ON THE STABILITY OF THE CELLULAR NEURAL NETWORKS WITH TIME LAGS

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Abstract: *Cellular neural networks (CNNs) are recurrent artificial neural networks. Due to their cyclic connections and to the neurons' nonlinear activation functions, recurrent neural networks are nonlinear dynamic systems, which display stable and unstable fixed points, limit cycles and chaotic behaviour. Since the field of neural networks is still a young one, improving the stability conditions for such systems is an obvious and quasi-permanent task. This paper focuses on CNNs affected by time delays. We are interested to obtain sufficient conditions for the asymptotical stability of a cellular neural network with time delay feedback and zero control templates. For this purpose we shall use a method suggested by Malkin [8], where the "exact" Liapunov-Krasovskii functional will be constructed according the procedure proposed by Kharitonov [6] for stability analysis of uncertain linear time delay systems.*

Keywords*: cellular neural network, time delays, Liapunov functional, absolute stability.*

1. INTRODUCTION

Cellular neural networks (CNNs), introduced in 1988 [2], are artificial recurrent neural networks displaying a multidimensional array of cells and local interconnections among the cells. CNNs have been successfully applied to signal and image processing, shape extraction and edge detection. In such applications stability and other problems of dynamical behaviour of the CNN are equally important. These properties are necessary for the network to achieve its goal and have to be checked on the mathematical model.

In the last ten years the research was oriented towards the dynamics of the networks affected by time delays due to the signal propagation at the synapses level of the biologic brain or the reacting lag in the case of the artificial neural network. These lags may introduce oscillations or may lead to instability of the network.

We are interested to obtain sufficient conditions for the asymptotical stability of a cellular neural network with time delay feedback and zero control templates. For this purpose we shall use a method suggested by Malkin [8] (see also [1]), where the "exact" Liapunov-Krasovskii functional will be constructed according the procedure proposed by Kharitonov [6] for stability analysis of uncertain linear time delay systems.

The sufficient conditions obtained here are independent of the delay parameter.

2. THE MATHEMATICAL MODEL AND PROBLEM STATEMENT

Consider a cellular neural network with time delay feedback and zero control templates

$$\dot{z}_i(t) = -a_i z_i(t) + \sum_{j\in N} c_{ij} g_j\left(z_j(t-\tau_j)\right) + I_i\,, \quad i=\overline{1,n} \tag{1}$$

where j is the index for the cells of the nearest neighborhood N of the i^{th} cell, a_i is a positive parameter, c_{ij} are synaptic weights (which can have an inhibitory effect if $c_{ij} < 0$, or an excitatory one if $c_{ij} > 0$), I_i is the bias and τ_j are positive delays.

The nonlinearities for the cellular neural networks are of the bipolar ramp type:

$$g_i(z_i) = \tfrac{1}{2}\left(|z_i+1| - |z_i - 1|\right) \tag{2}$$

what means they are bounded, monotonically increasing and globally Lipschitzian functions, with the Lipschitz constant $L_i = 1$.

Without loss of the generality, using a change of the coordinates, $x_i = z_i - z_i^*$, one can shift the equilibrium point z^* to the origin so that system (1) can be written into the form:

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j\in N} c_{ij} f_j\left(x_j(t-\tau_j)\right),\ i=\overline{1,n} \tag{3}$$

where we denoted

$$f_j(\sigma) = g_j(\sigma + z_j^*) - g_j(z_j^*),\ \forall j \tag{4}$$

Using a method proposed by Malkin (1952), we assume that there exists $k_i > 0$ such that the nonlinearities satisfy

$$0 \le k_i - \underline{k_i} < \frac{f_i(\sigma)}{\sigma} < k_i + \overline{k_i} \tag{5}$$

and that for $f_i(x_i) = k_i x_i$ the system

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j\in N} c_{ij} k_j x_j(t-\tau_j),\ i=\overline{1,n} \tag{6}$$

is exponentially stable. We underline that (5) is a normal condition taking into account the properties of the activation functions of CNN's neurons.

For instance, if $g_j(z)$ are given by (2) then since they are monotonically increasing and globally Lipschitzian with the Lipschitz constant $L_i = 1$ we shall have, taking also into account the above definition of f_i that

$$0 < \frac{f_i(\sigma)}{\sigma} \le 1 \tag{7}$$

Since $a_i > 0$ we may take in (5) $k_i = 0$, $\underline{k_i} = 0$, $\overline{k_i} = 1$. In the following we continue the general study which is valid also for other functions than defined by (5).

Denoting

$$A_0 = diag(-a_i)_1^n \tag{8}$$

$$C_j = \begin{pmatrix} 0 & \dots & 0 & c_{1j} & 0 & \dots & 0 \\ 0 & \dots & 0 & c_{2j} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & c_{nj} & 0 & \dots & 0 \end{pmatrix} \tag{9}$$

$$A_j = C_j \cdot \underset{i}{diag}(k_i)_1^n \tag{10}$$

system (6) may be written into the form

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{n} A_j x(t-\tau_j) \tag{11}$$

with the initial condition $x_i(\theta) = \varphi(\theta)$, for $\theta \in [-\tau, 0]$, where $\tau = \max_j \tau_j$, $\varphi \in \boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$.

Consider now the perturbed system:

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{n}\left(C_j \cdot \underset{i}{diag}(k_i + b_i)_1^n \right) y(t-\tau_j) \tag{12}$$

which can be written as

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{n}\left(A_j + \Delta_j\right) y(t-\tau_j) \tag{13}$$

with

$$\Delta_j = C_j \cdot \underset{i}{diag}(b_i)_1^n,\ j=\overline{1,n} \tag{14}$$

We are interested to find conditions such that the perturbed system (13) remains exponentially stable for all $b_i \in \left(-\underline{k_i}, \overline{k_i}\right)$, $\forall i = \overline{1,n}$. This is nothing more but robust stability of system (5) in the linear case. The idea of Malkin [1], [8] which will be described below will in fact give us more, the exponential stability of the nonlinear system (3) for which sufficient conditions will be obtained. In fact, $\left(-\underline{k_i}, \overline{k_i}\right)$ represents the proper interval for the nonlinear functions f_i attached to each cell of the network.

3. MAIN RESULTS

Given positive definite $n \times n$ matrices P_0, P_j, R_j, $j = \overline{1,n}$ let us define on $\boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$ the positive definite functional

$$W(\phi(\cdot)) = \phi^T(0)P_0\phi(0) + \sum_{j=1}^{n}\phi^T(-\tau_j)P_j\phi(\tau_j) + \sum_{j=1}^{n}\int_{-\tau_j}^{0}\phi^T(\theta)R_j\phi(\theta)d\theta \quad (15)$$

Since system (11) is exponentially stable, there exists a Liapunov-Krasovskii functional $V(\phi(\cdot))$ such that along the solutions of (11) we have the equality

$$\frac{d}{dt}V(x(t+\cdot)) = -W(x(t+\cdot)) \quad (16)$$

The "exact" Liapunov-Krasovskii functional is of the form

$$\begin{aligned} V(x(t+\cdot)) &= x^T(t)U(0)x(t) \\ &+ \sum_{j=1}^{n} 2x^T(t)\int_{-\tau_i}^{0} U(-\tau_i-\theta)C_j\left(\underset{i}{diag}(k_i)_1^n\right)x(t+\theta)d\theta \\ &+ \sum_{k=1}^{n}\sum_{j=1}^{n}\int_{-\tau_k}^{0} x^T(t+\theta_2)\left(\underset{i}{diag}(k_i)_1^n\right)C_j \cdot \\ &\cdot\left(\int_{-\tau_i}^{0} U(\theta_1-\theta_2+\tau_k+\tau_j)C_j\left(\underset{i}{diag}(k_i)_1^n\right)d\theta_1\right)d\theta_2 \\ &+ \sum_{j=1}^{n}\int_{-\tau_i}^{0} x^T(t+\theta)\left[(\tau_j+\theta)R_j+P_j\right]x(t+\theta)d\theta \end{aligned} \quad (17)$$

where, since the system (11) is exponentially stable, the matrix valued function

$$U(\tau) = \int_0^{\infty} K^T(t)\left[P_0 + \sum_{j=1}^{n}\left(P_j + \tau_j R_j\right)\right]K(t+\tau)dt \quad (18)$$

is well defined for all $\tau \in \boldsymbol{R}$; here K(t) is the fundamental matrix associated to the system (11) (see Kharitonov and Zhabko [6]).

Following the steps in Kharitonov [6], the time derivative of Liapunov-Krasovskii functional along the solutions of the perturbed system (13) is

$$\frac{d}{dt}V(y(t+\cdot)) = -W(y(t+\cdot)) + 2\left[\sum_{j=1}^{n}\Delta_j y(t+\tau_j)\right]^T \cdot \left[U(0)y(t) + \sum_{j=1}^{n}\int_{-\tau_j}^{0}U^T(\tau_j+\theta)A_j y(t+\theta)d\theta\right] \quad (19)$$

We assume that b_i, $i=\overline{1,n}$ are such that the matrices Δ_j, defined by (14), are constant and satisfy the condition

$$\Delta_j^T H_j \Delta_j \le \rho_j I, \quad j=\overline{1,n} \quad (20)$$

where H_j are definite positive matrices, ρ_j are given positive numbers and I is the identity matrix.

For the derivative of the functional (17) along the trajectories of the perturbed system (13) one obtains the following upper bound:

$$\begin{aligned} \frac{d}{dt}V(y(t+\cdot)) &\le -y^T(t)\left[P_0 - \mu U^T(0)\sum_{j=1}^{n}H_j^{-1}U(0)\right]y(t) \\ &- \sum_{j=1}^{n}\int_{-\tau_j}^{0} y^T(t+\theta)\left[R_j - \mu A_j^T U(\tau_j+\theta)\cdot\right. \\ &\left.\cdot\left(\sum_{k=1}^{n}H_k^{-1}\right)U^T(\tau_j+\theta)A_j\right]y(t+\theta)d\theta \\ &- \sum_{j=1}^{n} y^T(t-\tau_j)\left[P_j - \frac{2}{\mu}\rho_j I\right]y(t-\tau_j) \end{aligned} \quad (21)$$

where it is assumed that $H_j := \frac{1}{\mu}H_j$, $j=\overline{0,n}$.

We have constructed a Liapunov - Krasovskii quadratic functional which is strictly positive definite and with the derivative along linear system's solutions at least non-positive; this last property is preserved with respect to the uncertainties defined by (20) and this shows a possible robust exponential stability of the linearized system (6). But, as already mentioned, the idea of Malkin [1], [8] gives more - exponential stability of the nonlinear system (3). This will become clear from the short description of the method. Let $b_i(\sigma)$ be a nonlinear function defined from (5):

$$b_i(\sigma) = \frac{f_i(\sigma)}{\sigma} - k_i, \quad \forall i \quad (22)$$

Now, if the Liapunov function(al) and its derivative - both being quadratic forms - have good sign properties for all $b_i \in \left(-\underline{k_i}, \overline{k_i}\right)$, then for any fixed $x_i \ne 0$ one can obtain b_i from (22) and for $b_i(x_i) \in \left(-\underline{k_i}, \overline{k_i}\right)$ the properties of the Liapunov function(al) do not change.

Remark: *It is quite clear that the terms b_i may be even time varying what shows that f_i may be time varying within the interval $\left(-\underline{k_i}, \overline{k_i}\right)$ provided they are integrable with respect to t (Integrability is necessary just to secure existence of the solution for the Cauchy problem in the Carathéodory sense). Also the Lipschitz* *property has now to hold uniformly with respect to t.*

We are now in position to state the main mathematical result of the paper.

Theorem: *Let system (6) be exponentially stable. Then system (3) is exponentially stable for all nonlinearities of the form:*

$$f_i(x_i) = \left(k_i + b_i(x_i)\right)x_i \tag{23}$$

with $b_i(x_i)$ defined by (22), if there exist definite positive matrices P_0, P_j, R_j, $j=\overline{1,n}$ and a positive value μ, such that

$$P_0 > \mu U^T(0)\left(\sum_{j=1}^{n} H_j^{-1}\right)U(0) \tag{24}$$

$$R_j > \mu A_j^T U(\tau_j+\theta)\left(\sum_{k=1}^{n} H_k^{-1}\right)U^T(\tau_j+\theta)A_j \tag{25}$$

$$P_j > \frac{2}{\mu}\rho_j I\,. \tag{26}$$

Sketch of the proof: Indeed, for some constants $\delta > 0$ and $\gamma > 0$ the functional $V(\varphi)$ verifies the inequalities

$$\delta\|\varphi(0)\|^2 \le V(\varphi) \le \gamma\|\varphi\|_\tau^2 \tag{27}$$

where $\|\varphi\| = \max\limits_{-\tau\le s\le 0}\|\varphi(s)\|$.

The strictly positive lower bound of V (with $\delta > 0$) is not valid in general since $V(\varphi)$ is a positive definite quadratic functional on an infinite dimensional space and the spectrum of a positive operator does not meet in general the compacity assumption. In our case, however taking into account (17), the well delimitation of V from 0 i.e. $\delta > 0$ is secured.

Along the solutions of the perturbed system (13) the derivative of the functional is

$$\frac{d}{dt}V(y(t+\cdot)) = -\widetilde{W}(y(t+\cdot)) = \tag{28}$$

$$= -W(y(t+\cdot)) + 2\left[\sum_{j=1}^{n}\Delta_j y(t-\tau_j)\right]^T \cdot$$

$$\cdot\left[U(0)y(t) + \sum_{j=1}^{n}\int_{-\tau_i}^{0} U^T(\tau_j+\theta)A_j y(t+\theta)d\theta\right]$$

If matrices Δ_j, $j=\overline{1,n}$ depend on t and (or) on $y(t-\tau_j)$, $j=\overline{1,n}$ but they satisfy inequalities (20), then there exists $\varepsilon > 0$ such that

$$\widetilde{W}(\varphi) \ge \varepsilon W(\varphi) \tag{29}$$

Here also the lower bound on $\widetilde{W}$ may not be strictly positive in general but, if one takes into account (21) the delimitation of $\widetilde{W}$ from 0 is again secured.

Conditions (27) and (29) imply that the zero solution of the perturbed system (13) is global exponentially stable.

4. CONCLUDING REMARKS

The present paper extends our previous work ([3], [4], [5]), and states sufficient condition for the exponentially stability of a cellular neural network with time delay feedback and zero control templates. The result is independent of the delay parameter.

Since the Liapunov method gives only sufficient stability conditions, the urgent task is, as usual, improvement of these conditions making them sharper i.e. closer to the necessary ones. A normal way for this is to improve the Liapunov function(al).

Here we took into account the following. In the linear case a quadratic Liapunov functional may provide necessary and sufficient conditions for exponential stability, but in the time delay case the sharpest most general quadratic Liapunov function (as suggested by the papers of Datko and Infante with Castelan - their exact references are to be found in [7]) is rather difficult to manipulate. On the other hand, the simplified versions which are currently used (including our earlier reference [3] - [5]) deserve improvement. This fact lead us to the approach of Kharitonov [6], [7] which gives a constructive approach to linear time lag robust systems which is sufficiently sharp.

The natural nonlinear extension of linear robustness is robustness with respect to sector restricted nonlinearities. This is of course the standard absolute stability problem and if we want to take advantage of all properties of a quadratic Liapunov function(al), the approach of Malkin is the simplest and also the most reasonable. Worth mentioning that we used it in both Malkin and Barbashin expositions which are formulated for the single nonlinearity case. Here we obtained *a result for several nonlinearities, also in the delay case*. Of course the conditions are sufficient and still require improvement. Such improvement can be obtained only by adequate application of the entire set of procedures and results of the Liapunov method; moreover, the Liapunov method remains the basic one in coping with such problems as oscillations and several equilibria.

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