Optimal Tuning of PI/PID/PID$^{(n-1)}$ Controllers in Active Disturbance Rejection Control

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Abstract: A new method based on the LQR approach is proposed for the optimal tuning of the Active Disturbance Rejection Control (ADRC). The input-output behavior of a complex system is approximated by a low order local model that allows designing PI and PID controllers into the ADRC framework. By imposing the condition of guaranteed dominant pole placement, it can be proposed a criterion for selecting the $Q$ and $R$ matrices in order to have a desired percentage overshoot and settling time of the closed loop response. It is also considered the extension of the method when the local model has a general order $n$, in this case, the controller will be of the PID$^{(n-1)}$ type. Some numerical examples are considered to show the effectiveness of the approach.

Keywords: ADRC, ESO, LQR, PI, PID, PID$^{(n-1)}$.

1. INTRODUCTION

Let us consider a Single-Input-Single-Output (SISO) plant which can be described exactly in its operating range by the following implicit input-output equation

$$
P(t, w(t), y(t), \dot{y}(t), \ldots, y^{(n_y)}(t), u(t), \dot{u}(t), \ldots, u^{(n_u)}(t)) = 0, \quad (1)
$$

where the derivatives orders satisfy the relation $n_y \geq n_u$ and $F(\cdot)$ is a sufficiently smooth function of the external disturbance $w(t)$, the input $u(t)$ and the output $y(t)$. Assume that for some integer $n$, such that $0 < n \leq n_y$, it is verified \( \frac{\partial F}{\partial y^{(n)}} \neq 0 \). The implicit function theorem yields then locally

$$
y(t)^{(n)} = G(t, w(t), y(t), \dot{y}(t), \ldots, y^{(n_y)}(t), u(t), \dot{u}(t), \ldots, u^{(n_u)}(t)) \cdot (2)
$$

By setting $G(t) = \ddot{f}(\cdot) + au(t)$ in (2), being $a$ a real unknown scaling factor of the system that can be approximated by $\dot{a}$, one has

$$
y(t)^{(n)} = f + \dot{a}u(t), \quad (3)
$$

where $f = \ddot{f}(\cdot) + (a - \dot{a})u(t)$. Finding simple and reliable differential equations for describing a particular system is a difficult task. In deriving a reasonably mathematical model, it is frequently necessary to ignore certain inherent properties of the system. In particular, phenomena like frictions, change of load, heat effects, aging, dispersions due to mass productions, environment and others, are not easy to take into account. Because of this, the equation (1) is only partially known.
ESO and DAC, UIO and DOB is that ESO was conceived to deal with nonlinear systems with mixed uncertainties (i.e. unmodeled dynamics and disturbances). The ADRC technique assumes that the control law is of the form (3) can be considered as one of the states of the system. An estimate of this state, provided by an ESO can be used in the control signal (4) to compensate for the real perturbation in the plant. ADRC has been applied with success to many practical problems: web tension regulation (You et al. 2001), motion control (Gao et al., 2001b), electric power assist steering system (Dong et al. 2010), chemical process control (Zheng et al. 2009), industrial motion control platform (Tian and Gao, 2009), uncertain multivariable system with time delay (Xia et al., 2007), MEMS gyroscopes (Zheng et al., 2008), and linear parameter varying systems (Teppa-Garran and Garcia, 2013a), to cite few of them.

The tuning procedure in ADRC was originally proposed in a nonlinear form (Han, 1998, 1999, 2009; Gao et al., 2001a), but the large number of gains made tuning an art. The structure was simplified to its linear form (Gao, 2003) and parameterized into a few gains. In its linear form, the tuning is essentially a pole-placement technique and the desired performance is indirectly achieved through the location of the closed-loop poles (controller and ESO). However, the final choice of these poles becomes a trial-and-error strategy that may be difficult for practicing engineers to fully understand and to competently apply to real systems. Linear Quadratic Regulator (LQR) (Kwakernaak and Sivan, 1972; Anderson and Moore, 1989) is a well-known design technique in modern optimal control theory and has been widely used in many applications. In contrast with pole-placement, the desired performance objectives are directly addressed by the LQR method. Finally, in section 6, some numerical examples are considered to show the effectiveness of the approach.

### 2. ADRC FUNDAMENTALS

By doing (5) a Proportional-Integral (PI) control law is also shown how to extend the optimal tuning method to the case of an ADRC method. For this, the control law must be of the PID type given by

\[ u_0(t) = k_p e(t) + k_i \int e(t) dt + k_d \frac{de(t)}{dt}, \]

where \( e(t) = r - y(t) \) is the tracking error, \( r \) is the set point and \( k_p, k_i, k_d \) are the usual PID gains. The LQR approach is then used to find an optimal PI/PID controller tuning algorithm for the ADRC method. It is proposed a criterion for selection of the \( Q \) and \( R \) matrices which lead to the desired settling time and percentage overshoot of the closed-loop system. Beside considering the use of a low order ADRC \((n = 1 \text{ or } n = 2)\) for controlling a complex general plant.
\( \dot{z}(t) = Az(t) + Bu_1(t) + L(y(t) - \hat{y}(t)) \) \tag{9} \\
\( \hat{y}(t) = Cz(t) \)

where \( z = [\dot{x}, f]^T \in \mathbb{R}^{n+1} \) is the estimated of the state vector \( x(t) \) and the generalized perturbation \( f \) and \( L = [\beta_1, \beta_2, \ldots, \beta_{n+1}]^T \) is the observer gain vector. As \( z_{n+1}(t) \rightarrow f(\cdot) \), it is used to actively cancel \( f(\cdot) \) in (6) by applying

\[
\begin{align*}
    u_1(t) &= u_0(t) - z_{n+1}(t),
\end{align*}
\]

this reduces the plant to \( y^{(n)} = (f - z_{n+1}) + u_0 \approx u_0 \), a unit gain integral chain system, allowing a PD type controller to be used, that is

\[
    u_0(t) = k_1(r - z_1(t)) - k_2z_2(t) - \cdots - k_nz_n(t). \tag{11}
\]

The set point \( r \) is only present in the proportional term; an approximate closed-loop transfer function is computed as

\[
    y(s) = \frac{k_1}{r(s)} = \frac{k_1}{s^n + k_2s^{n-1} + \cdots + k_1}.
\]

Where \( K = [k_1, k_2, \ldots, k_n] \) is the controller gain. The configuration of the ADRC is shown in Figure 1.

![Fig. 1. Standard ADRC configuration.](image)

### 3. LQR Design of the ADRC

The idea of ADRC is to divide the process of controller design into two parts: one is to compensate for the generalized perturbation, which is reconstructed by the input-output data via an ESO (GESO); the other is to realize the desired performance for the compensated system. Moreover, ADRC does not set strict mathematical constraints on the uncertainties to be estimated. Equation (14) is a result of the separation principle for a controller designed using an observer and a state-feedback constant-gain. It states, that the observer gain and state-feedback gain can be designed separately since the overall closed-loop eigenvalues are the union of those due to the observer alone and those due to the state-feedback controller alone. It can be proved that the separation principle holds for dynamic controllers and not just constant-gain controllers.

**Theorem 1**: In Fig. 1, if the constant-gain controller \( K = [K_1 \ 1] \) is replaced by the dynamic controller \( C(s) = [A_c \ B_c; C_c \ D_c] \) then the separation principle still holds.

**Proof**: See (Davison et al., 2003).

Theorem 1 allows to stay in the ADRC framework where one first compensate for the generalized perturbation by an ESO (the optimal computation of the ESO gain vector will be considered further in this section) and then, it is employed the LQR method for optimal tuning the gains of a PI/PID/PID(0-1) control law.

### 3.1 Optimal PI-ADRC Design

The real plant (1) is approximated continuously by the first order equation

\[
    \dot{y}(t) = f + u_1(t). \tag{15}
\]

Where \( f \) contains the whole structural information of the system, in the general case, is a nonlinear time-varying function of the variables of the system, including disturbances. It should be noted that (15) is a model that is used only for the purpose of the controller design. The controller, once designed should be applied to the process.

The extended-state space model of (15) is

\[
    \begin{align*}
        \dot{x}_1(t) &= [0 \ 1] \begin{bmatrix} x_1(t) \ x_2(t) \end{bmatrix} + [1 \ 0] u_1(t) + [0 \ 1] f \\
        y(t) &= [1 \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},
    \end{align*}
\]

where \( [x_1(t), x_2(t)]^T = [y(t), f]^T \) and the ESO is
\[
\begin{bmatrix}
\dot{z}_1(t) \\
\dot{z}_2(t)
\end{bmatrix} = 
\begin{bmatrix}
\beta_1 & 1 \\
\beta_2 & 0
\end{bmatrix}
\begin{bmatrix}
z_1(t) \\
z_2(t)
\end{bmatrix} + 
\begin{bmatrix}
1 \\
0
\end{bmatrix} u_1(t) + 
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} \gamma(t)
\]

\[
\dot{f} = [0 \quad 1]
\begin{bmatrix}
z_1(t) \\
z_2(t)
\end{bmatrix}
\]

for [\(z_1(t) \quad z_2(t)\)] = [\(\dot{\gamma}(t) \quad \dot{f}\)]. This ESO is of second order and there is a redundancy when the first component of the state may be measured directly. To reduce complexity, a reduced order ESO (RESO) (Tian, 2007; Zheng et al., 2011, Zheng et al., 2012) is employed to estimate \(x_y\), that is

\[
\dot{z}_2(t) = -\beta_1 z_2(t) - \beta_1 u_1(t) + \beta_1 \dot{\gamma}(t),
\]

where \(\beta_1\) is the RESO gain. The first derivative of \(\gamma(t)\) can be approximated from the difference of two neighboring \(y(t)\) sample values. However, in order to avoid intensifying measurement noise by direct numerical differentiation on signal \(y(t)\), a RESO without output derivative (Teppa-Garran and Garcia, 2013b) can be defined as

\[
\dot{z}_1(t) = -\beta_1 z_1(t) - \beta_1 u_1(t) - \beta_1^2 \gamma(t),
\]

(17)

Together with

\[
z_2(t) = \dot{z}_1(t) + \beta_1 \gamma(t).
\]

(18)

Using \(u_1(t) = u_0(t) - z_2(t)\) in (15) yields

\[
\dot{\gamma}(t) = u_0(t).
\]

(19)

It is defined \(u_0\) as a PI control law

\[
u_0(t) = k_1 \int e(t) dt + k_p e(t),
\]

where

\[
e(t) = r - y(t),
\]

(20)

is the tracking error. In the case of set-point reference \(r\), (20) can be expressed as

\[
\dot{e}(t) = -\dot{y}(t).
\]

(21)

Taking the derivative of (19) and using (21) gives

\[
\ddot{e}(t) = -\ddot{u}_0(t).
\]

(22)

Let \(v_1(t) = e(t)\) and \(v_2(t) = \dot{e}(t)\) then (22) can be expressed in state-space as

\[
\dot{\mathbf{v}}(t) = F \mathbf{v}(t) + G \ddot{u}_0(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \ddot{u}_0(t).
\]

(23)

In order to have a LQR formulation of the ADRC problem, the following quadratic cost is considered

\[
J = \int_0^\infty (\mathbf{v}(t)^T Q \mathbf{v}(t) + \rho \ddot{u}_0^2(t)) dt.
\]

(24)

where \(Q\) is a positive semi-definite matrix and \(\rho > 0\). It is well known that the minimization of (24) gives the state-feedback control

\[
\ddot{u}_0(t) = -K \mathbf{v}(t) = -k_1 e(t) - k_2 \dot{e}(t)
\]

(25)

where \(K = \rho^{-1} G^T P\) and \(P\) is the symmetric positive definite solution of the Continuous Algebraic Riccati Equation (CARE) given by

\[
F^T P + PF + Q - \rho^{-1} P G G^T P = 0.
\]

(26)

Taking integration on both sides of (25), the optimal PI-ADRC is obtained as

\[
u_0(t) = k_1 \int e(t) dt + k_p e(t),
\]

(27)

that is

\[
[k_1 \quad k_p] = [-k_1 \quad -k_2].
\]

Now, explicit expressions for the gains \(k_1\) and \(k_p\) are found. Substituting \(Q = \text{diag}(q_1 \quad q_2)\) and the symmetric matrix

\[
\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}
\]

into the CARE (26) yields

\[
\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}.
\]

(28)

Its positive definite solution is

\[
\begin{bmatrix} p_{12} = \sqrt{q_1 \rho} \\
p_{22} = \sqrt{2 \rho q_1 \rho + q_2 \rho} \\
p_{11} = \frac{p_{12} p_{22}}{\rho}
\end{bmatrix}
\]

(29)

and

\[
\mathbf{K} = \rho^{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -\frac{p_{12}}{\rho} & -\frac{p_{22}}{\rho} \\
\end{bmatrix}
\]

(30)

\textbf{Remark 1:} Because of the structure of matrices \(F\) and \(G\) in (23). The PI controller depends only on \(p_{12}\) and \(p_{22}\) in (30). The choice of a non-diagonal matrix \(Q\) in (28) produces the same result that a diagonal \(Q\). This is, there is not loss of generality in using a diagonal \(Q\).

In LQR design, the elements of the matrices \(Q\) and \(R\) are usually selected by trial and error, moreover, every different value of \(Q\) and \(R\) will eventually end up with a different system response, making the word optimal ambiguous. In order to overcome this difficulty, it is now derived a direct relationship between the \(Q\) matrix and the 2 % settling time (Ts) criterion and percentage overshoot (PO) of the closed-loop system.

\textbf{Theorem 2:} The closed-loop response of the ADRC system (23) with quadratic cost (24) and PI control law (27) has desired Ts and PO if and only if the elements of the matrix \(Q = \text{diag}(q_1 \quad q_2)\) in (28) are chosen as

\[
\begin{bmatrix} q_1 = \rho w_n^4 \\ q_2 = 2 \rho (2 \xi^2 - 1) w_n^2 \end{bmatrix}
\]

(31)

where \(w_n\) is the natural frequency and \(\xi\) is the damping ratio of the closed-loop response.

\textbf{Proof:} From PO \(= 100 e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}}\) it is obtained the damping ratio \(\xi\) and from 2% Ts criterion, the natural frequency is \(w_n = 4 / \text{CTs}\). By using (23) the characteristic polynomial is found as \(\det (\xi^2 - F + GK)\). Imposing the desired
The characteristic polynomial as the usual second order form \( s^2 + 2\xi w_n s + w_n^2 \), equating coefficients on both sides of the previous polynomials and then using (28) establishes the theorem.

**Remark 2:** The positive-definiteness of \( Q \) implies \( \xi > \sqrt{2}/2 \) in (31).

In Fig. 2, it is shown the optimal PI-ADRC.

**3.2 Optimal PID-ADRC design**

The real plant (1) is now continuously approximated by the local model

\[
\ddot{x}(t) = f + u_1(t). \tag{32}
\]

The extended state-space model of (32) for \( x = [x_1 \ x_2 \ x_3]^T = [y \ \dot{y} \ f]^T \) is

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t)
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} u_1(t) + \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} \dot{f}
\]

\[
y(t) = [1 \ 0 \ 0] x(t)
\]

It is supposed again that the controlled output \( y \) is available for feedback; the second-order RESO (Tian, 2007; Zheng et al., 2011, Zheng et al., 2012) is then

\[
\ddot{z}_1(t) = \begin{bmatrix}
-\beta_1 & 1 & 0 \\
-\beta_2 & 0 & 1 \\
-\beta_3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z_1(t) \\
z_2(t) \\
z_3(t)
\end{bmatrix}
+ \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} u_1(t) + \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} \dot{f}
\]

\[
\dot{f} = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
z_1(t) \\
z_2(t)
\end{bmatrix}
\]

where \( [z_1 \ z_2]^T = [\dot{y} \ \dot{f}]^T \).

**Remark 3:** In order to avoid numerical differentiation in computing the second-order RESO it is employed the result (Teppa-Garran and Garcia, 2013b) to express the RESO as

\[
\begin{bmatrix}
\ddot{z}_1 \\
\ddot{z}_2
\end{bmatrix} = \begin{bmatrix}
-\beta_1 & 1 & 0 \\
-\beta_2 & 0 & 1 \\
-\beta_3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
+ \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} u_1 + \begin{bmatrix}
\beta_1 & \beta_2 & \beta_3
\end{bmatrix} y
\]

combined with

\[
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix}
+ \begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} y.
\]

The control signal \( u_0 \) is now chosen of the PID type

\[ u_0(t) = k_1 \int e(t) dt + k_p e(t) + k_d \dot{e}(t) \]

Using \( u_1 = u_0 - z_2 \) in (32) yields

\[ \ddot{y}(t) = u_0(t). \tag{33} \]

Employing (20) and taking the derivative of (33) results in

\[ \dddot{y}(t) = -\ddot{u}_0(t). \tag{34} \]

Let \( v_1(t) = e(t), v_2(t) = \dot{e}(t) \) and \( v_3(t) = \ddot{e}(t) \) then (34) can be expressed in state-space as

\[
\begin{pmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
\dot{v}_3
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} v + \begin{pmatrix}
0 \\
0 \\
-1
\end{pmatrix} \dddot{u}_0.
\]

The minimization of the LQR formulation of the ADRC problem (24) gives now the state-feedback control

\[ u_0(t) = -K v(t) = -k_1 e(t) - k_2 \dot{e}(t) - k_3 \ddot{e}(t). \tag{36} \]

Taking integration on both sides of (36) results in the optimal PID-ADRC

\[ u_0(t) = k_1 \int e(t) dt + k_p e(t) + k_d \dot{e}(t), \tag{37} \]

that is

\[ [k_1 \ k_p \ k_d] = [-k_1 \ -k_2 \ -k_3]. \]

Now, explicit expressions for \( k_p \) and \( k_d \) are found. Considering \( Q = \text{diag}(q_1 \ q_2 \ q_3) \) and the symmetric matrix

\[
\begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{12} & p_{22} & p_{23} \\
p_{13} & p_{23} & p_{33}
\end{pmatrix}
\]

the optimal state feedback gain is

\[
K = \rho^{-1} G^T P = \begin{pmatrix}
-p_{11}/\rho & -p_{12}/\rho & -p_{13}/\rho
\end{pmatrix}
\]

This is

\[ [k_1 \ k_p \ k_d] = \begin{pmatrix}
p_{11}/\rho & p_{12}/\rho & p_{13}/\rho
\end{pmatrix}. \tag{38} \]

The closed-loop polynomial is

\[ \det(sI - F + G K) = s^3 + \frac{p_{11}}{\rho} s^2 + \frac{p_{12}}{\rho} s + \frac{p_{13}}{\rho}. \tag{39} \]

The desired closed-loop polynomial is chosen as

\[ (s + \lambda w_n)(s^2 + 2\xi w_n s + w_n^2) \]

where \( \lambda \gg \xi w_n \) for guaranteed dominant pole placement (GDPP). Equating coefficients on both sides of polynomials (39) and (40) produces

\[ p_{23} = \rho(\lambda + 2) \xi w_n \tag{41} \]

\[ p_{23} = \rho(2\lambda \xi^2 + 1) w_n^2. \]

\[ p_{13} = \rho \lambda \xi w_n \]

Solving the CARE (26) yields the nonlinear system of equations

\[ \begin{pmatrix}
q_1 \\
p_{11} \\
p_{12} \\
p_{13} \\
p_{21} \\
p_{22} \\
p_{23} \\
p_{31} \\
p_{32} \\
p_{33}
\end{pmatrix} = 0. \tag{42} \]

**Remark 4:** Again, there is not loss of generality in the choice of a diagonal \( Q \) in (42).
As the values $p_{13}, p_{23}$ and $p_{33}$ are already known in (41) one has from (42)

$$p_{11} = \frac{p_{13} p_{23}}{p_0},$$  
$$p_{12} = \frac{p_{13} p_{22}}{p_0},$$  
$$p_{22} = \frac{p_{22} p_{23}}{p_0} - p_{13}.$$  

And the matrix $Q$ becomes

$$Q = \text{diag}(q_1, q_2, q_3) = \text{diag}\left(\frac{p_{11}}{p_0}, \frac{p_{12}}{p_0} - 2p_{12}, \frac{p_{22}}{p_0} - p_{23}\right).$$

Using the closed-loop parameters defined previously in (40) allows expressing the matrix $Q$ components as

$$q_1 = \rho \lambda^2 \xi^2 w_n^2,$$
$$q_2 = \rho[1 + 2\lambda^2 \xi^2(2\xi^2 - 1)]w_n^2,$$
$$q_3 = \rho[\xi^2(\lambda^2 + 4) - 2]w_n^2.$$  

From the above results it can be summarized the procedure in the following theorem.

**Theorem 3:** The closed-loop response of the ADRC system (35) with quadratic cost (24), PID control law (37) and GDPP condition in (40) has desired Ts and PO if and only if the elements of the matrix $Q$ are chosen as established in (44).

**Remark 5:** The positive-definiteness of $Q$ implies again $\zeta > \sqrt{2}/2$.

In Fig.3, it is shown the PID-ADRC implementation.

3.3 Optimal RESO design

As stated, ADRC is based on the separation principle; this allows treating the unknown dynamic and disturbances in a physical process as the generalized disturbance, built an ESO to estimate it in real-time, and then cancelling its effect using the estimate as part of the control signal.

In the proposed PID-ADRC method, the second-order RESO equations are given by

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu_z(t) + L(y - \bar{y}) \\
\bar{y} &= Cz(t)
\end{align*}
\]

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$  

where $[z_1 \ z_2]^T = [\bar{y} \ f]^T$. The duality principle is now employed in order to have a LQR formulation of the RESO design. This is easily done by replacing $A \leftarrow A^T, B \leftarrow C^T$ and $K \leftarrow L^T$. The quadratic cost is chosen as

$$J = \int_0^\infty (z(t)^T Q_o z(t) + \rho_o u(t)^T) dt.$$  

The CARE equation becomes

$$AM + MA^T + Q_o - \rho_o^{-1}MC^T CM = 0.$$  

Substituting $Q_o = \text{diag}(Q_o_1, Q_o_2)$ and the symmetric matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix},$$

into (47) gives

$$2m_{12} + q_o_1 - m_{11}^2 = 0,$$

$$m_{22} - m_{11}m_{12} = 0,$$

$$q_o_2 = \frac{m_{12}^2}{\rho_o}.$$  

Its positive definite solution is

$$m_{12} = \sqrt{q_o_2 \rho_o},$$
$$m_{11} = \sqrt{2 \rho_o \sqrt{q_o_2 \rho_o} + q_o_1 \rho_o},$$
$$m_{22} - m_{11}m_{12} = 0.$$  

and the optimal RESO gain is found as

$$L = \rho_o^{-1}MC^T = [\beta_1 \ \beta_2]^T = \begin{bmatrix} m_{11} & m_{12} \end{bmatrix}^T.$$  

The RESO characteristic polynomial is computed as

$$s^2 + \beta_1 s + \beta_2 = s^2 + \frac{m_{11}}{\rho_o} s + \frac{m_{12}}{\rho_o},$$

and the desired RESO polynomial is imposed as

$$(s + w_0)^2 = s^2 + 2w_0 s + w_0^2,$$

where the closed-loop RESO bandwidth $w_0$ must satisfy the relation

$$w_0 \gg \lambda \gg \xi w_n.$$  

Equating coefficients of polynomials (49) and (50) yields

$$m_{11} = 2\rho_o w_0^2,$$

$$m_{12} = \rho_o w_0^2.$$  

Replacing (52) in (48) allows computing the components of the matrix $Q_o$ through

$$q_o_1 = 2\rho_o w_0^2,$$

$$q_o_2 = \rho_o w_0^2,$$

and the optimal RESO gains are obtained doing

$$\beta_1 = 2w_0,$$

$$\beta_2 = w_0.$$  

In the case of PI-ADRC design, it is easy to show that for the first order RESO (17-18), one has

$$\beta_1 = w_0.$$
Where \( w_0 \) must satisfy
\[
w_0 \gg \zeta w_n, \tag{55}\]
From the above procedure it can be established the following theorem.

**Theorem 4**: Let a RESO be described as in (16) for PI-ADRC (resp. (45) for PID-ADRC) with closed-loop bandwidth \( w_0 \) that satisfies (55) (resp. (51) for PID-ADRC), then the optimal RESO gains that minimizes the quadratic cost (46) are given by (54) (resp. (53) for PID-ADRC).

4. OPTIMAL PI/PID-ADRC ALGORITHMS

It is summarized the entire development of design in two algorithms given in Table 1 for PI-ADRC and in Table 2 for the case of PID-ADRC.

**Table 1. Algorithm for Optimal PI-ADRC design.**

| Input: PO ← Desired closed-loop percent overshoot, Ts ← desired closed-loop settling time. |
| 1: From PO obtain the closed-loop damping ratio as \( \zeta = \frac{\sqrt{(lnPO/100)^2}}{\pi^2 + (lnPO/100)^2} \). From Ts obtain the closed-loop natural frequency as \( w_n = 4/CTs \). |
| 2: Compute \( w_0 \) in the RESO from (55). |
| 3: Compute the RESO gain \( \beta_1 \) from (54). |
| 4: Compute \( q_1 \) and \( q_3 \) from (31). |
| 5: Compute \( p_{12} \) and \( p_{22} \) from (29). |
| 6: Compute \( k_p \) and \( k_i \) from (30). |

**Output:** PI parameters \( k_p \) and \( k_i \) and RESO gain \( \beta_1 \).

**Table 2. Algorithm for Optimal PID-ADRC design.**

| Input: PO ← Desired closed-loop percent overshoot, Ts ← desired closed-loop settling time. |
| 1: From PO obtain the closed-loop damping ratio as \( \zeta = \frac{\sqrt{(lnPO/100)^2}}{\pi^2 + (lnPO/100)^2} \). From Ts obtain the closed-loop natural frequency as \( w_n = 4/CTs \). |
| 2: Compute \( w_0 \) in the RESO from (51). |
| 3: Compute the RESO gains \( \beta_1 \) and \( \beta_2 \) from (53). |
| 4: Compute \( q_1, q_2 \) and \( q_3 \) from (44). |
| 5: Compute \( p_{13}, p_{23} \) and \( p_{43} \) from (41). |
| 6: Compute \( k_p, k_i \) and \( k_d \) from (38). |

**Output:** PID parameters \( k_p \), \( k_i \) and \( k_d \) and RESO gain \( L = [\beta_1 \beta_2]^T \).

For ease of reference in Table 3 are given the optimal values of the gains of the PI/PID – ADRC. The relation (55), that allows to select the RESO bandwidth \( w_0 \) depending on the desired real part of the closed-loop poles \( \zeta w_n \) is fixed as
\[
w_0 = \alpha_2 \alpha_4 \zeta w_n, \tag{56}\]
where \( \alpha_2 \in \mathbb{R}^+. \) In the case of relation (51) it is done
\[
w_0 = \alpha_2 \alpha_4 \zeta w_n, \tag{57}\]
with \( \alpha_2 \in \mathbb{R}^+, \lambda = \alpha_1 \zeta w_n \) and \( w_0 = \alpha_2 \lambda. \) Some guidelines to choose \( \alpha_1 \) and \( \alpha_2 \) will be given in the next section.

**Table 3. Optimal PI/PID – ADRC gains.**

<table>
<thead>
<tr>
<th></th>
<th>( k_i )</th>
<th>( k_p )</th>
<th>( k_d )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PI ADRC</td>
<td>( w_n^2 )</td>
<td>( 2\zeta w_n )</td>
<td>( \alpha_1 \zeta w_n )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PID ADRC</td>
<td>( \alpha_1 \zeta \zeta w_n )</td>
<td>( 2\zeta \zeta w_n + 1 )</td>
<td>( \alpha_1 \zeta w_n )</td>
<td>( \alpha_2 \zeta \zeta w_n )</td>
<td>( \alpha_2^2 \zeta \zeta w_n )</td>
</tr>
</tbody>
</table>

5. LQR DESIGN OF THE ADRC: GENERAL CASE

Now it is shown how to extend the method to the case when the real plant (1) is approximated continuously by the \( n \)th order equation (6). The extended state space model of (6) is given by (8). The \( n \)th order RESO (Tian, 2007; Zheng et al., 2011, Zheng et al., 2012) is
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\vdots \\
\dot{z}_{n-1} \\
\dot{z}_n
\end{bmatrix} =
\begin{bmatrix}
\beta_1 & 1 & 0 & \cdots & 0 \\
\beta_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
\beta_{n-1} & 0 & \cdots & 1 & 0 \\
\beta_n & 0 & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{n-1} \\
z_n
\end{bmatrix}
\tag{58}
\]

where
\[
z = [z_1 \ z_2 \ \cdots \ z_{n-1} \ z_n]^T = \left[ \begin{array}{c} \hat{y} \\ \dot{y} \\ \ddots \\ \dot{y}^{(n-1)} \\ f \end{array} \right]^T.
\]

**Remark 6**: In order to avoid numerical differentiation in computing the \( n \)th-order RESO in (58) it is used the result (Teppa-Garran and Garcia, 2013b) to express the RESO as
\[
\dot{\bar{z}}(t) =
\begin{bmatrix}
-\beta_1 & 1 & 0 & \cdots & 0 \\
-\beta_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
-\beta_{n-1} & 0 & \cdots & 1 & 0 \\
-\beta_n & 0 & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\bar{z}(t) \\
\dot{\bar{z}}(t) \\
\vdots \\
\ddots \\

\end{bmatrix}
\tag{59}
\]
combined with
\[
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_n
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \cdots & 0 & \beta_1 \\
0 & 1 & \cdots & 0 & \beta_2 \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & \cdots & 1 & \beta_{n-1} \\
0 & 0 & \cdots & 0 & \beta_n
\end{bmatrix}
\begin{bmatrix}
\bar{y}(t) \\
\dot{\bar{y}}(t) \\
\vdots \\
\ddots \\
\end{bmatrix}
\tag{60}
\]

Using \( u_1 = u_0 - z_n \) in (6) yields
\[
y(t)^{(n)} = u_0(t).
\tag{59}
\]

Employing (20) and taking the derivative of (59) results in
\[
e(t)^{(n+1)} = -\dot{u}_0(t).
\tag{60}
\]
Doing
\[
v = [v_1, v_2, \ldots, v_n, v_{n+1}]^T
\]
\[ \dot{v}(t) = Fv(t) + G\dot{u}_0(t). \]  
(61)

The state space model given by (60) gives

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
-1 
\end{bmatrix}_{x(n+1)}
\]

The minimization of the LQR formulation of the ADRC problem (24) gives

\[
\dot{x}_0(t) = -P_1 \begin{bmatrix} k_1 & k_p & k_d_1 & \ldots & k_{d_{n-1}} \end{bmatrix} v(t)
\]
(62)

Taking integration on both sides of (62) results in the optimal PID \((n-1)\)-ADRC

\[
\begin{align*}
\int \left( \begin{array}{c}
\dot{x}_0(t) + k_p e(t) + k_d \dot{e}(t) + \ldots + k_{d_{n-1}} e^{(n-1)} \\
k_{d_{n-1}} e^{(n-1)}
\end{array} \right) dt &= 0 \\
\int k_i & \begin{bmatrix} k_1 & k_p & k_d_1 & \ldots & k_{d_{n-1}} \end{bmatrix} v(t) dt = 0
\end{align*}
\]
(63)

that is

\[
\begin{bmatrix}
k_1 & k_p & k_d_1 & \ldots & k_{d_{n-1}} \\
-k_1 & -k_2 & -k_3 & \ldots & -k_{n+1}
\end{bmatrix}
\]

Now, explicit expressions for the controller gains \(k_p, k_{d_1}, \ldots, k_{d_{n-1}}\) are found. Considering

\[
Q = \text{diag}(q_1, q_2, \ldots, q_{n+1})
\]

and the symmetric matrix

\[
\begin{align*}
\left( \begin{array}{cc}
(n-1) & 0 \\
0 & n-1 
\end{array} \right) s^{n+1} + & \left( \begin{array}{cc}
(n-1) & 2 \zeta \omega_n \\
0 & 1 \end{array} \right) \omega_n^2 s^n \\
+ & \left( \begin{array}{cc}
(n-1) & 1 \\
0 & 3 \end{array} \right) \zeta \omega_n^2 s^{n-1} \\
+ & \left( \begin{array}{cc}
(n-1) & 2 \\
0 & 4 \end{array} \right) \zeta \omega_n^2 s^{n-2} \\
+ & \left( \begin{array}{cc}
(n-1) & 3 \\
0 & 5 \end{array} \right) \zeta \omega_n^2 s^{n-3} \\
& \vdots \\
+ & \left( \begin{array}{cc}
(n-1) & 4 \\
0 & 6 \end{array} \right) \zeta \omega_n^2 s^{n-4} \\
+ & \left( \begin{array}{cc}
(n-1) & 5 \\
0 & 7 \end{array} \right) \zeta \omega_n^2 s^{n-5} \\
+ & \left( \begin{array}{cc}
(n-1) & 6 \\
0 & 8 \end{array} \right) \zeta \omega_n^2 s^{n-6} \\
+ & \left( \begin{array}{cc}
(n-1) & 7 \\
0 & 9 \end{array} \right) \zeta \omega_n^2 s^{n-7} \\
+ & \left( \begin{array}{cc}
(n-1) & 8 \\
0 & 10 \end{array} \right) \zeta \omega_n^2 s^{n-8} \\
& \vdots \\
+ & \left( \begin{array}{cc}
(n-1) & 9 \\
0 & 11 \end{array} \right) \zeta \omega_n^2 s^{n-9} \\
& \vdots \\
+ & \left( \begin{array}{cc}
(n-1) & 10 \\
0 & 12 \end{array} \right) \zeta \omega_n^2 s^{n-10}
\end{align*}
\]

where \(n \geq 2\) and the binomial coefficient for \(k \leq n\) is

\[
\begin{bmatrix} n \end{bmatrix}_k = \frac{n!}{k!(n-k)!}
\]

Using (64) and equating coefficients on both sides of polynomials (65) and (67) give the optimal gains of the PID\((n-1)\)-ADRC, that is

\[
\begin{align*}
k_1 &= \left( \begin{array}{c}
(n-1) \\
0 \\
\vdots \\
0 \\
-1 
\end{array} \right) \omega_n^2 \zeta \omega_n^{n-1} \\
k_p &= \left( \begin{array}{c}
(n-1) \\
0 \\
\vdots \\
0 \\
-1 
\end{array} \right) \omega_n^2 \zeta \omega_n^{n-2} \\
k_d_1 &= \left( \begin{array}{c}
(n-1) \\
0 \\
\vdots \\
0 \\
-1 
\end{array} \right) \omega_n^2 \zeta \omega_n^{n-3} \\
& \vdots \\
k_{d_{n-1}} &= \left( \begin{array}{c}
(n-1) \\
0 \\
\vdots \\
0 \\
-1 
\end{array} \right) \omega_n^2 \zeta \omega_n^{n-1}
\end{align*}
\]
Remark 7: As expected, if (68) is evaluated for \( n = 2 \) one obtains the PID-ADRC gains computed previously.

The optimal RESO gain \( L = [\beta_1 \ldots \beta_n]^T \) in (58) is found by minimizing the quadratic cost (46), that is

\[
L = \rho_0^{-1}MC^T = [\beta_1 \ldots \beta_n]^T = \left[ \frac{m_{11}}{\rho_0} \ldots \frac{m_{n,n+1}}{\rho_0} \right]^T.
\]

Following a similar procedure, it can be obtained

\[
\beta_i = \frac{n!}{(n-i)!} w_0^2 \quad i = 1, \ldots, n.
\]

(69)

If it is considered the presence of measurement noise \( \eta(t) \) in the controlled output \( y(t) \), the estimation error dynamics of the ESO (12) takes the form

\[
\dot{e}(t) = (A - LC)e(t) + E\dot{f}(t) - L\eta(t).
\]

It is evident that a large value for \( L \) will enhance the effect of the measurement noise, since this is usually a high-frequency signal. It is needed a compromise between speed of response and noise immunity. In PI/PID/PID\(^{(n-1)}\)-ADRC, for controller design, the requirements of the closed-loop performance in time domain (PO and Ts) are converted into a pair of conjugate poles \( \lambda = -\alpha \pm jb \) where \( a = \xi w_n \). When one has a derivative order \( n \geq 2 \) in (6) the condition (66) requires that the ratio of the real part of any other poles to \( -\alpha \) exceeds \( \lambda \). For GDPP, the constant \( \lambda \) is usually chosen following the rule

\[
\lambda = 3 \text{ to } 5 \text{ times } (\xi w_n).
\]

(70)

The larger \( \xi w_n \), the faster the response, the larger the control signal and a system more susceptible to noise. For RESO design, it can be chosen the bandwidth \( w_0 \) by fixing \( \alpha_1 \) in (56) for \( n = 1 \) in (6) or \( \alpha_2 \) in (57) for \( n \geq 2 \) in (6). In the case of \( \alpha_1 \) one fixes

\[
\alpha_1 = 2 \text{ to } 6 \text{ times } (\xi w_n).
\]

(71)

And for \( \alpha_2 \) it is used

\[
\alpha_2 = 2 \text{ to } 6 \text{ times } (\lambda).
\]

(72)

This ensures the observer errors decay faster than the desired closed-loop dynamics allowing the controller poles to dominate the total response. If sensor noise if a problem then the observer poles may be chosen slower than two times \( (\xi w_n \text{ or } \lambda) \). This would yield a system with lower bandwidth, more noise smoothing and less control energy expenditure.

6. SOME EXAMPLES

In this section, some numerical examples are considered to show the effectiveness of the method. In all the computer simulations, the values of the controller’s gains predicted by the design equations are employed without doing any post tuning refinement.

6.1 Mass-spring system with dynamic friction

The motion equation of a mass-spring system with dynamic friction is adapted from (Fliess et al., 2011) as

\[
\ddot{y} = -6y - 20y^3 - 5\dot{y} + 2F(\dot{y}) + 2u.
\]

(73)

Where the dynamic friction is given by

\[
F(\dot{y}) = \begin{cases} 0.3 + 0.4(\dot{y} + 0.25)^2 - 5\dot{y} & \text{if } \dot{y} < 0 \\ -0.3 - 0.4(\dot{y} + 0.25)^2 - 5\dot{y} & \text{if } \dot{y} > 0 \end{cases}
\]

A PI-ADRC is designed for (6) considering \( PO = 0.1, Ts = 4 \) and \( \alpha_1 = 5 \) and a PID-ADRC for (32) when the condition \( a_2 = 3 \) is added. The Fig. 4 shows the results for a step set point and Fig. 5 the resulting control inputs. In Fig. 6, it is added a constant disturbance of amplitude 0.1 applied since \( t = 10 \). Finally, in Fig. 7 a zero mean Gaussian white noise of variance 0.01 is added at the output for testing the robustness property of the design. It is evident from the results, the good performance of both controllers.

Fig. 4. PI/PID-ADRC tracking for a mass-spring system with dynamic friction.

Fig. 5. PI/PID-ADRC command inputs for the tracking problem of a mass-spring system with dynamic friction.

Fig. 6. PI/PID-ADRC tracking and disturbance rejection for a mass-spring system with dynamic friction.
6.2 Unstable linear system with a large spectrum

The system is characterized by the transfer function

\[
\frac{s^5}{(s + 10)(s + 1)(s + 0.1)(s - 0.2)(s - 2)(s - 20)}
\]

A PI-ADRC is designed for \( P_0 = 0.1, Ts = 0.4 \) and \( \alpha_i = 5 \) when the system is locally described by (6). The Fig. 8 shows the effectiveness of the tracking of a square signal of amplitude 1 and frequency 0.1 Hz.

6.3 Inverted pendulum

The motion equation of the inverted pendulum of Fig. 9 is given by

\[
J\ddot{y}(t) - mgl\sin[y(t)] + b\dot{y}(t) = u(t),
\]

where \( y(t) \) describes the angular deviation from the upright position. It is assumed a damping term proportional to the angular velocity and that it is possible to affect the pendulum by a torque \( u(t) \) at its base. The parameters of the system are in Table 4. The controller’s gains computed in the design of the mass-spring control system are used for the inverted pendulum. The Fig. 10 exhibits the tracking performance of a square signal of amplitude 1 and frequency 0.01 Hz. The Fig. 11 (zoomed) considers the disturbance rejection of a 0.1 amplitude step signal applied since \( t = 10 \) in the output channel. The results show that PI-ADRC has a little better performance in the case of tracking and PID-ADRC has a better one in the case of step disturbance rejection. If it is done a post tuning adjustment of the controllers gains or it is modified the estimate of the scaling factor \( \hat{a} \), the results may be distinctly different.

Table 4. Parameters of the inverted pendulum.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>Mass of the sphere</td>
<td>0.1 Kg</td>
</tr>
<tr>
<td>( g )</td>
<td>Gravity on the surface of the earth</td>
<td>9.8 m/s²</td>
</tr>
<tr>
<td>( l )</td>
<td>Length of the rigid rod</td>
<td>1 m</td>
</tr>
<tr>
<td>( J )</td>
<td>Moment of the inertia</td>
<td>0.1 Kg.m²</td>
</tr>
</tbody>
</table>

7. CONCLUSIONS

It has been developed a method that allows tuning a PI/PID/PID\(^{(n-1)}\) controller into the ADRC framework. The
method makes use of the LQR approach and by imposing the condition of guaranteed dominant pole placement; it is defined a criterion for selecting the $Q$ and $R$ matrices in order to have a desired percentage overshoot and settling time of the closed loop response. Some numerical examples are considered to show the effectiveness of the approach.

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REFERENCES