(.)$. The Kolmogorov backward operator $L'(.)$ becomes

$$L'(.) = \sum_i f_i\frac{\partial(.)}{\partial y_i} + \frac{1}{2}\sum_{i,\ j}(GG^T)_{ij}(y,t)\frac{\partial^2(.)}{\partial y_i\,\partial y_j}$$

$$= y_2\frac{\partial(.)}{\partial y_1} - \left(\frac{D}{M}y_2\right)\frac{\partial(.)}{\partial y_2} - \left(\frac{VE'_a}{M\ X}\right)\sin y_1\frac{\partial(.)}{\partial y_2} + \frac{1}{2}\frac{\sigma_1^2}{M^2}\frac{\partial^2(.)}{\partial y_2^2}$$

$$+\frac{1}{2}\frac{\sigma_2^2V^2E_a'^2}{M^2X^2}\sin^2 y_1\frac{\partial^2(.)}{\partial y_2^2}. \tag{5}$$

The Fokker-Planck operator in combination with the Kolmogorov backward operator is utilized to derive the conditional moment evolution equation. The conditional moment evolution $d\langle\phi(y_t)\rangle$ for the stochastic system is

$$d\langle\phi(y_t)\rangle = \langle L'\phi,\ p\rangle dt = \langle\phi, Lp\rangle dt$$

$$= \left(\left\langle f^T(y_t,t)\frac{\partial\phi(y_t)}{\partial y_t}\right\rangle + \frac{1}{2}\left\langle tr((GG^T)(y_t,t)\frac{\partial^2\phi(y_t)}{\partial y_t\partial y_t^T})\right\rangle\right)dt, \tag{6}$$

Note that the notations $\langle\ \rangle$ and $\langle,\rangle$ are different and denote the conditional expectation operator and the inner product respectively. After considering $\phi(y_t) = y_i$ as well as $\phi(y_t) = y_iy_j$ where, $1 \le i \le n, 1 \le j \le n,$ we get the exact conditional mean and conditional variance evolutions, i.e. (Jazwinski, 1970), pp. 136-137),

$$d\langle y_i\rangle = \langle f_i(y_t,t)\rangle dt, \tag{7}$$

$$dP_{ij} = d\langle y_iy_j\rangle - d\langle y_i\rangle\langle y_j\rangle,$$

$$= (\langle f_iy_j\rangle - \langle f_i\rangle\langle y_j\rangle + \langle y_if_j\rangle - \langle y_i\rangle\langle f_j\rangle$$

$$+\langle (GG^T)_{ij}(y_t,t)\rangle)dt, \tag{8}$$

Where

$$\langle y_i\rangle = E(y_i(t)|y_{t_0},t_0),\ P_{ij} = \langle (y_i - \langle y_i\rangle)(y_j - \langle y_j\rangle)\rangle.$$

A system of exact evolution equations, equation (7) and (8), is not convenient form for numerical experiments. For this reason, we utilize the following higher-order conditional mean and variance equations (Patel and Sharma, 2014).

$$d\langle y_i(t)\rangle = \left(f_i(\langle y_t\rangle,t) + \frac{1}{2}\sum_{p,q}P_{pq}\frac{\partial^2 f_i(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle}\right)dt, \tag{9}$$

$$dP_{ij} = \left(\sum_p P_{ip}\frac{\partial f_j(\langle y_t\rangle,t)}{\partial\langle y_p\rangle} + \frac{1}{2}\sum_{p,q,r}P_{ip}P_{qr}\frac{\partial^3 f_j(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\,\partial\langle y_q\rangle\,\partial\langle y_r\rangle}\right.$$

$$+\sum_p P_{jp}\frac{\partial f_i(\langle y_t\rangle,t)}{\partial\langle y_p\rangle} + \frac{1}{2}\sum_{p,q,r}P_{jp}P_{qr}\frac{\partial^3 f_i(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\,\partial\langle y_q\rangle\,\partial\langle y_r\rangle}$$

$$\left.+(GG^T)_{ij}(\langle y_t\rangle,t) + \frac{1}{2}\sum_{p,q}P_{pq}\frac{\partial^2(GG^T)_{ij}(\langle y_t\rangle,t)}{\partial\langle y_p\rangle\partial\langle y_q\rangle}\right)dt. \tag{10}$$

After combining equations (3), (9) and (10), we get the following set of coupled conditional moment evolution equations, the conditional mean evolution and the variance evolutions for the stochastic problem of concern here:

$$d\langle y_1\rangle = \langle y_2\rangle dt, \tag{11}$$

$$d\langle y_2\rangle = \left(-\frac{D}{M}\langle y_2\rangle + \left(\frac{P_{11}}{2} - 1\right)\frac{VE'_a}{M\ X}\sin\langle y_1\rangle + \frac{P_m}{M}\right)dt, \tag{12}$$

$$dP_{11} = 2P_{12}dt, \tag{13}$$

$$dP_{22} = (2(-P_{12}\frac{VE'_a}{M\ X}\cos\langle y_1\rangle - P_{22}\frac{D}{M}) + \frac{\sigma_2^2 V^2 E_a'^2}{2M^2\ X^2} + \frac{\sigma_1^2}{M^2}$$

$$+ (P_{11} - \frac{1}{2})\frac{\sigma_2^2 V^2 E_a'^2}{M^2\ X^2}\cos 2\langle y_1\rangle + P_{12}P_{11}\frac{VE'_a}{M\ X}\cos\langle y_1\rangle)dt, \quad (14)$$

$$dP_{12} = dP_{21} = (-P_{11}\frac{VE'_a}{M\ X}\cos\langle y_1\rangle - P_{12}\frac{D}{M} + P_{22}$$

$$+ \frac{1}{2}P_{11}P_{11}\frac{VE'_a}{M\ X}\cos\langle y_1\rangle)dt, \quad (15)$$

In the integral setting, the mean evolutions can be recast as

$$\langle y_1(t)\rangle = \langle y_1(t_0)\rangle + \int_{t_0}^{t}\langle y_2(\tau)\rangle d\tau, \quad (16)$$

$$\langle y_2(t)\rangle = \langle y_2(t_0)\rangle + \int_{t_0}^{t}(-\frac{D}{M}\langle y_2(\tau)\rangle + (\frac{P_{11}(\tau)}{2} - 1)$$

$$\frac{VE'_a}{M\ X}\sin\langle y_1(\tau)\rangle + \frac{P_m}{M})d\tau. \quad (17)$$

The conditional variance evolutions can be restated as

$$P_{11}(t) = P_{11}(t_0) + \int_{t_0}^{t} 2P_{12}(\tau)d\tau, \quad (18)$$

$$P_{22}(t) = P_{22}(t_0) + \int_{t_0}^{t}(2(-P_{12}(\tau)\frac{VE'_a}{M\ X}\cos\langle y_1(\tau)\rangle - P_{22}(\tau)\frac{D}{M})$$

$$+ \frac{\sigma_2^2 V^2 E_a'^2}{2M^2\ X^2} + \frac{\sigma_1^2}{M^2} + (P_{11}(\tau) - \frac{1}{2})\frac{\sigma_2^2 V^2 E_a'^2}{M^2\ X^2}\cos 2\langle y_1(\tau)\rangle$$

$$+ P_{12}(\tau)P_{11}(\tau)\frac{VE'_a}{M\ X}\cos\langle y_1(\tau)\rangle)d\tau, \quad (19)$$

$$P_{12}(t) = P_{12}(t_0) + \int_{t_0}^{t}(-P_{11}(\tau)\frac{VE'_a}{M\ X}\cos\langle y_1(\tau)\rangle - P_{12}(\tau)\frac{D}{M}$$

$$+ P_{22}(\tau) + \frac{1}{2}P_{11}(\tau)P_{11}(\tau)\frac{VE'_a}{M\ X}\cos\langle y_1(\tau)\rangle)d\tau. \quad (20)$$

4. NUMERICAL SIMULATIONS

We consider the following first set of intial conditions and system parameters (Ghanavati et al., 2013) for the numerical simulations of the stochastic SMIB system:

$V = 1.0$ pu, $E'_a = 1.2$ pu, $D = 0.03$ pu/rad/sec, $X'_d = 0.15$ pu,

$H = 4$ MW/MVA, $X_l = 0.1$ pu, $X = 0.25$ pu, $\sigma_1 = 0.08$,

$\omega_s = 2\pi \times 60 = 376.8$ rad/sec, $M = \frac{2H}{\omega_s} = 0.02123, \sigma_2 = 0.06$,

$\langle y_1(0)\rangle = 1$ rad, $\langle y_2(0)\rangle = 2$ rad/sec, $P_m = 1$ pu,

$P_{11}(0) = 1\ \text{rad}^2$, $P_{12}(0) = 0\ \text{rad}^2/\text{sec}$, $P_{22}(0) = 1\ \text{rad}^2/\text{sec}^2$.

Note that the above system parameters are associated with figure (1) and equation (3) of the paper. Since numerical simulations of the SDEs are relatively scarce, we explain the method and intent of simulations in some detail. Numerical simulations of the SDEs can be achieved by discretizing stochastic differential equations as well as ordinary differential equations. The stochastic system of the paper assumes the structure of a vector stochastic differential equation. It is important to note that the Fokker-Planck equation of the stochastic system of this paper is a partial differential equation and the conditional moment of the stochastic system is also described as a system of coupled conditional mean and variance equations. The coupled conditional mean and variance equations assume the structure of Ordinary Differential Equations. Numerical simulations of stochastic differential equations can be found in an authoritative book authored by (Kloeden and Platen, 1999).

The intent of numerical simulations of stochastic differential equations as well as their estimation equations is the two-fold. First, the SDE simulations show the qualitative characteristics of the stochastic SMIB system. That reveal the noisy trajectories are apart from the noise-free state trajectories. Thus, the problem of analyzing stochasticity of dynamical systems is imperative and suggest 'respect the stochasticity of real physical systems'. Secondly, 'examine the efficacy of estimation equations of the stochastic systems'. Generally, we examine the efficacy of estimation equations in the theoretical and numerical studies for the stochastic systems with bounded state trajectories. Stochatsic systems with unbounded state trajectories are of little interest. Thus, we choose the system parameters of stochastic systems that reveal bound state trajectories. A comparison between the state trajectories of noisy and noise-free cases of the stochastic system considered here is demonstrated in figures (3) and (4) of the paper. In figures (3) and (4), the solid line (--) line trajectories denote noise-free trajectories and the dotted line (…) trajectories denote noisy trajectories of the states. In figures (5)-(6), the solid line (-) trajectories denote the lower-order conditional variance state trajectories of the stochastic system of the paper. On the other hand, the dotted line trajectories denote the higher-order conditional variance state trajectories of the stochastic system of the paper. The first-order estimation, a lower-order, preserves some of the qualitative characteristics of the non-linear stochastic systems with greater non-linearities. On the other hand, the higher-order estimation, e.g. third-order estimations, preserves qualitative characteristics of square and cubic non-linearities completely. Thus, the higher-order estimation equations of the paper, equations (11)-(15), lead to the better state estimates in the conditional variance sense. Equations (16)-(20) denote the higher-order estimation equations in the integral setting.

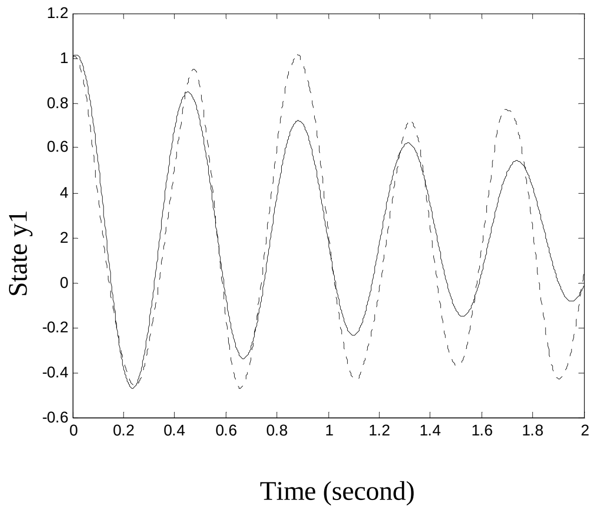

Fig. 3. A comparison between perturbed and unperturbed trajectories.

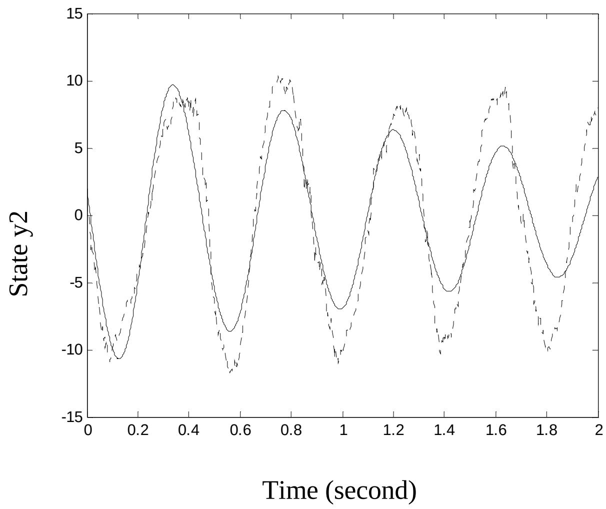

Fig. 4. A comparison between perturbed and unperturbed trajectories.

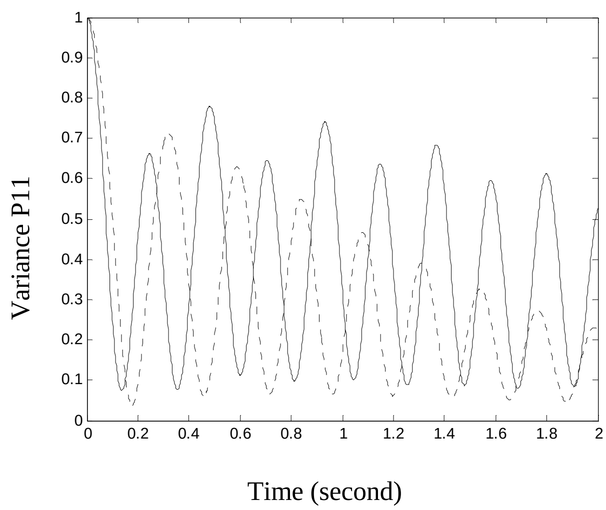

Fig. 5. A comparison between the two variance trajectories.

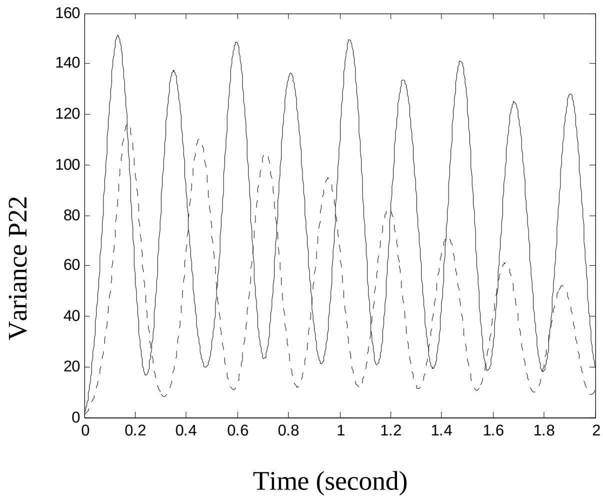

Fig. 6. A comparison between the two variance trajectories.

Here, we demonstrate the numerical simulations of the stochastic system of this paper by considering the second set of system parameters, i.e.

$$V = 1.0 \text{ pu}, E'_a = 1.2 \text{ pu}, D = 0.1 \text{ pu/rad/sec}, X'_d = 0.15 \text{ pu},$$

$$H = 4 \text{ MW/MVA}, X_l = 0.1 \text{ pu}, X = 0.25 \text{ pu}, \sigma_1 = 0.08,$$

$$\omega_s = 2\pi \times 60 = 376.8 \text{ rad/sec}, \ M = \frac{2H}{\omega_s} = 0.02123, \sigma_2 = 0.06,$$

$$\langle y_1(0) \rangle = 1 \text{ rad}, \langle y_2(0) \rangle = 2 \text{ rad/sec}, \ P_m = 1 \text{ pu},$$

$$P_{11}(0) = 1 \text{ rad}^2, P_{12}(0) = 0 \text{ rad}^2/\text{sec}, \ P_{22}(0) = 1 \text{ rad}^2/\text{sec}^2.$$

A careful observation reveals that the second set is different from the first set in the damping sense of the Single Machine-Infinite Bus system. The second set is intended to achieve the numerical simulations of a ‘relatively’ larger damping machine. Figures (7) and (8) show a comparison between the noisy and noise-free trajectories that utilize the second set of data.

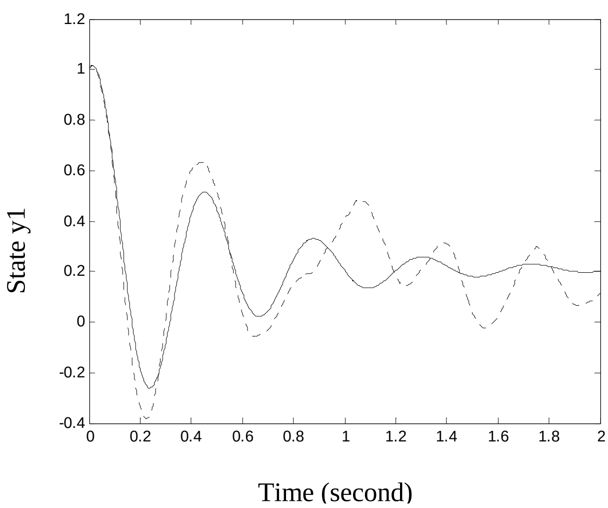

Fig. 7. A comparison between perturbed and unperturbed trajectories.

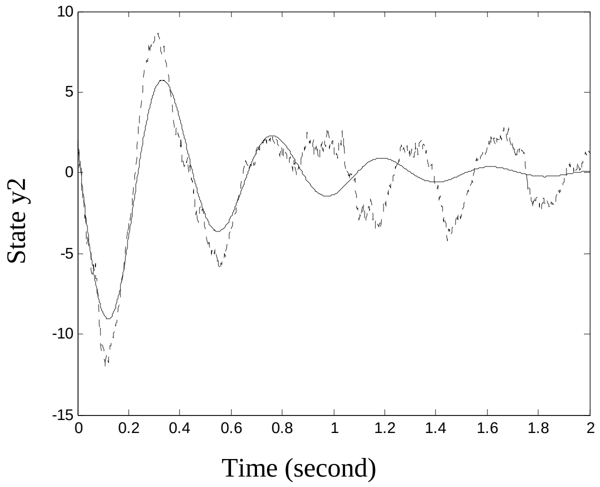

Fig 8. A comparison between perturbed and unperturbed trajectories.

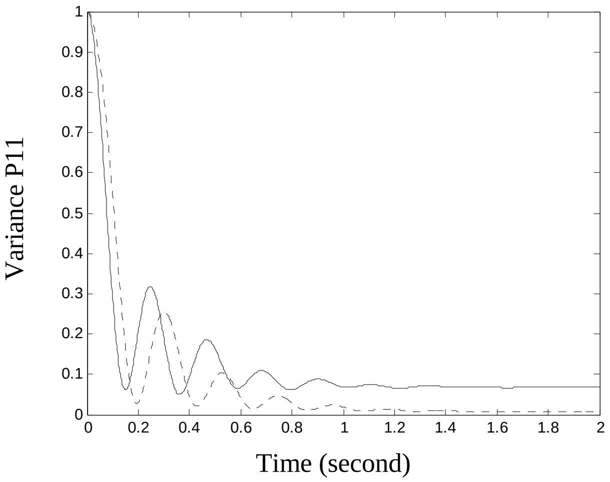

Fig. 9. A comparison between the two variance trajectories.

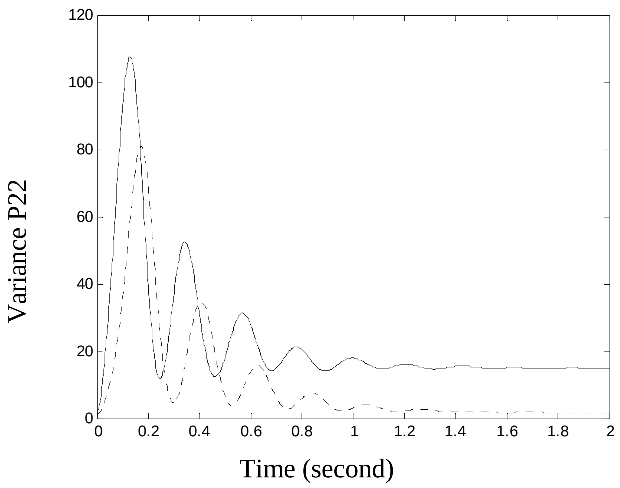

Fig. 10. A comparison between the two variance trajectories.

Figures (9)-(10) display the conditional variance trajectories of the higher-order and lower estimation equations. Note that the graphical notations of the second set of data are same as that of the first set of data. The numerical simulations of figures (9)-(10) reveal the efficacy of estimation equations in the conditional variance sense, see the dotted line trajectories of figures (9)-(10). That can be argued in a similar manner as adopted in the first case.

5. CONCLUSION

The main contribution of this paper is to analyse the stochasticity of the SMIB system using the Itô stochastic differential equation framework. Numerical simulations suggest the efficacy of equations (11) to (15) for the noise analysis of the SMIB system for the case in which 'observation are valueless'. Note that the Fokker-Planck equation is an appealing mathematical method for the noise analysis of dynamical systems, where observations are valueless. This paper does that.

Another theoretical contribution of this paper is to reveal connections between stochastic differential equations, partial differential equations and power systems dynamics. Thus, this paper is revealing. That brings the notions of stochastic differential equations and power systems dynamics together for first time. The contributions of this paper are philosophical as well as theoretical, since this paper introduces formal stochastic processes, which are very less explored, into power system dynamics.

Finally, this paper demonstrates a connection between the notion of conditional characteristic function and the parameters of the SMIB system. That is not available in literature yet, see the *appendix* of the paper.

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APPENDIX

Here, we explain briefly the machine swing equation, then stochasticity of the single machine-infinite bus system and finally, their conditional characteristic function evolution equation. The dynamic equation governing the motion of the machine rotor of a three-phase synchronous generator is called the swing equation. In rotational systems, the net accelerating torque acting on a rotating body is the product of the moment of inertia of the rotor times its angular acceleration. That hinges on Newtonian mechanics. The equation for the rotor motion is given by

$$J\alpha_m = T_m - T_e - T_d = T_a,$$

where J is the moment of inertia of the rotating masses in kgm^2, α_m is rotor angular acceleration, $rad/\sec^2$, T_m is the mechanical torque in Nm, T_e is the electrical torque in Nm, T_d is the damping torque in Nm and T_a is the net accelerating torque in Nm. The rotor angular acceleration is defined as

$$\alpha_m = \frac{d\omega_m}{dt} = \frac{d^2\theta_m}{dt^2},$$

where $\omega_m = \dfrac{d\theta_m}{dt}$ is the rotor angular velocity in $rad/\sec$ and θ_m is the rotor angular position with respect to a stationary axis in rad. It is convenient to measure the rotor angular position with respect to a synchronously rotating reference axis and accordingly, it is defined as

$$\theta_m = \omega_{ms}t + \delta_m,$$

where ω_{ms} is the synchronous angular velocity of the rotor $rad/\sec$ and δ_m is the rotor angular position with respect to a synchronously rotating axis in 'rad'. Then, by taking the second order derivative of θ_m with respect to time, we get

$$\frac{d^2\theta_m}{dt^2} = \frac{d^2\delta_m}{dt^2}.$$

After combining the above preceding equations, we get

$$J\frac{d^2\theta_m}{dt^2} = J\frac{d^2\delta_m}{dt^2} = T_m - T_e - T_d = T_a.$$

Accordingly, by multiplying the both sides of equation by angular velocity ω_m, the above equation can be written as

$$J\omega_m\frac{d^2\delta_m}{dt^2} = \omega_m T_m - \omega_m T_e - \omega_m T_d.$$

Recall that power is the product of torque and angular velocity, we get

$$J\omega_m \frac{d^2\delta_m}{dt^2} = P_m - P_e - P_d,$$

where P_m is the mechanical power in watt, P_e is the electrical power in watt, and P_d is the damping power in watt. At the synchronous speed ω_{ms}, the angular momentum $M = J\omega_{ms}$. Thus, the above acceleration equation is recast as

$$M\frac{d^2\delta_m}{dt^2} = P_m - P_e - P_d.$$

Here, we introduce the notion of the normalized inertia constant H, i.e.

$$H = \frac{\text{stored kinetic energy at synchronous speed}}{\text{three-phase rating of the genrator}}$$

$$= \frac{J\omega_{ms}^2}{2S_B} = \frac{M\omega_{ms}}{2S_B}.$$

Thus, the swing equation in the per unit setting can be re-written as

$$\frac{2H}{\omega_{ms}}\frac{d^2\delta_m}{dt^2} = \frac{P_m - P_e - P_d}{S_B}, \tag{A.1}$$

where P_m, P_e, P_d are per-unit mechanical, electrical and damping powers respectively. The relationship between electrical angular velocity and mechanical angular velocity is $\omega_s = \frac{p}{2}\omega_{ms}$. After substituting these relationships in equation (A. 1), the machine swing equation boils down to

$$\frac{2H}{\omega_s}\frac{d^2\delta}{dt^2} = P_m - P_e - P_d.$$

Alternatively,

$$M\frac{d^2\delta}{dt^2} = P_m - P_e - P_d. \tag{A.2}$$

Considering a non-salient pole machine, where $P_e = \frac{VE'_a}{X}\sin\delta$, $P_d = D\frac{d\delta}{dt}$. Suppose the white noise processes η_t and ξ_t denote randomnesses in electrical and mechanical powers respectively. Consider the stochastic processes η_t and ξ_t are independent random variables. Under the above assumptions and restrictions, equation (A.2) is modified as

$$\ddot{\delta} = -\frac{D}{M}\dot{\delta} - \frac{(1+\sigma_2\eta_t)VE'_a}{M\ X}\sin\delta + \frac{P_m}{M} + \frac{\sigma_1}{M}\xi_t.$$

By utilizing the notion of phase space formulations, we get

$$\dot{\delta} = \omega,$$

$$\dot{\omega} = \left(-\frac{D}{M}\dot{\delta} - \frac{VE'_a}{M\ X}\sin\delta - \sigma_2\eta_t\frac{VE'_a}{M\ X}\sin\delta + \frac{\sigma_1}{M}\xi_t + \frac{P_m}{M}\right).$$

Alternatively,

$$\frac{d}{dt}\begin{pmatrix}\delta\\ \omega\end{pmatrix} = \begin{pmatrix}\omega\\ -\frac{D}{M}\dot{\delta} - \frac{VE'_a}{M\ X}\sin\delta + \frac{P_m}{M}\end{pmatrix} + \begin{pmatrix}0 & 0\\ \frac{\sigma_1}{M} & -\frac{\sigma_2 VE'_a}{M\ X}\sin\delta\end{pmatrix}\begin{pmatrix}\xi_t\\ \eta_t\end{pmatrix}.$$

Since the white noise process is an informal stochastic process, we introduce the Itô framework, a formal stochastic interpretation. In the Itô framework, we utilize the differential of the Brownian motion. Thus,

$$d\begin{pmatrix}\delta\\ \omega\end{pmatrix} = \begin{pmatrix}\omega\\ -\frac{D}{M}\dot{\delta} - \frac{VE'_a}{M\ X}\sin\delta + \frac{P_m}{M}\end{pmatrix}dt + \begin{pmatrix}0 & 0\\ \frac{\sigma_1}{M} & -\frac{\sigma_2 VE'_a}{M\ X}\sin\delta\end{pmatrix}\begin{pmatrix}dB_1(t)\\ dB_2(t)\end{pmatrix}. \tag{A.3}$$

Note that equation (A.3) is an alternative interpretation of equation (3) of the paper and $dB_1 = \xi_t dt$, $dB_2 = \eta_t dt$.

In this paper, we adopt the Fokker-Planck setting for the stochastic system considered here. The Fokker-Planck equation has a connection with the conditional characteristic function evolution equation. That has found applications to derive the Fokker-Planck equation and Central Limit Theorem of stochastic processes. Thus, it is worthwhile to write the conditional characteristic function evolution for the stochastic system considered here. Thus, the evolution $d\langle\phi(y_t)\rangle$ of conditional moment becomes

$$d\langle\phi(y_t)\rangle = (\left\langle f^T(y_t,t)\frac{\partial\phi(y_t)}{\partial y_t}\right\rangle + \frac{1}{2}\left\langle tr((GG^T)(y_t,t)\frac{\partial^2\phi(y_t)}{\partial y_t\partial y_t^T})\right\rangle)dt.$$

In the component-wise description, the above evolution is recast as

$$d\langle\varphi(y_t)\rangle = \left\langle\sum_i f_i(y_t,t)\frac{\partial\varphi(y_t)}{\partial y_i} + \frac{1}{2}\sum_i (GG^T)_{ii}(y_t,t)\frac{\partial^2\varphi(y_t)}{\partial y_i^2} + \sum_{i\langle j}(GG^T)_{ij}(y_t,t)\frac{\partial^2\varphi(y_t)}{\partial y_i\partial y_j}\right\rangle dt.$$

Suppose $\phi(y_t) = e^{s^T y_t}$, the above becomes the conditional characteristic function evolution, i.e.

$$d\left\langle e^{s^T y_t}\right\rangle = \left\langle \sum_i f_i(y_t,t)\, s_i e^{s^T y_t} + \frac{1}{2}\sum_i (GG^T)_{ii}(y_t,t)\, s_i^2 e^{s^T y_t}\right.$$

$$\left. + \sum_{i\langle j} (GG^T)_{ij}(y_t,t) s_i s_j\, e^{s^T y_t}\right\rangle dt.$$

Since the conditional expectation operator $\langle\ \rangle$ is a linear operator, we get

$$d\left\langle e^{s^T y_t}\right\rangle = (\left\langle \sum_i f_i(y_t,t) s_i e^{s^T y_t}\right\rangle$$
$$+\frac{1}{2}\left\langle \sum_i (GG^T)_{ii}(y_t,t) s_i^2\, e^{s^T y_t}\right\rangle$$
$$+\left\langle \sum_{i\langle j} (GG^T)_{ij}(y_t,t) s_i s_j\, e^{s^T y_t}\right\rangle)dt.$$

In the matrix-vector format, the above can be recast as

$$d\left\langle e^{s^T y_t}\right\rangle = (\left\langle s^T f(y_t,t) e^{s^T y_t}\right\rangle$$
$$+\frac{1}{2}\left\langle tr(ss^T (GG^T)(y_t,t)) e^{s^T y_t}\right\rangle)dt.$$

For the stochastic SMIB system of the paper, the evolution of conditional characteristic function becomes

$$d\left\langle e^{s^T y_t}\right\rangle = (\left\langle y_2 s_1 e^{s^T y_t}\right\rangle$$

$$+\left\langle (-\frac{D}{M} y_2 - \frac{VE_a'}{M\ X}\sin y_1 + \frac{P_m}{M}) s_2 e^{s^T y_t}\right\rangle$$

$$+\frac{1}{2}\left\langle (\frac{\sigma_1^2}{M^2} + \frac{\sigma_2^2 V^2 E_a'^2}{M^2\ X^2}\sin^2 y_1) s_2^2 e^{s^T y_t}\right\rangle)dt. \tag{A.4}$$

After restating equation (A.4) by adopting the notations of state vector of the SMIB system, $y_t = (y_1, y_2)^T = (\delta, \omega)^T$, we get

$$d\left\langle e^{s_1\delta + s_2\omega}\right\rangle = (\langle \omega\, s_1\ \exp(s_1\delta + s_2\omega)\rangle$$

$$+\left\langle (-\frac{D}{M}\omega - \frac{VE_a'}{M\ X}\sin\delta + \frac{P_m}{M}) s_2\ \exp(s_1\delta + s_2\omega)\right\rangle$$

$$+\frac{1}{2}\left\langle (\frac{\sigma_1^2}{M^2} + \frac{\sigma_2^2 V^2 E_a'^2}{M^2\ X^2}\sin^2\delta) s_2^2\ \exp(s_1\delta + s_2\omega)\right\rangle)dt.$$

The above expression has theoretical importance in the sense that the expression reveals a connection between the notion of conditional characteristic function of stochastic processes and the single machine-infinite bus system.