CEAI, Vol.18, No.2 pp. 11-19, 2016 Printed in Romania

Stabilization of a Second Order System with a Time Delay Controller

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Abstract: In this paper stabilisation of a second order system $\ddot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)=\mathbf{B}u(t)$ with a time delay output feedback $u(t)=Ky(t-h)$ is analysed. Considered class of second order systems is described, that can be physically modeled as LC ladder networks, and at the same time can be used as an approximation of a distributed parameter system with undamped oscillations. It is followed with stability analysis of resulting infinite dimensional system. It is shown that application of transfer functions is justified and apply the Padé approximation in order to obtain approximated stability regions via constrained optimisation. Then the formulas for derivatives are given, along with their numerical effectiveness comparison. Finally the obtained stability regions are used to optimise the impulse response of closed loop system determining the appropriate values of performance index with James-Nichols-Philips theorem. All these results are illustrated with simulations and optimisation results for different sizes of LC ladder. Also, the merits and limitations of Padé approximation are briefly discussed.

Keywords: Stabilisation, time delay feedback, stability regions, Padé approximation , discrete spectrum

1. INTRODUCTION

Second order matrix systems are important mathematical models in analysis of vibrations and oscillations. They can be described by the following equation

$$\begin{aligned}\ddot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)&=\mathbf{B}u(t)\\ y(t)&=\mathbf{C}\mathbf{x}(t)\end{aligned}\tag{1}$$

In this paper the following properties are considered $\mathbf{x}(t)\in R^n$, $u(t)\in R$, $y(t)\in R$, $\mathbf{A}\in R^{n\times n}$, $\mathbf{B}\in R^{n\times 1}$ and $\mathbf{C}\in R^{1\times n}$. The solution stylof (1) is given by (see for example (Turowicz, 2005):

$$\begin{aligned}\mathbf{x}(t)=\ &\cos(\sqrt{\mathbf{A}}t)\mathbf{x}(0)+(\sqrt{\mathbf{A}})^{-1}\sin(\sqrt{\mathbf{A}}t)\dot{\mathbf{x}}(0)+\\ &+(\sqrt{\mathbf{A}})^{-1}\int_0^t\sin(\sqrt{\mathbf{A}}(t-s))\mathbf{B}u(s)\mathrm{d}s\end{aligned}\tag{2}$$

where matrix sine and cosine function are given by (see for example Higham, 2008):

$$\cos(\mathbf{X})=\mathbf{I}-\frac{\mathbf{X}^2}{2!}+\frac{\mathbf{X}^4}{4!}-\frac{\mathbf{X}^6}{6!}+\ldots$$

$$\sin(\mathbf{X})=\mathbf{X}-\frac{\mathbf{X}^3}{3!}+\frac{\mathbf{X}^5}{5!}-\frac{\mathbf{X}^7}{7!}+\ldots$$

As it can be observed when substituted to (2) matrix $\sqrt{\mathbf{A}}$ is present either in even powers or as a product of its odd power and its inverse, resulting in even power. That is why $\sqrt{\mathbf{A}}$ does need to be explicitly computed when using series expansion.

Systems with positive definite matrix $\mathbf{A}$, are of special interest, as they exhibit undamped oscillations. Systems of this kind were considered among the others by (Kaczorek, 2007; Mitkowski and Skruch, 2009).

In this paper the following matrices are considered

$$\mathbf{A}=\frac{1}{LC}\begin{bmatrix}2 & -1 & 0 & \dots & 0\\ -1 & 2 & -1 & \dots & 0\\ 0 & -1 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \dots & 2 & -1 & 0\\ 0 & \dots & -1 & 2 & -1\\ 0 & \dots & 0 & -1 & 2\end{bmatrix}_{n\times n}$$
$$\mathbf{B}=\frac{1}{LC}\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}0\\\vdots\\0\\0\\1\end{bmatrix}^{\mathrm{T}}\tag{3}$$

As it can be easily verified $\mathbf{A}$ is positive definite.

These matrices correspond naturally to the mathematical model of uniform LC ladder network such as one given in the figure 1. Such networks are often used as electrical analogs used in analysis of spacial discretisations of hyperbolic partial differential equations.

For example consider lossless transmission cable, which can be described by the following equations

$$lc\frac{\partial^2 x(t,z)}{\partial t^2} = \frac{\partial^2 x(t,z)}{\partial z^2}$$

$$x(t,0) = u(t)$$

$$x(t,1) = 0$$

$$t \ge 0, \quad 0 \le z \le 1$$

After applying difference approximation

$$\frac{\partial^2 x(t,z)}{\partial z^2} \approx \frac{1}{\Delta}\left(\frac{x(t,z+\Delta) - 2x(t,z) + x(t,z-\Delta)}{\Delta}\right)$$

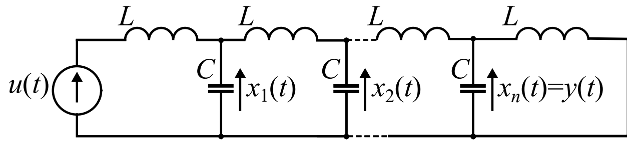


Fig. 1. LC ladder network.

where $\Delta = 1/n$ and $z = (2k-1)\Delta/2$ for $k = 1, 2, \ldots, n$ one obtains system

$$\ddot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) = \mathbf{B}u(t) \tag{4}$$

where

$$\mathbf{A} = \frac{n^2}{lc}\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 2 & -1 & 0 \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{bmatrix}_{n\times n} \quad \mathbf{B} = \frac{n^2}{lc}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

2. STABILISATION

System (1) is oscillatory. It has $2n$ imaginary eigenvalues from the set $\{s : s = \sqrt{z}, \forall z \in \lambda(\mathbf{A})\}$. For similar LC networks many schemes of stabilisation were discussed in (Mitkowski, 2004), with a conclusion, that only dynamic output feedback can stabilise this system. In this paper an infinite dimensional feedback in the form of proportional, time delayed controller will be considered

$$u(t) = Ky(t-h) \tag{6}$$

where K is the gain and $h > 0$ is the time delay. Usually, in control application focus is on elimination of the influence of delay (which is usually negative), what leads to difficult control problems. On the other hand, introducing or increasing a delay to the system is very simple - it can be implemented with appropriate buffers. That is why such controller can be easily applied. This control system structure is presented in the figure 2.

Results presented in this paper are a substantial development of those obtained in (Baranowski and Mitkowski, 2009) and represent an alternative approach to the one presented in (Baranowski and Mitkowski, 2012b).

2.1 Stability of time delay systems

System (1) with feedback (6) can be equivalently written as:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-h) + \mathbf{B}u(t) \tag{7}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{A} & 0 \end{bmatrix} \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ \mathbf{B}K\mathbf{C} & 0 \end{bmatrix} \tag{8}$$

with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ given by (3).

System (7) is exponentially stable iff roots s_i of the equation

$$\det(s\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1 e^{-sh}) = 0 \tag{9}$$

fulfill

$$\mathrm{Re}\, s_i < 0, \forall i \tag{10}$$

(see for example (Klamka, 1990 p. 166)). More detailed analysis can be found in (Baranowski and Mitkowski, 2012b). Different approaches to time delay system stability can be seen for example in (Duda, 2010, 2013; Iotga, 2014; Yeroglu, 2015).

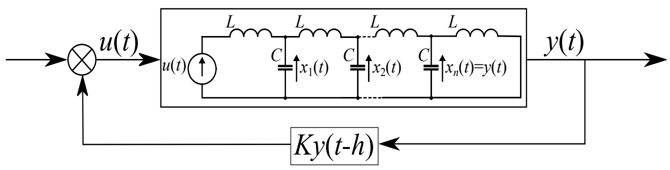


Fig. 2. Control loop structure.

3. STABILITY REGIONS

In previous works regarding applications of (6) to oscillatory systems two approaches were dominant. The first one, based on matrix pencils was considered in (Abdallah et al., 1993; Niculescu and Abdallah, 2000) where some general results were derived. The other approach used Nyquist stability criterion and its results were shown in (Mitkowski and Skruch, 2009). As it was shown in cited works, for simple cases an analytical solution can be obtained. Let us consider system (1) with controller (6) with $n = 1$. System

$$\ddot{x}(t) + \frac{1}{LC}x(t) - \frac{K}{LC}x(t-h) = 0$$

Using Nyquist criterion one can determine the stability region analytically for $l = 0, 1, 2\ldots$

$$0 < K < \frac{1+4l}{1+4l+8l^2}$$
$$\frac{2l\pi\sqrt{LC}}{\sqrt{1-K}} < h < \frac{(2l+1)\pi\sqrt{LC}}{\sqrt{1-K}}$$

Both of these directions were characterized by two aspects:

- only positive feedbacks were considered ($K > 0$),

- the focus was on finding such h for which given K will cause the closed loop system to be asymptotically stable.

Approach presented in this paper considers both positive and negative feedbacks and represents K as a function of h. It leads to some interesting geometric properties of stability regions which will be discussed in following sections. For analysis of stability the transfer function of system (1) will be considered, i.e.

$$Y(s) = G(s)U(s) \tag{11}$$

$$G(s) = \mathbf{C}(s^2\mathbf{I}+\mathbf{A})^{-1}\mathbf{B} = \frac{l(s)}{m(s)} \tag{12}$$

$$U(s) = K\mathrm{e}^{-hs} \tag{13}$$

where $l(s) = l_0 > 0$ and $m(s)$ is a polynomial of order $2n$ with only positive coefficients, and imaginary roots. It can be verified that $m(\lambda)$ is a characteristic polynomial of $\mathbf{A}_0$ and also that $m(s) = w(LCs^2)$, where $w(\lambda)$ is a characteristic polynomial of $\mathbf{A}$ (see (Baranowski and Mitkowski, 2012b)).

As it was shown in section 2.1, stability analysis is equivalent to appropriate location of roots of characteristic quasipolynomial. In earlier research Padé approximations of (13) were considered in (Baranowski et al., 2009) in order to reduce the problem to simple polynomial stability (for example see (Kaczorek, 2007)).

3.1 Padé approximation

Padé approximations of the exponential function $s \to \mathrm{e}^{-sh}$ of order $[q,q]$ with $q \in N$ are considered. These approximations are rational functions in a form

$$\frac{Q_q(-sh)}{Q_q(sh)}$$

with Q_q is a polynomial of q-th order given by

$$Q_q(sh) = \int_0^\infty t^q (t+sh)^q \mathrm{e}^{-t}\mathrm{d}t =$$
$$= \sum_{j=0}^{q} \frac{(2q-j)!q!}{j!(q-j)!}(sh)^j =$$
$$= \sum_{j=0}^{q} \frac{(q+j)!q!}{j!(q-j)!}(sh)^{q-j}$$

also recurrence formulas are possible. Padé approximations have two interesting properties:

- $Q_q(\lambda)$ has roots in left open complex half plane for any q.

- frequency characteristics of system in a form (13) can be approximated with arbitrary precision in the sense of $L^\infty(R)$ norm.

One can then approximate the controller (13)

$$\hat{U}(s) = K\frac{Q_q(-sh)}{Q_q(sh)} \tag{14}$$

Closing the feedback loop of (12) and (14) one will get a closed loop system transfer function

$$G_c(s) = \frac{G(s)\hat{U}(s)}{1-G(s)\hat{U}(s)} \tag{15}$$

with a denominator

$$d(s) = m(s)Q_q(sh) - Kl_0Q_q(-sh) \tag{16}$$

Analysis of stability of (16) will allow to find approximate region of stability in the set of possible h and K.

Investigation of numerical properties of Padé approximation leads to interesting results. Noticeably the approximated system gives a very good approximation of dominating poles, which can be seen in the figure 3 (true spectrum was computed with approximation of infinitesimal generator of appropriate semigroup as in (Breda et al., 2004)). Moreover, the approximation of low order is correct only for short delays. It introduces an effect of collapsing stability regions (see figure 4) - it means that for longer delays low order approximations introduce larger (and false) regions of stability. However convergence is observed. One cannot rise the order of approximation too high, because it leads to ill conditioned polynomial, which roots are not reliable (see figure 5).

3. DETERMINATION OF STABILITY REGIONS

For this analysis it is assumed that for given h the set of stabilizing K is connected or empty - this is based on numerical analysis, because there are no general theorems allowing to assure connectedness of Hurwitz sectors. From this assumption and because for $K = 0$ system will never be asymptotically stable system can be stabilised either by positive or by negative feedback, but never by both. It is also assumed, that the value of maximal possible gain ensuring stability depends continuously on h, which also can be justified numerically. In that case it is only necessary to determine boudary values of K and it can be performed through constrained optimisation.

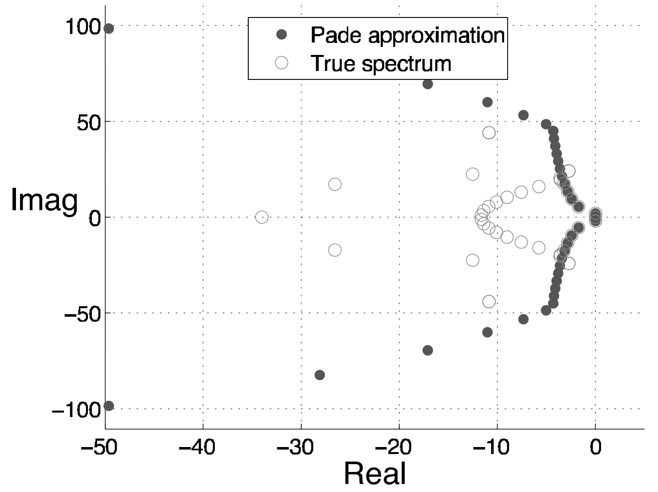


Fig. 3. Comparison of approximated (roots of (16)) and true spectrum of system (12)-(13).

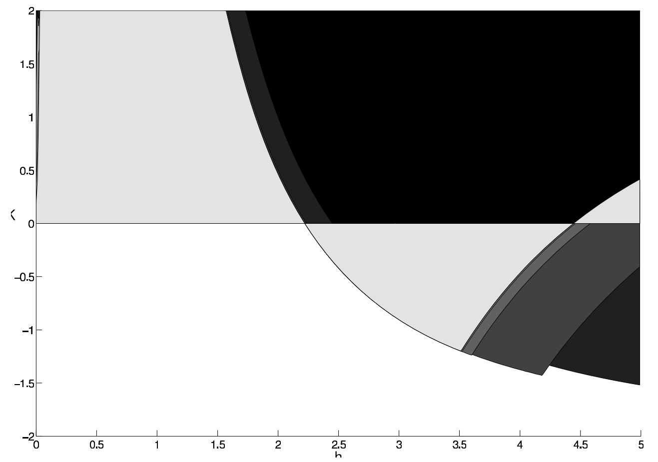


Fig. 4. Collapsing of approximate stability regions - the lighter colors correspond to higher orders.

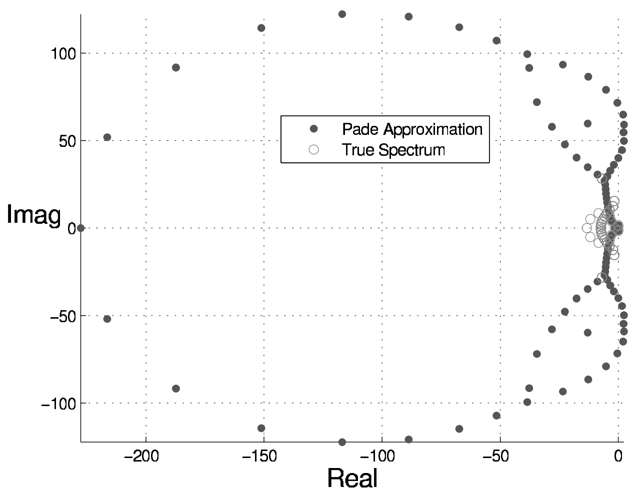


Fig. 5. Comparison of approximated (roots of (16)) and true spectrum of system (12)-(13) – ill conditioned approximation (approximation of order $n > 50$).

To determine the region of stability, cyclic minimisation of performance index will be performed, for different h

$$\begin{array}{ll} J(K) = & -K^2 \\ \text{s.t.} & \\ \mathbf{a}K \le & \mathbf{c} \\ h_i(K) \le & 0, i = 1, 2, \ldots \end{array} \tag{17}$$

The constraints are devised from the variant of Hurwitz criterion and are represented as follows. $\mathbf{a}$ and $\mathbf{c}$ are used to ensure that last $q+1$ coefficients of $d(s)$ are positive (first $2n$ are always positive and not influenced by K) - $\mathbf{a}$ is a vector consisting of coefficients of $Q_q(-sh)$ multiplied by l_0, $\mathbf{c}$ is vector of last $q+1$ coefficients of $m(s)Q_q(sh)$. Inequalities $h_i(K)$ are appropriately even or odd principal minors of Hurwitz matrix, depending on oddity of q , multiplied by -1 . Because usually order of Padé approximation is high the number of constraints is about 30.

Because the constraints h_i are highly nonlinear, especially for high $2n+q$ it is necessary to compute their gradients in some other way than through forward differences. A method of such computation can be devised from the formulas for the derivative of determinant. Principal minors are essentially determinants of lower order matrices. Because of that only the reasoning for derivative of determinant of Hurwitz matrix $\mathbf{H}(K)$ is presented. One wants to find

$$\frac{\mathrm{d}}{\mathrm{d}K} \det \mathbf{H}(K) \tag{18}$$

Let us denote $\mathbf{H}'(K)$ the matrix, whose elements are derivatives of elements of $\mathbf{H}(K)$ with respect to K . It can be easily seen, that this is a Hurwitz matrix of polynomial $-l_0 Q_q(-sh)$ treated as the polynomial of order $2n+q$. Then from Jacobi's Formula

$$\frac{\mathrm{d}}{\mathrm{d}K} \det \mathbf{H}(K) = \mathrm{tr}(\mathrm{Adj}(\mathbf{H}(K))\mathbf{H}'(K)) \tag{19}$$

where $\mathrm{tr}(\mathbf{A})$ is the trace of matrix $\mathbf{A}$, and $\mathrm{Adj}(\mathbf{A})$ is the adjugate matrix of $\mathbf{A}$. Because of astronomical computational complexity of adjugate matrices this formula has little use in gradient computations. It can be however reformulated into

$$\frac{\mathrm{d}}{\mathrm{d}K} \det \mathbf{H}(K) = \mathrm{tr}(\det(\mathbf{H}(K))\mathbf{H}(K)^{-1}\mathbf{H}'(K)) \tag{20}$$

which is much easier to compute but has a one serious drawback - it is useless for singular matrices. Other method free from that drawback is the following formula (see (Turowicz, 2005)):

$$\frac{\mathrm{d}}{\mathrm{d}K} \det \mathbf{H}(K) = \sum_{i=1}^{2n+q} \det \mathbf{H}_i(K) \tag{21}$$

where $\mathbf{H}_i(K)$ is a matrix $\mathbf{H}(K)$ in which the i-th column was replaced by the i-th column of $\mathbf{H}'(K)$. Both these formulas were tested statistically regarding their computation time. Such comparison for derivatives of determinant of 50 $\times$ 50 matrices is illustrated in the figure 6.

As it can be seen, matrix inversion algorithm is faster, and the difference rises with the matrix dimension. That is why a compromise solution was proposed. Because determinants of $\mathbf{H}(K)$ or its minors can become 0 – which corresponds to reaching the desired boundary value – matrix inversion algorithm (20) cannot be used directly.

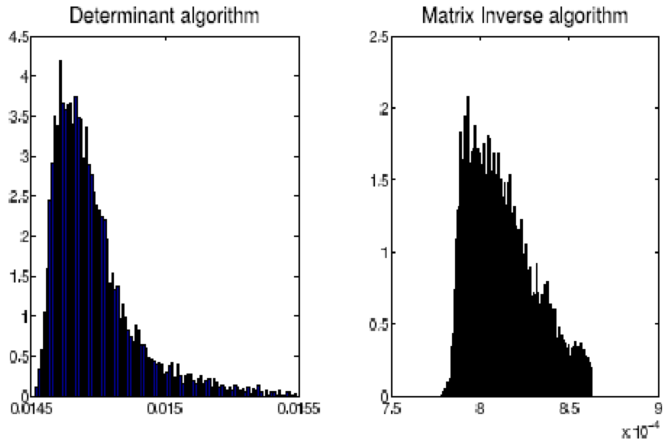


Fig. 6. Histogram of gradient computation speed - determinant and matrix inversion algorithms. The horizontal axis is the time of computation, and the vertical is percentage of occurrence. Both graphs have outliers removed.

So in order to quickly compute necessary gradients, an algorithm was constructed that used matrix inversion (20) in nonsingular cases and determinants (21) in singular ones.

To ensure fast convergence for different h from the assumption of continuous dependence of boundary K on h a linear extrapolation was used as a way of finding next initial value for optimisation along with necessary contractions to stay in the feasible set. Also maximal line search step was strictly bounded, to avoid leaving the feasible set by accident. It was observed in some cases that constraints resulting from Hurwitz criterion can become arbitrarily small, but positive - keeping algorithm in the desired, even very strict constraint violence level but with unstable closed loop system.

4.1 Examples of results

In figures 7 to 9 the stability regions for ladders with $n = 1, 2, 3$ are presented. For those $h \in [0,10]$ is analysed, and for this interval approximation with $q = 16$ shown appropriate correctness. Results were verified through simulation of a system (1) with a delay controller (6). As it was mentioned before some interesting graphical properties can be observed. For $n = 1$ almost for every h there exists a stabilising controller. Moreover areas of stability are bounded by the continuous curves.

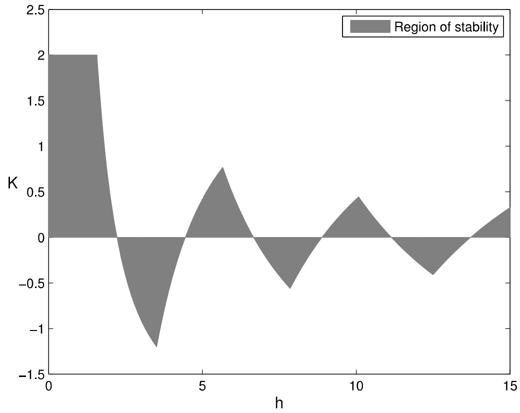


Fig. 7. Stability region for $n = 1$.

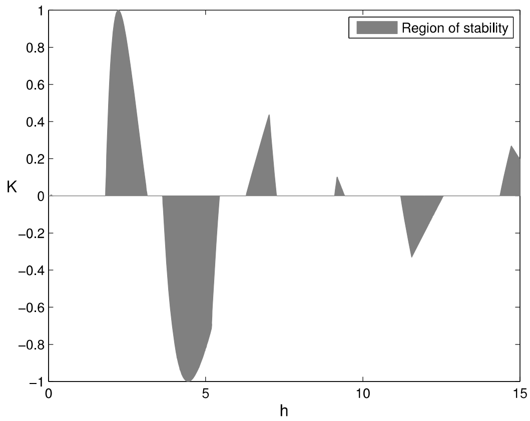


Fig. 8. Stability region for $n = 2$.

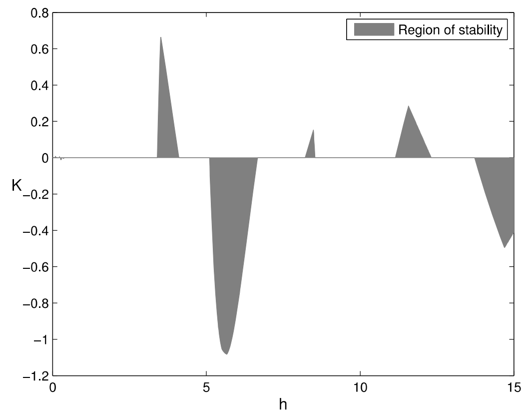


Fig. 9. Stability region for $n = 3$.

The most interesting aspect occurs for $n \geq 2$. One can clearly see, that in order to stabilise system, one have to use delays with a high value. Small delays, which were usefull in $n = 1$ now can lead to destabilisation. It leads to nearly philosophical conclusion that in some situations you have to act no sooner than after some time - too fast reaction can be disastrous. Moreover, there are now entire intervals where stabilisation is not possible - it leads to conclusion that information from some periods in the past is useless. This approach for finding stability regions was considered in (Baranowski et al., 2009).

For $n = 3$ and higher the stability regions become more and more irregular, with very 'thin' regions where stabilisation is possible, and generally smaller gains are available.

5. OPTIMISATION OF IMPULSE RESPONSE

Determination of stability region is not sufficient to obtain a good stabilising controller for the system. The choice of appropriate values of K and h becomes a problem of controller tuning. One of the criteria that can be used for such tuning is the requirement that impulse response of the system should consist of only brief transitional behaviour. One of the method of ensuring such situation is the minimisation of $L^2[0,\infty)$ norm of output (also known as ISE criterion):

$$J = \int_0^\infty y^2(t)\mathrm{d}t \tag{22}$$

Finding the optimal pair (h, K) requires minimising the performance index (22) for impose response. Having information regarding stability regions one can perform such minimisation. A classical result can be considered (see for example (Grabowski, 1996)).

Theorem 1 (James-Nichols-Philips) Let

$$Y(s) = \int_0^\infty y(t)\mathrm{e}^{-st}\mathrm{d}t = \frac{L(s)}{M(s)}$$

with

$$L(s) = \sum_{i=0}^{n-1} b_i s^i, \quad M(s) = \sum_{i=0}^{n} a_i s^i$$

then

$$\int_0^\infty y^2(t)\mathrm{d}t = (-1)^{n^{-1}} \frac{\det(\widetilde{\mathbf{H}})}{a_n \det(\mathbf{H})}$$

where $\mathbf{H}$ is the Hurwitz matrix of $M(s)$ polynomial given by

$$\mathbf{H} = \begin{bmatrix} a_{n-1} & 1 & 0 & \dots & \dots & \dots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & \dots & \dots & \dots & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & \dots & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & \dots & \dots & 0 & a_0 \end{bmatrix} \in R^{n\times n}$$

and

$$\widetilde{\mathbf{H}} = \left[\frac{1}{2}\mathbf{H}_L b, (n-1) \text{ last columns of } \mathbf{H}\right]$$

$$\mathbf{b}^{\mathrm{T}} = \left\lfloor(-1)^{n-1} b_{n-1} \dots -b_1\ b_0\right\rfloor$$

and $\mathbf{H}_L$ is the Hurwitz matrix of $L(s)$ polynomial treated as a n-th order polynomial (for $i > m$, $b_i = 0$). This theorem provides means of determining performance index of finite dimensional systems without simulation. Closed loop system is infinite dimensional, however can be approximated with the Padé approximation of the controller in order to get the approximation of impulse response

$$Y(s) = \frac{Q_q(sh)l(s)}{m(s)Q_q(sh) - Kl_0 Q_q(-sh)}$$

for which theorem 1 can be used. Now one can use the constraints of stability regions and formula for performance index to construct a nonlinear programing problem. In order to improve finding the minimiser one can present the analytical formula for gradient of (22).

$$\nabla J = \frac{1}{(-1)^{n-1}(a_n \det(\mathbf{H}))^2}\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \tag{23}$$

$$\begin{aligned} J_1 = {} & \frac{\partial}{\partial K}\det(\widetilde{\mathbf{H}})a_n \det(\mathbf{H}) + \\ - {} & \det(\widetilde{\mathbf{H}})a_n \frac{\partial}{\partial K}\det(\mathbf{H}) \end{aligned} \tag{24}$$

$$\begin{aligned} J_2 = {} & \frac{\partial}{\partial h}\det(\widetilde{\mathbf{H}})a_n \det(\mathbf{H}) + \\ - {} & \det(\widetilde{\mathbf{H}})\left(a_n \frac{\partial}{\partial h}\det(\mathbf{H}) + \alpha_n \det(\mathbf{H})\right) \end{aligned} \tag{25}$$

where $\alpha_n = q \cdot q!\cdot h^{q-1}$. Derivatives of determinants are given by formulas (20) and (21). Arguments of $\mathbf{H}(h,K)$ and $a_n(h)$ were dropped to increase clarity.

Because of the nature of the process one needs to search for optimum only inside the stability region. Moreover, optimal value is never on the boundary, because then $\det(\mathbf{H}) = 0$. That is why one can simplify the optimisation using the spline approximations of the boundary as constraints, which is much easier than multiple constraints arising from Hurwitz criterion. Moreover stability regions are not connected. That is why natural approach is to optimise in every connected set separately and then choose the global minimiser.

5.1 Examples of results

The results of optimisation are presented in figures 10, 11 and 12 for $n = 1, 2$ and 3 respectively.

Optimisation of system with $n = 1$ provides most regular results. In the figure 10a one can observe, that the "valley" of optimal values of performance index (22) is smooth, much smoother than the boundary of stability region. As one can see in the figure 10b the optimal results are located in the interior of stability region approaching the boundary leads to generally higher values of (22) approaching infinity for such

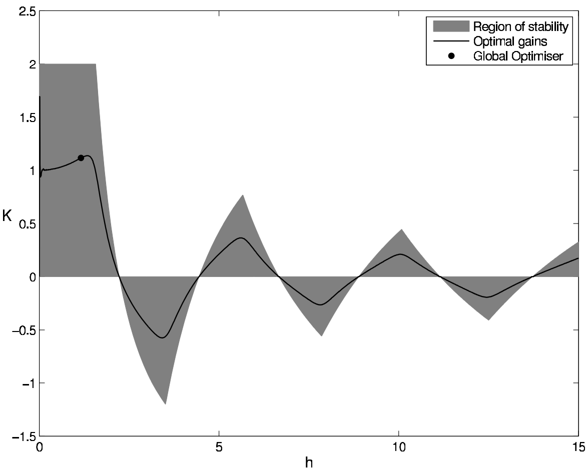
h that no stabilising gains exist (points of changing the sign of stabiliser). Impulse response presented in the figure 10c is well damped, with just a few oscillations before steady state. What is interesting, the optimal stabilisation is obtained by using a positive feedback loop.

For higher orders ($n = 2, 3$) more complicated stability regions influence the optimisation. However global minimisers are located in the interior of stability region. This is advantageous, because errors in determination of stability region caused by Padé apprximation are located mostly at the boundary. What is interesting, is that global minimiser is located in the second stability region, and while for $n = 2$ values are close, for $n = 3$ global optimum is significantly smaller for larger h. Impulse responses are however much more oscillating than for the $n = 1$ case. It can be justified, by the fact that frequencies of natural oscillations in systems are not multiplicities of each other. For $n = 1$ there was only one frequency, and stabilisation by delayed feedback could be compared to influencing the system with phase shifted sine wave. One can see it as similar to wave interference. In case of higher orders there is no such possibility.

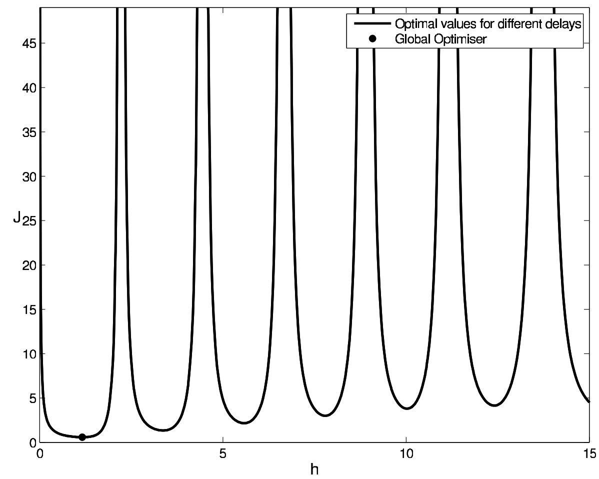
6. CONCLUSIONS

In this paper a new methodology of time delay controller parameter analysis was presented. This methodology allows to compute stability regions easily using Padé approximation. Determined stability regions were also used for determination of optimal impulse response (for the approximation). In this optimisation a new application of classical James-Nichols-Philips theorem. Also new methods of gradient computations are presented along with numerical analysis of their effectiveness.

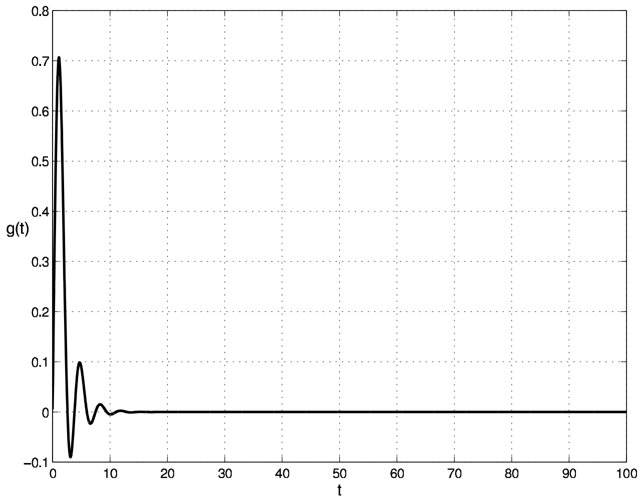
It should be noted, that validity of Padé approximation is limited, as it is correct only for delays relatively small with respect to order. Increasing order on the other hand leads to ill conditioning of polynomials and instability.



(a) Minimal values of performance index in relation to stability regions.

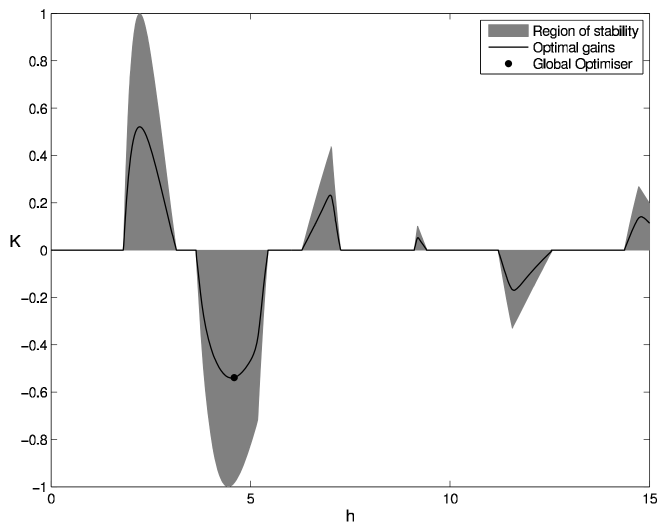


(b) Optimal performance index values for differing delay *h*.

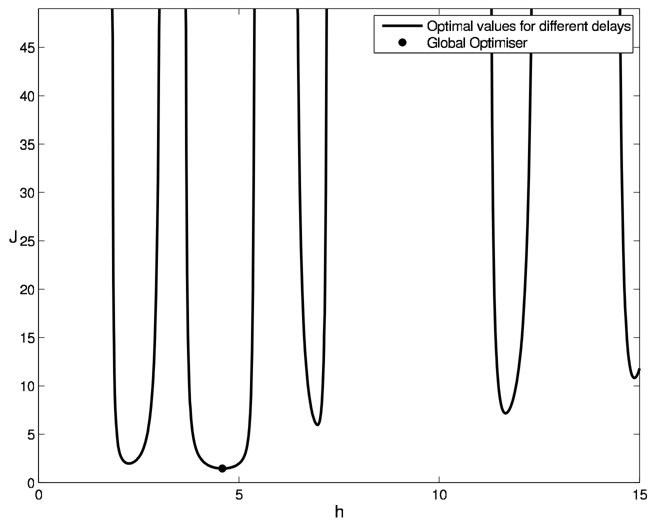


(c) The impulse response of global minimiser.

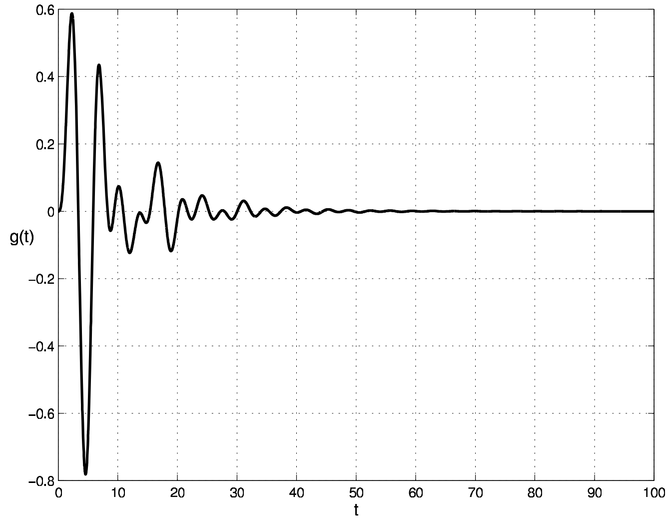
Fig. 10. Optimisation of impulse response for n = 1.



(a) Minimal values of performance index in relation to stability regions.

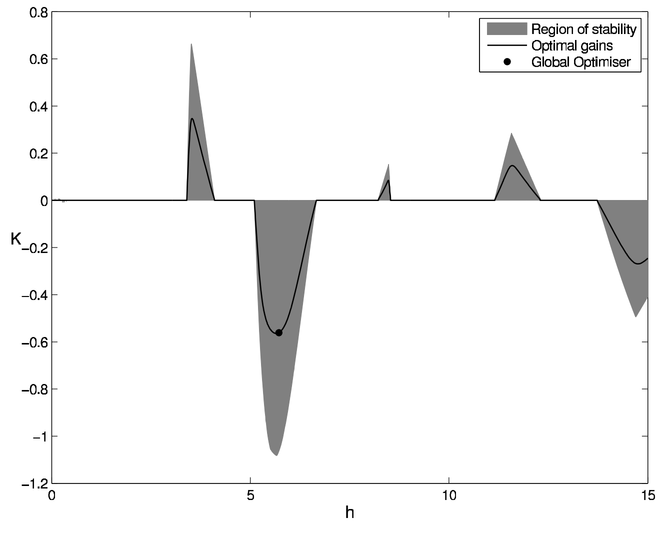


(b) Optimal performance index values for differing delay *h*.

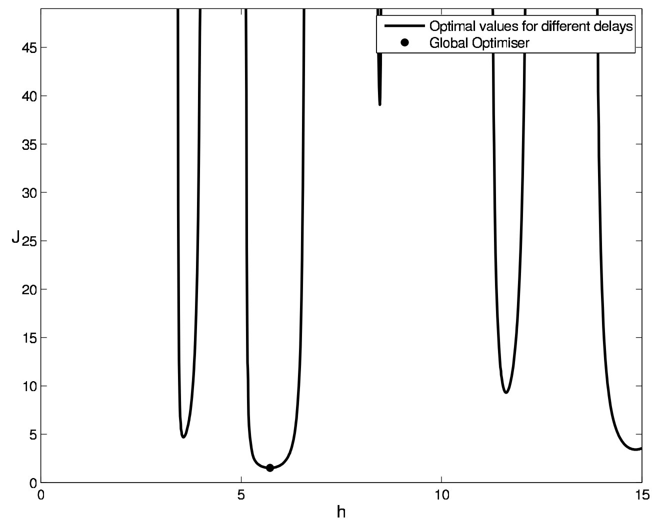


(c) The impulse response of global minimiser.

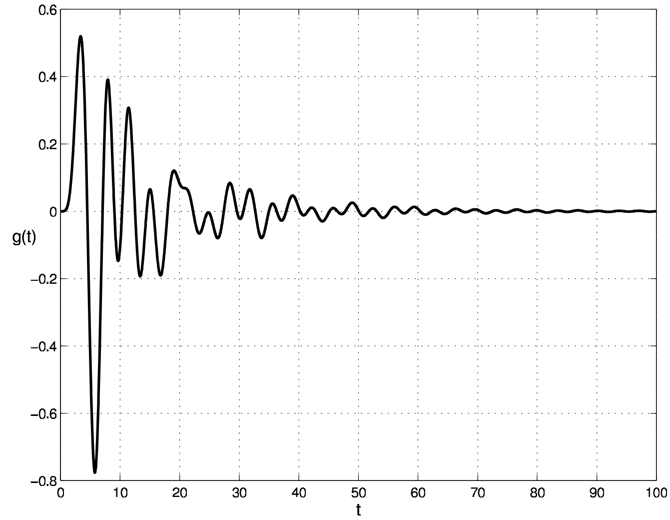
Fig. 11. Optimisation of impulse response for $n = 2$.



(a) Minimal values of performance index in relation to stability regions.



(b) Optimal performance index values for differing delay h.



(c) The impulse response of global minimiser.

Fig. 12. Optimisation of impulse response for $n = 3$.

In order to obtain exact stability regions one should consider other author's work (Baranowski and Mitkowski, 2012b) where D-partition method was analysed. Optimisation of the non approximated system is more complicated, as the formulas require either computing complex variable integrals or solving Lyapunov operator equations.

Further research will also attempt different approximations. Very good results were obtained for approximation of delay in nonlinear glucose dynamics system (see (Baranowski, 2010)) using Legendre polynomials. This approach can also be very useful for stability analysis, as the same basis was used in (Breda et al., 2004) in analysis of true spectrum of time delay system. Different approximation approach is based on impulse response approximation. This approach approximates impulse responses that are located in $L^1[0,\infty)\times L^2[0,\infty)$ with Laguerre orthonormal functions. This approach cannot be used for stability analysis, but for stable systems can be a great method for optimisation -

coefficients of approximation are the L^2 norm of the impulse response. This approach was used with much success in analysis of non-integer order systems (see (Bania and Baranowski, 2013; Baranowski et. al, 2014; 2016). For other results on ladder network application for control of distributed parameter systems see (Baranowski and Mitkowski, 2012a).

Work partially realized in the scope of project titled ”Design and application of non-integer order subsystems in control systems”. Project was financed by National Science Centre on the base of decision no. DEC-2013/09/D/ST7/03960.

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