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Optimal Tuning of PI/PID/PID$^{(n-1)}$ Controllers in Active Disturbance Rejection Control

Pedro Teppa-Garran*, Germain Garcia**

** Departamento de Procesos y Sistemas, Universidad Simón Bolívar, Valle de Sartenejas, Municipio Baruta, Estado Miranda, Caracas, Venezuela (pteppa@usb.ve)*
***CNRS, LAAS, 7 Avenue du Colonel Roche, F-31400 Toulouse, France and Université de Toulouse, LAAS, INSA, Toulouse, France (garcia@laas.fr)*

Abstract: A new method based on the LQR approach is proposed for the optimal tuning of the Active Disturbance Rejection Control (ADRC). The input-output behavior of a complex system is approximated by a low order local model that allows designing PI and PID controllers into the ADRC framework. By imposing the condition of guaranteed dominant pole placement, it can be proposed a criterion for selecting the Q and R matrices in order to have a desired percentage overshoot and settling time of the closed loop response. It is also considered the extension of the method when the local model has a general order n, in this case, the controller will be of the PID $^{(n-1)}$ type. Some numerical examples are considered to show the effectiveness of the approach.

Keywords: ADRC, ESO, LQR, PI, PID, PID$^{(n-1)}$.

1. INTRODUCTION

Let us consider a Single-Input-Single-Output (SISO) plant which can be described exactly in its operating range by the following implicit input-output equation

$$F\begin{pmatrix} t, w(t), y(t), \dot{y}(t), \ldots, y(t)^{(n_y)}, \\ u(t), \dot{u}(t), \ldots, u(t)^{(n_u)} \end{pmatrix} = 0, \tag{1}$$

where the derivatives orders satisfy the relation $n_y \geq n_u$ and $F(\cdot)$ is a sufficiently smooth function of the external disturbance $w(t)$, the input $u(t)$ and the output $y(t)$. Assume that for some integer n, such that $0 < n \leq n_y$, it is verified $\frac{\partial F}{\partial y(t)^{(n)}} \neq 0$. The implicit function theorem yields then locally

$$y(t)^{(n)} = G\begin{pmatrix} t, \mathrm{w}(t), y(t), \dot{y}(t), \ldots \\ y(t)^{(n-1)}, y(t)^{(n+1)}, \ldots \\ \ldots y(t)^{(n_y)}, u(t), \dot{u}(t), \ldots, u(t)^{(n_u)} \end{pmatrix}. \tag{2}$$

By setting $G(\cdot) = \bar{f}(\cdot) + \alpha u(t)$ in (2), being α a real unknown scaling factor of the system that can be approximated by $\hat{\alpha}$, one has

$$y(t)^{(n)} = f + \hat{\alpha} u(t), \tag{3}$$

where $f = \bar{f}(\cdot) + (\alpha - \hat{\alpha})u(t)$. Finding simple and reliable differential equations for describing a particular system is a difficult task. In deriving a reasonably mathematical model, it is frequently necessary to ignore certain inherent properties of the system. In particular, phenomena like frictions, change of load, heat effects, aging, dispersions due to mass productions, environment and others, are not easy to take into account. Because of this, the equation (1) is only partially known.

Active Disturbance Rejection Control (ADRC) (Han, 1998, 1999, 2009; Gao et al., 2001a) is a method that does not require a detailed mathematical description of the system. The basic idea is to model the system with an input disturbance that represents any difference between the model and the actual system, including external disturbances, this input is them lumped in the term f in (3). By estimating the disturbance in real time, the information is feedback to cancel the disturbance using the control law

$$u(t) = (u_0(t) - \hat{f})/\hat{\alpha}, \tag{4}$$

where $\hat{f}$ is the estimated of the generalized disturbance f. Replacing (4) in (3) forces the plant to behave as a desired simple plant, a series of cascaded integrators

$$y(t)^{(n)} = u_0(t). \tag{5}$$

The control law u_0 is chosen in ADRC of the Proportional-Derivative (PD) type.

One of the main issues in control is to deal with uncertainties including internal (parameter and unmodeled dynamics) and external (disturbances). However, most uncertainties are not measurable. Hence, how to estimate uncertainties by using the control input and output of the system is a significant problem. Many approaches such as, disturbance accommodation control (DAC) (Johnson, 1971, 1976), the unknown input observer (UIO) (Basile and Marro, 1969; Hostetter and Meditch, 1973), the disturbance observer (DOB) (Profeta et al., 1990; Bickel and Tomizuka, 1999; Schrijver and Van Dijk, 2002) and the extended state observer (ESO) (Han, 1998, 1999, 2009; Gao et al., 2001a) have been proposed to estimate uncertainties from the input-output data. In DAC, UIO and DOB the external disturbance of a linear time-invariant system is estimated and then rejected. DAC and UIO can be viewed as a special case of DOB (Profeta et al., 1990). The main difference between

ESO and DAC, UIO and DOB is that ESO was conceived to deal with nonlinear systems with mixed uncertainties (i.e. unmodeled dynamics and disturbances). The ADRC technique assumes that f in (3) can be considered as one of the states of the system. An estimate of this state, provided by an ESO can be used in the control signal (4) to compensate for the real perturbation in the plant. ADRC has been applied with success to many practical problems: web tension regulation (You et al., 2001), motion control (Gao et al., 2001b), electric power assist steering system (Dong et al. 2010), chemical process control (Zheng et al., 2009), industrial motion control platform (Tian and Gao, 2009), uncertain multivariable system with time delay (Xia et al., 2007), MEMS gyroscopes (Zheng et al., 2008), and linear parameter varying systems (Teppa-Garran and Garcia, 2013a), to cite few of them.

The tuning procedure in ADRC was originally proposed in a nonlinear form (Han, 1998, 1999, 2009; Gao et al., 2001a), but the large number of gains made tuning an art. The structure was simplified to its linear form (Gao, 2003) and parameterized into a few gains. In its linear form, the tuning is essentially a pole-placement technique and the desired performance is indirectly achieved through the location of the closed-loop poles (controller and ESO). However, the final choice of these poles becomes a trial-and-error strategy that may be difficult for practicing engineers to fully understand and to competently apply to real systems. Linear Quadratic Regulator (LQR) (Kwakernaak and Sivan, 1972; Anderson and Moore, 1989) is a well-known design technique in modern optimal control theory and has been widely used in many applications. In contrast with pole-placement, the desired performance objectives are directly addressed by minimizing a quadratic function of the state and control input. On the other hand, one major criticism on LQR design is that the selection of matrices Q and R is not clear in order to meet the closed-loop performance specifications.

The main contribution of this this paper is to develop an optimal tuning of the ADRC method that guarantees some closed-loop specifications by employing the LQR technique and a reduced tuning parameters set. The input-output behaviour of the system will be approximated by the local model (3) considering a derivative order of $n=1$ or $n=2$. This is the case when in (1) it is verified $\frac{\partial F}{\partial y(t)^{(i)}} \neq 0$ for $i=1,\ 2$. Then, for a derivative order $n=1$ in (3) it is proposed in (5) a Proportional-Integral (PI) control law

$$u_0(t) = k_p e(t) + k_i \int e(t)dt,$$

and for $n=2$ a PID control law

$$u_0(t) = k_p e(t) + k_i \int e(t)dt + k_d \frac{de(t)}{dt},$$

where $e(t) = r - y(t)$ is the tracking error, r is the set point and k_p, k_i and k_d are the usual PID gains. The LQR approach is then used to find an optimal PI/PID controller tuning algorithm for the ADRC method. It is proposed a criterion for selection of the Q and R matrices which lead to the desired settling time and percentage overshoot of the closed-loop system. Beside considering the use of a low order ADRC ($n=1$ or $n=2$) for controlling a complex general plant. It is also shown how to extend the optimal tuning method to the case of nth order ADRC. For this, the control law must be of the PID $^{(n-1)}$ type given by

$$u_0(t) = k_p e(t) + k_i \int e(t)dt + k_{d_1}\dot{e}(t) + k_{d_2}\ddot{e}(t) + \cdots + k_{d_{n-1}} e^{(n-1)}(t).$$

The article is organized as follows. Section 2 considers the fundamentals of the ADRC theory. In section 3, it is employed the LQR method for optimal tuning of the PI/PID - ADRC gains and for designed an ESO by the duality principle. In section 4 are summarized the results into some design algorithms. The extension of the method to the nth order ADRC optimal tuning is considered in section 5, where it is shown how to select the gains of the PID $^{(n-1)}$ controller via the LQR method. Finally, in section 6, some numerical examples are considered to show the effectiveness of the approach.

Notation: Capital bold typeface letters denote matrices and small bold typeface letters denote vectors. $\dot{a} = da/dt$, $a^{(i)} = d^i a/dt^i$, $A^T = transpose\ of\ A$, $\mathbb{R}$ is the set of real numbers, $\hat{a} = estimate\ of\ vector\ a$, $rk(A) = rank\ of\ A$, $\det(A) = determinant\ of\ matrix\ A$.

2. ADRC FUNDAMENTALS

By doing $\hat{\alpha}u(t) = u_1(t)$, the plant equation (3) changes to

$$y(t)^{(n)} = f + u_1(t). \tag{6}$$

Let the state vector be

$$\bar{\boldsymbol{x}}(t) = [x_1(t),\ x_2(t), \dots, x_n(t)]^T = [y(t),\ \dot{y}(t), \dots, y(t)^{(n-1)}]^T.$$

The state space model of the plant can be written as

$$\begin{cases} \dot{\bar{\boldsymbol{x}}}(t) = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{B}}(f + u_1(t)) \\ \quad y(t) = \bar{\boldsymbol{C}}\bar{\boldsymbol{x}}(t) \end{cases}, \tag{7}$$

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(n\,x\,n)} \quad \bar{\boldsymbol{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(n\,x\,1)}$$

$$\bar{\boldsymbol{C}} = [1 \quad 0 \quad \dots \quad 0]_{(1\,x\,n)}.$$

The ESO regards all factors affecting the plant, including the nonlinear dynamics, uncertainties and the external disturbances as a generalized disturbance f that will be treated as an extended state to be observed. Let $x_{n+1} = f$, the state vector is now $\boldsymbol{x} = [\bar{\boldsymbol{x}},\ f]^T \in \mathbb{R}^{n+1}$ and the state space model of the plant becomes

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u_1(t) + \boldsymbol{E}\dot{f}(t) \\ \quad y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}, \tag{8}$$

$$\boldsymbol{A} = \begin{bmatrix} \bar{\boldsymbol{A}}_{n\,x\,n} & \bar{\boldsymbol{B}}_{n\,x\,1} \\ \boldsymbol{0}_{1\,x\,n} & 0_{1\,x\,1} \end{bmatrix}_{(n+1)x(n+1)} \quad \boldsymbol{B} = \begin{bmatrix} \bar{\boldsymbol{B}}_{n\,x\,1} \\ 0_{1\,x\,1} \end{bmatrix}_{(n+1)x1}$$

$$\boldsymbol{C} = [\bar{\boldsymbol{C}}_{1\,x\,n} \quad 0_{1\,x\,1}]_{1x(n+1)} \quad \boldsymbol{E} = \begin{bmatrix} 0_{n\,x\,1} \\ 1_{1\,x\,1} \end{bmatrix}_{(n+1)x1}.$$

For system (8) the ESO is designed as follows

$$\begin{cases} \dot{\boldsymbol{z}}(t) = \boldsymbol{A}\boldsymbol{z}(t) + \boldsymbol{B}u_1(t) + \boldsymbol{L}(y(t) - \hat{y}(t)) \\ \hat{y}(t) = \boldsymbol{C}\boldsymbol{z}(t) \end{cases}, \tag{9}$$

where $\boldsymbol{z} = [\hat{\boldsymbol{x}}, \hat{f}]^T \in \mathbb{R}^{n+1}$ is the estimated of the state vector $\boldsymbol{x}(t)$ and the generalized perturbation f and $\boldsymbol{L} = [\beta_1, \beta_2, \dots, \beta_{n+1}]^T$ is the observer gain vector. As $z_{n+1}(t) \to f(\cdot)$, it is used to actively cancel $f(\cdot)$ in (6) by applying

$$u_1(t) = u_0(t) - z_{n+1}(t), \tag{10}$$

this reduces the plant to $y^{(n)} = (f - z_{n+1}) + u_0 \approx u_0$, a unit gain integral chain system, allowing a PD type controller to be used, that is

$$u_0(t) = k_1(r - z_1(t)) - k_2 z_2(t) - \dots - k_n z_n(t). \tag{11}$$

The set point r is only present in the proportional term; an approximate closed-loop transfer function is computed as

$$\frac{y(s)}{r(s)} = \frac{k_1}{s^n + k_n s^{n-1} + \dots + k_1}.$$

Where $\boldsymbol{K} = [k_1, k_2, \dots, k_n]$ is the controller gain. The configuration of the ADRC is shown in figure 1.

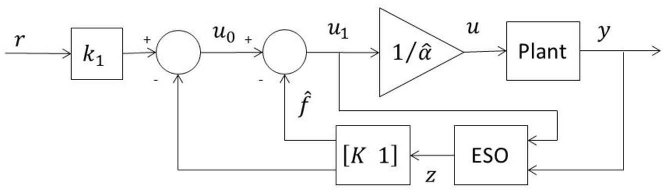

Fig. 1. Standard ADRC configuration.

The estimation error is

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{z}(t).$$

Using (8) and (9) the dynamics of the state estimation error of the ESO is given as

$$\dot{\boldsymbol{e}}(t) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}(t) + \boldsymbol{E}\dot{f}(t). \tag{12}$$

Expressing u_1 in (10 - 11) as

$$u_1(t) = k_1 r - [\boldsymbol{K} \quad 1]\boldsymbol{z}(t), \tag{13}$$

and then combining (7), (12) and (13) yields the closed-loop ADRC equations

$$\begin{bmatrix} \dot{\bar{\boldsymbol{x}}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{A}} - \bar{\boldsymbol{B}}\boldsymbol{K} & \bar{\boldsymbol{B}}(\boldsymbol{K} \quad 0) \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{e}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}k_1 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ \boldsymbol{E} \end{bmatrix} \dot{f}. \tag{14}$$

From (14), it is straightforward to verify that the eigenvalues of the system matrix of the closed-loop ADRC equations are given by the eigenvalues of $(\bar{\boldsymbol{A}} - \bar{\boldsymbol{B}}\boldsymbol{K})$ and $(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})$. Since it can be shown that the pair $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}})$ is controllable and the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, the stability of (14) can always be ensured by placing the controller and observer poles appropriately. Moreover, under the assumption of boundedness of $\dot{f}(\cdot)$, the BIBO stability of (14) is assured (Zheng et al., 2007). This is the case when $\dot{f}(\cdot) = 0$ or its rate of change is small. If the rate of change is not negligible, one can design a generalized extended state observer (GESO) of pth order (Miklosovic et al., 2006) to estimate the state together with $f, \dot{f}, \ddot{f}, \dots, f^{(p-1)}$. If $f^{(p)}$ is negligible, the closed-loop ADRC system (14) will be BIBO stable, enabling to dealt the problem of fast varying generalized perturbation within the ADRC framework.

3. LQR DESIGN OF THE ADRC

The idea of ADRC is to divide the process of controller design into two parts: one is to compensate for the generalized perturbation, which is reconstructed by the input-output data via an ESO (GESO); the other is to realize the desired performance for the compensated system. Moreover, ADRC does not set strict mathematical constraints on the uncertainties to be estimated. Equation (14) is a result of the separation principle for a controller designed using an observer and a state-feedback constant-gain. It states, that the observer gain and state-feedback gain can be designed separately since the overall closed-loop eigenvalues are the union of those due to the observer alone and those due to the state-feedback controller alone. It can be proved that the separation principle holds for dynamic controllers and not just constant-gain controllers.

Theorem 1: In Fig. 1, if the constant-gain controller $\tilde{\mathbf{K}} = [\mathbf{K} \quad 1]$ is replaced by the dynamic controller $\mathrm{C}(s) = \begin{bmatrix} \mathbf{A_c} & \mathbf{B_c} \\ \mathbf{C_c} & \mathbf{D_c} \end{bmatrix}$ then the separation principle still holds.

Proof: See (Davison et al., 2003).

Theorem 1 allows to stay in the ADRC framework where one first compensate for the generalized perturbation by an ESO (the optimal computation of the ESO gain vector will be considered further in this section) and then, it is employed the LQR method for optimal tuning the gains of a PI/PID/PID$^{(n-1)}$ control law.

3.1 Optimal PI-ADRC design

The real plant (1) is approximated continuously by the first order equation

$$\dot{y}(t) = f + u_1(t). \tag{15}$$

Where f contains the whole structural information of the system, in the general case, is a nonlinear time-varying function of the variables of the system, including disturbances. It should be noted that (15) is a model that is used only for the purpose of the controller design. The controller, once designed should be applied to the process.

The extended-state space model of (15) is

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{f} \\ y(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \end{cases}$$

where $[x_1(t) \quad x_2(t)]^T = [y(t) \quad f]^T$ and the ESO is

$$\begin{cases}\begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_2(t)\end{bmatrix}=\begin{bmatrix}-\beta_1 & 1\\ -\beta_2 & 0\end{bmatrix}\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u_1(t)+\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}y(t)\\ \hat{f}=[0 \quad 1]\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix}\end{cases}$$

for $[z_1(t) \quad z_2(t)]^T=[\hat{y}(t) \quad \hat{f}]^T$. This ESO is of second order and there is a redundancy when the first component of the state may be measured directly. To reduce complexity, a reduced order ESO (RESO) (Tian, 2007; Zheng et al., 2011, Zheng et al., 2012) is employed to estimate x_2, that is

$$\dot{z}_2(t)=-\beta_1 z_2(t)-\beta_1 u_1(t)+\beta_1\dot{y}(t), \tag{16}$$

where β_1 is the RESO gain. The first derivative of $y(t)$ can be approximated from the difference of two neighboring $y(t)$ sample values. However, in order to avoid intensifying measurement noise by direct numerical differentiation on signal $y(t)$, a RESO without output derivative (Teppa-Garran and Garcia, 2013b) can be defined as

$$\dot{\bar{z}}_1(t)=-\beta_1\bar{z}_1(t)-\beta_1 u_1(t)-\beta_1^2 y(t), \tag{17}$$

together with

$$z_2(t)=\bar{z}_1(t)+\beta_1 y(t). \tag{18}$$

Using $u_1(t)=u_0(t)-z_2(t)$ in (15) yields

$$\dot{y}(t)=u_0(t). \tag{19}$$

It is defined u_0 as a PI control law

$$u_0(t)=k_i\int e(t)dt+k_p e(t),$$

where

$$e(t)=r-y(t), \tag{20}$$

is the tracking error. In the case of set-point reference r, (20) can be expressed as

$$\ddot{e}(t)=-\ddot{y}(t). \tag{21}$$

Taking the derivative of (19) and using (21) gives

$$\ddot{e}(t)=-\dot{u}_0(t). \tag{22}$$

Let $v_1(t)=e(t)$ and $v_2(t)=\dot{e}(t)$ then (22) can be expressed in state-space as

$$\dot{\boldsymbol{v}}(t)=\boldsymbol{F}\boldsymbol{v}(t)+\boldsymbol{G}\dot{u}_0(t)=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\boldsymbol{v}(t)+\begin{bmatrix}0\\ -1\end{bmatrix}\dot{u}_0(t). \tag{23}$$

In order to have a LQR formulation of the ADRC problem, the following quadratic cost is considered

$$J=\int_0^\infty (\boldsymbol{v}(t)^T\boldsymbol{Q}\boldsymbol{v}(t)+\rho\dot{u}_0^2(t))\,dt, \tag{24}$$

where $\boldsymbol{Q}$ is a positive semi-definite matrix and $\rho>0$. It is well known that the minimization of (24) gives the state-feedback control

$$\dot{u}_0(t)=-\boldsymbol{K}\boldsymbol{v}(t)=-k_1 e(t)-k_2\dot{e}(t) \tag{25}$$

where $\boldsymbol{K}=\rho^{-1}\boldsymbol{G}^T\boldsymbol{P}$ and $\boldsymbol{P}$ is the symmetric positive definite solution of the Continuous Algebraic Riccati Equation (CARE) given by

$$\boldsymbol{F}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{F}+\boldsymbol{Q}-\rho^{-1}\boldsymbol{P}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{P}=0. \tag{26}$$

Taking integration on both sides of (25), the optimal PI-ADRC is obtained as

$$u_0(t)=k_i\int e(t)\,dt+k_p e(t), \tag{27}$$

that is

$$[k_i \quad k_p]=[-k_1 \quad -k_2].$$

Now, explicit expressions for the gains k_i and k_p are found. Substituting $\boldsymbol{Q}=diag(q_1 \quad q_2)$ and the symmetric matrix

$$\boldsymbol{P}=\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}$$

into the CARE (26) yields

$$\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}+\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+\begin{bmatrix}q_1 & 0\\ 0 & q_2\end{bmatrix} \tag{28}$$
$$-\rho^{-1}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0\\ -1\end{bmatrix}[0 \quad -1]\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}=0.$$

Its positive definite solution is

$$\begin{cases}p_{12}=\sqrt{q_1\rho},\\ p_{22}=\sqrt{2\rho\sqrt{q_1\rho}+q_2\rho}\\ p_{11}=\dfrac{p_{12}p_{22}}{\rho}.\end{cases} \tag{29}$$

and

$$\boldsymbol{K}=\rho^{-1}[0 \quad -1]\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}=\left[-\frac{p_{12}}{\rho} \quad -\frac{p_{22}}{\rho}\right]. \tag{30}$$

Remark 1: Because of the structure of matrices **F** and **G** in (23). The PI controller depends only on p_{12} and p_{22} in (30). The choice of a non-diagonal matrix **Q** in (28) produces the same result that a diagonal **Q**. This is, there is not loss of generality in using a diagonal **Q**.

In LQR design, the elements of the matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ are usually selected by trial and error, moreover, every different value of $\boldsymbol{Q}$ and $\boldsymbol{R}$ will eventually end up with a different system response, making the word optimal ambiguous. In order to overcome this difficulty, it is now derived a direct relationship between the $\boldsymbol{Q}$ matrix and the 2 % settling time (Ts) criterion and percentage overshoot (PO) of the closed-loop system.

Theorem 2: The closed-loop response of the ADRC system (23) with quadratic cost (24) and PI control law (27) has desired Ts and PO if and only if the elements of the matrix $\mathbf{Q}=\mathrm{diag}(q_1 \quad q_2)$ in (28) are chosen as

$$\begin{cases}q_1=\rho w_n^4\\ q_2=2\rho(2\zeta^2-1)w_n^2,\end{cases} \tag{31}$$

where $\mathrm{w_n}$ is the natural frequency and ζ is the damping ratio of the closed-loop response.

Proof: From $\mathrm{PO}=100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ it is obtained the damping ratio ζ and from 2% Ts criterion, the natural frequency is $\mathrm{w_n}=4/\zeta\mathrm{Ts}$. By using (23) the characteristic polynomial is found as $\det(s\mathbb{I}_2-\mathbf{F}+\mathbf{GK})$. Imposing the desired characteristic polynomial as the usual second order form $s^2 + 2\zeta w_n s + w_n^2$, equating coefficients on both sides of the previous polynomials and then using (28) establishes the theorem.

Remark 2: The positive-definiteness of $\mathbf{Q}$ implies $\zeta > \sqrt{2}/2$ in (31).

In Fig. 2, it is shown the optimal PI-ADRC.

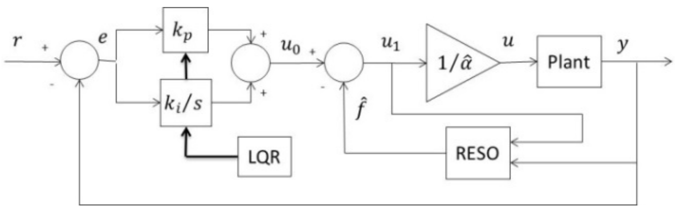

Fig. 2. Optimal PI-ADRC configuration.

3.2 Optimal PID-ADRC design

The real plant (1) is now continuously approximated by the local model

$$\ddot{y}(t) = f + u_1(t). \quad (32)$$

The extended state-space model of (32) for $\boldsymbol{x} = [x_1 \quad x_2 \quad x_3]^T = [y \quad \dot{y} \quad f]^T$ is

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{f} \\ y(t) = [1 \quad 0 \quad 0] x(t) \end{cases}$$

It is supposed again that the controlled output y is available for feedback; the second–order RESO (Tian, 2007; Zheng et al., 2011, Zheng et al., 2012) is then

$$\begin{cases} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -\beta_1 & 1 \\ -\beta_2 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \dot{y}(t) \\ \hat{f} = [0 \quad 1] \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \end{cases}$$

where $[z_1 \quad z_2]^T = [\hat{y} \quad \hat{f}]^T$.

Remark 3: In order to avoid numerical differentiation in computing the second-order RESO it is employed the result (Teppa-Garran and Garcia, 2013b) to express the RESO as

$$\begin{bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \end{bmatrix} = \begin{bmatrix} -\beta_1 & 1 \\ -\beta_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \beta_2 - \beta_1^2 \\ -\beta_1 \beta_2 \end{bmatrix} y$$

combined with

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} y.$$

The control signal u_0 is now chosen of the PID type

$$u_0(t) = k_i \int e(t)dt + k_p e(t) + k_d \dot{e}(t)$$

Using $u_1 = u_0 - z_2$ in (32) yields

$$\ddot{y}(t) = u_0(t). \quad (33)$$

Employing (20) and taking the derivative of (33) results in

$$\dddot{e}(t) = -\dot{u}_0(t). \quad (34)$$

Let $v_1(t) = e(t)$, $v_2(t) = \dot{e}(t)$ and $v_3(t) = \ddot{e}(t)$ then (34) can be expressed in state-space as

$$\dot{\boldsymbol{v}}(t) = \boldsymbol{F}\boldsymbol{v}(t) + \boldsymbol{G}\dot{u}_0(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{v} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \dot{u}_0. \quad (35)$$

The minimization of the LQR formulation of the ADRC problem (24) gives now the state-feedback control

$$\dot{u}_0(t) = -\boldsymbol{K}\boldsymbol{v}(t) = -k_1 e(t) - k_2 \dot{e}(t) - k_3 \ddot{e}(t). \quad (36)$$

Taking integration on both sides of (36) results in the optimal PID-ADRC

$$u_0(t) = k_i \int e(t)dt + k_p e(t) + k_d \dot{e}(t), \quad (37)$$

that is

$$[k_i \quad k_p \quad k_d] = [-k_1 \quad -k_2 \quad -k_3].$$

Now, explicit expressions for k_i, k_p and k_d are founf. Considering $\boldsymbol{Q} = diag(q_1 \quad q_2 \quad q_3)$ and the symmetric matrix

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix},$$

the optimal state feedback gain is

$$\boldsymbol{K} = \rho^{-1} \boldsymbol{G}^T \boldsymbol{P} = \left[-\frac{p_{13}}{\rho} \quad -\frac{p_{23}}{\rho} \quad -\frac{p_{33}}{\rho}\right].$$

This is

$$[k_i \quad k_p \quad k_d] = \left[\frac{p_{13}}{\rho} \quad \frac{p_{23}}{\rho} \quad \frac{p_{33}}{\rho}\right]. \quad (38)$$

The closed-loop polynomial is

$$\det(s\boldsymbol{I} - \boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}) = s^3 + \frac{p_{33}}{\rho} s^2 + \frac{p_{23}}{\rho} s + \frac{p_{13}}{\rho}. \quad (39)$$

The desired closed-loop polynomial is chosen as

$$\begin{aligned} &(s + \lambda \zeta w_n)(s^2 + 2\zeta w_n s + w_n^2) \\ &= s^3 + (\lambda + 2)\zeta w_n s^2 + (2\lambda\zeta^2 + 1) w_n^2 s + \lambda \zeta w_n^3 \end{aligned} \quad (40)$$

where $\lambda \gg \zeta w_n$ for guaranteed dominant pole placement (GDPP). Equating coefficients on both sides of polynomials (39) and (40) produces

$$\begin{aligned} p_{33} &= \rho(\lambda + 2)\zeta w_n \\ p_{23} &= \rho(2\lambda\zeta^2 + 1) w_n^2. \\ p_{13} &= \rho \lambda \zeta w_n^3 \end{aligned} \quad (41)$$

Solving the CARE (26) yields the nonlinear system of equations

$$\begin{bmatrix} q_1 & p_{11} & p_{12} \\ p_{11} & 2p_{12} + q_2 & p_{13} + p_{22} \\ p_{12} & p_{13} + p_{22} & 2p_{23} + q_3 \end{bmatrix} - \varrho^{-1} \begin{bmatrix} p_{13}^2 & p_{13}p_{23} & p_{13}p_{33} \\ p_{13}p_{23} & p_{23}^2 & p_{23}p_{33} \\ p_{13}p_{33} & p_{23}p_{33} & p_{33}^2 \end{bmatrix} = 0. \quad (42)$$

Remark 4: Again, there is not loss of generality in the choice of a diagonal Q in (42).

As the values p_{13}, p_{23} and p_{33} are already known in (41) one has from (42)

$$p_{11} = \frac{p_{13}p_{23}}{\rho}. \quad (43)$$
$$p_{12} = \frac{p_{13}p_{33}}{\rho}.$$
$$p_{22} = \frac{p_{23}p_{33}}{\rho} - p_{13}.$$

And the matrix Q becomes

$$\boldsymbol{Q} = \text{diag}(q_1 \quad q_2 \quad q_3) = diag\left(\frac{p_{13}^2}{\rho}, \quad \frac{p_{23}^2}{\rho} - 2p_{12}, \quad \frac{p_{33}^2}{\rho} - 2p_{23}\right).$$

Using the closed-loop parameters defined previously in (40) allows expressing the matrix $\boldsymbol{Q}$ components as

$$q_1 = \rho\lambda^2\zeta^2 w_n^6. \quad (44)$$
$$q_2 = \rho[1 + 2\lambda^2\zeta^2(2\zeta^2 - 1)]w_n^4.$$
$$q_3 = \rho[\zeta^2(\lambda^2+4) - 2]w_n^2.$$

From the above results it can be summarized the procedure in the following theorem.

Theorem 3: The closed-loop response of the ADRC system (35) with quadratic cost (24), PID control law (37) and GDPP condition in (40) has desired Ts and PO if and only if the elements of the matrix **Q** are chosen as established in (44).

Remark 5: The positive-definiteness of **Q** implies again $\zeta > \sqrt{2}/2$.

In Fig.3, it is shown the PID-ADRC implementation

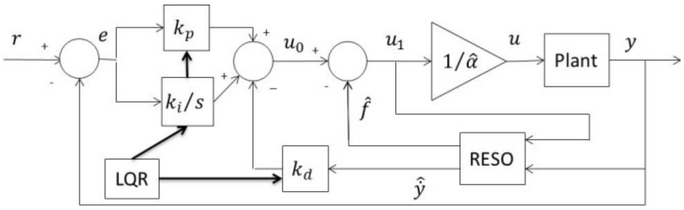

Fig. 3. Optimal PID-ADRC implementation.

3.3 Optimal RESO design

As stated, ADRC is based on the separation principle; this allows treating the unknown dynamic and disturbances in a physical process as the generalized disturbance, built an ESO to estimate it in real-time, and then cancelling its effect using the estimate as part of the control signal.

In the proposed PID-ADRC method, the second-order RESO equations are given by

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= \boldsymbol{A}z(t) + \boldsymbol{B}u_1(t) + \boldsymbol{L}(\dot{\boldsymbol{y}} - \widehat{\boldsymbol{y}}) \\ \widehat{\boldsymbol{y}} &= \boldsymbol{Cz}(t) \end{aligned}, \quad (45)$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{C} = [1 \quad 0],$$
$$\boldsymbol{L} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

where $[z_1 \quad z_2]^T = [\hat{y} \quad \hat{f}]^T$. The duality principle is now employed in order to have a LQR formulation of the RESO design. This is easily done by replacing $A \leftarrow A^T, B \leftarrow C^T$ and $K \leftarrow L^T$. The quadratic cost is chosen as

$$J = \int_0^\infty (\boldsymbol{z}(t)^T \boldsymbol{Q_o} \boldsymbol{z}(t) + \rho_o u_1^2(t))\, dt. \quad (46)$$

The CARE equation becomes

$$\boldsymbol{AM} + \boldsymbol{MA}^T + \boldsymbol{Q_o} - \rho_o{}^{-1}\boldsymbol{MC}^T\boldsymbol{CM} = 0. \quad (47)$$

Substituting $\boldsymbol{Q_o} = diag(q_{o1} \quad q_{o2})$ and the symmetric matrix

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix},$$

into (47) gives

$$2m_{12} + q_{o1} - \frac{m_{11}^2}{\rho_o} = 0, \quad (48)$$
$$m_{22} - \frac{m_{11}m_{12}}{\rho_o} = 0,$$
$$q_{o2} = \frac{m_{12}^2}{\rho_o}.$$

Its positive definite solution is

$$m_{12} = \sqrt{q_{o2}\rho_o},$$
$$m_{11} = \sqrt{2\rho_o\sqrt{q_{o2}\rho_o} + q_{o1}\rho_o},$$
$$m_{22} - \frac{m_{11}m_{12}}{\rho_o} = 0.$$

and the optimal RESO gain is found as

$$\boldsymbol{L} = \rho_o{}^{-1}\boldsymbol{MC}^T = [\beta_1 \quad \beta_2]^T = \begin{bmatrix} \frac{m_{11}}{\rho_o} & \frac{m_{12}}{\rho_o} \end{bmatrix}^T.$$

The RESO characteristic polynomial is computed as

$$s^2 + \beta_1 s + \beta_2 = s^2 + \frac{m_{11}}{\rho_o}s + \frac{m_{12}}{\rho_o}, \quad (49)$$

and the desired RESO polynomial is imposed as

$$(s + w_0)^2 = s^2 + 2w_0 s + w_0^2, \quad (50)$$

where the closed-loop RESO bandwidth w_0 must satisfy the relation

$$w_0 \gg \lambda \gg \zeta w_n, \quad (51)$$

Equating coefficients of polynomials (49) and (50) yields

$$m_{11} = 2\rho_o w_o, \quad (52)$$
$$m_{12} = \rho_o w_o^2.$$

Replacing (52) in (48) allows computing the components of the matrix $\boldsymbol{Q_o}$ through

$$q_{o1} = 2\rho_o w_0^2,$$
$$q_{o2} = \rho_o w_o^4,$$

and the optimal RESO gains are obtained doing

$$\beta_1 = 2w_0, \quad (53)$$
$$\beta_2 = w_0^2.$$

In the case of PI-ADRC design, it is easy to show that for the first order RESO (17-18), one has

$$\beta_1 = w_0. \quad (54)$$

Where w_0 must satisfy

$$w_0 \gg \zeta w_n, \tag{55}$$

From the above procedure it can be established the following theorem.

Theorem 4: Let a RESO be described as in (16) for PI-ADRC (resp. (45) for PID-ADRC) with closed-loop bandwidth w_0 that satisfies (55) (resp. (51) for PID-ADRC), then the optimal RESO gains that minimizes the quadratic cost (46) are given by (54) (resp. (53) for PID-ADRC).

4. OPTIMAL PI/PID-ADRC ALGORITHMS

It is summarized the entire development of design in two algorithms given in Table 1 for PI-ADRC and in Table 2 for the case of PID-ADRC.

Table 1. Algorithm for Optimal PI-ADRC design.

Input: *PO ← Desired closed-loop percent overshoot,* *Ts ← desired closed-loop settling time.*
1: From PO obtain the closed-loop damping ratio as $\zeta = \sqrt{\frac{(lnPO/100)^2}{\pi^2+(lnPO/100)^2}}$*.From Ts obtain the closed-loop natural frequency as* $w_n = 4/\zeta Ts$*.*
2: Compute w_0 *in the RESO from* (55).
3: Compute the RESO gain β_1 *from* (54).
4: Compute q_1 *and* q_2 *from* (31).
5: Compute p_{12} *and* p_{22} *from* (29).
6: Compute k_p *and* k_i *from* (30).
Output: PI parameters k_p *and* k_i *and RESO gain* β_1*.*

Table 2. Algorithm for Optimal PID-ADRC design.

Input: *PO ← Desired closed-loop percent overshoot,* *Ts ← desired closed-loop settling time.*
1: From PO obtain the closed-loop damping ratio as $\zeta = \sqrt{\frac{(lnPO/100)^2}{\pi^2+(lnPO/100)^2}}$*. From Ts obtain the closed-loop natural frequency as* $w_n = 4/\zeta Ts$*.*
2: Compute w_0 *in the RESO from* (51).
3: Compute the RESO gains β_1 and β_2 from (53).
4: Compute q_1, q_2 and q_3 from (44).
5: Compute p_{13}, p_{23} and p_{33} from (41).
6: Compute k_p, k_i and k_d from (38).
Output: PID parameters k_p, k_i and k_d and RESO gain $L = [\beta_1 \quad \beta_2]^T$.

For ease of reference in Table 3 are given the optimal values of the gains of the PI/PID – ADRC. The relation (55), that allows to select the RESO bandwidth w_0 depending on the desired real part of the closed-loop poles ζw_n is fixed as

$$w_0 = \alpha_1 \zeta w_n, \tag{56}$$

where $\alpha_1 \in \mathbb{R}^+$. In the case of relation (51) it is done

$$w_0 = \alpha_2 \alpha_1 \zeta w_n, \tag{57}$$

with $\alpha_2 \in \mathbb{R}^+$, $\lambda = \alpha_1 \zeta w_n$ and $w_0 = \alpha_2 \lambda$. Some guidelines to choose α_1 and α_2 will be given in the next section.

Table 3. Optimal PI/PID – ADRC gains.

	k_i	k_p	k_d	β_1	β_2
PI ADRC	w_n^2	$2\zeta w_n$		$\alpha_1 \zeta w_n$	
PID ADRC	$\alpha_1 \zeta^2 w_n^4$	$(2\alpha_1 \zeta^3 w_n + 1) w_n^2$	$(\alpha_1 \zeta w_n + 2)\zeta w_n$	$2\alpha_2 \alpha_1 \zeta w_n$	$\alpha_2^2 \alpha_1^2 \zeta^2 w_n^2$

5. LQR DESIGN OF THE ADRC: GENERAL CASE

Now it is shown how to extend the method to the case when the real plant (1) is approximated continuously by the *n*th order equation (6). The extended state space model of (6) is given by (8). The *n*th order RESO (Tian, 2007; Zheng et al., 2011, Zheng et al., 2012) is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \beta_1 & 1 & 0 & \dots & 0 \\ \beta_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & 0 & 0 & \dots & 1 \\ \beta_n & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \dot{y} \tag{58}$$

where

$$z = [z_1 \quad z_2 \quad \cdots \quad z_{n-1} \quad z_n]^T = \left[\hat{y} \quad \hat{\dot{y}} \quad \dots \quad \hat{y}^{(n-1)} \quad \hat{f}\right]^T.$$

Remark 6: In order to avoid numerical differentiation in computing the nth-order RESO in (58) it is used the result (Teppa-Garran and Garcia, 2013b) to express the RESO as

$$\dot{\bar{\boldsymbol{z}}}(t) = \begin{bmatrix} -\beta_1 & 1 & 0 & \dots & 0 \\ -\beta_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_{n-1} & 0 & 0 & \dots & 1 \\ -\beta_n & 0 & 0 & \dots & 0 \end{bmatrix} \bar{\boldsymbol{z}}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} \beta_2 - \beta_1^2 \\ \beta_3 - \beta_1 \beta_2 \\ \vdots \\ \beta_n - \beta_1 \beta_{n-1} \\ -\beta_1 \beta_n \end{bmatrix} y(t).$$

combined with

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} y.$$

Using $u_1 = u_0 - z_n$ in (6) yields

$$y(t)^{(n)} = u_0(t). \tag{59}$$

Employing (20) and taking the derivative of (59) results in

$$e(t)^{(n+1)} = -\dot{u}_0(t). \tag{60}$$

Doing

$$\boldsymbol{v} = [v_1, v_2, \dots, v_n, v_{n+1}]^T$$

$= \left[e, \dot{e}, \quad \dots, \quad e^{(n-1)}, e^{(n)}\right]^T$

in (60) gives the state space model

$$\dot{\boldsymbol{v}}(t) = \boldsymbol{F}\boldsymbol{v}(t) + \boldsymbol{G}\dot{u}_0(t). \tag{61}$$

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(n+1) \, x \, (n+1)} \quad \boldsymbol{G} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}_{(n+1) \, x \, 1}$$

The minimization of the LQR formulation of the ADRC problem (24) gives

$$\dot{u}_0(t) = -[k_1, k_2, \dots, k_n, k_{n+1}]\boldsymbol{v}(t) \tag{62}$$

Taking integration on both sides of (62) results in the optimal $\mathrm{PID}^{(n-1)}$ – ADRC

$$u_0(t) = k_i \int e\,(t)dt + k_p e(t) + k_{d_1}\dot{e}(t) + k_{d_2}\ddot{e}(t) + \dots + k_{d_{n-1}}e^{(n-1)}, \tag{63}$$

that is

$$[k_i \quad k_p \quad k_{d_1} \quad \cdots \quad k_{d_{n-1}}] = [-k_1 \quad -k_2 \quad -k_3 \quad \cdots \quad -k_{n+1}].$$

Now, explicit expressions for the controller gains k_i, $k_p, k_{d_1}, \dots, k_{d_{n-1}}$ are found. Considering

$$\boldsymbol{Q} = diag(q_1, q_1, \dots, q_{n+1})$$

and the symmetric matrix

$$\boldsymbol{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n+1} \\ p_{1,2} & p_{2,2} & p_{2,3} & \cdots & p_{2,n+1} \\ p_{1,3} & p_{2,3} & p_{3,3} & \cdots & p_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{1,n+1} & p_{2,n+1} & p_{3,n+1} & \cdots & p_{1,n+1} \end{bmatrix},$$

The optimal feedback gain is

$$\boldsymbol{K} = \rho^{-1}\boldsymbol{G}^T\boldsymbol{P} = \left[-\frac{p_{1,n+1}}{\rho} \quad -\frac{p_{2,n+1}}{\rho} \quad \cdots \quad -\frac{p_{n+1,n+1}}{\rho}\right],$$

that is

$$[k_i \quad k_p \quad k_{d_1} \quad \cdots \quad k_{d_{n-1}}] = \left[-\frac{p_{1,n+1}}{\rho} \quad -\frac{p_{2,n+1}}{\rho} \quad \cdots \quad -\frac{p_{n+1,n+1}}{\rho}\right]. \tag{64}$$

The closed-loop polynomial is computed through

$$\det(s\boldsymbol{I} - \boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}) \tag{65}$$
$$= s^{n+1} + \frac{p_{n+1,n+1}}{\rho}s^n + \dots + \frac{p_{2,n+1}}{\rho} + \frac{p_{1,n+1}}{\rho}$$

The desired closed-loop polynomial is chosen as

$$(s + \lambda\zeta w_n)^{n-1}(s^2 + 2\zeta w_n s + w_n^2), \tag{66}$$

where $\lambda \gg \zeta w_n$ for GDPP. Using the binomial theorem, the polynomial (66) becomes

$$\begin{aligned} &\binom{n-1}{0}s^{n+1} + \left[\binom{n-1}{0}2\zeta w_n + \binom{n-1}{1}\lambda\zeta w_n\right]s^n \\ &+ \left[\binom{n-1}{0}w_n^2 + \binom{n-1}{1}(2\zeta w_n)(\lambda\zeta w_n) + \binom{n-1}{2}(\lambda\zeta w_n)^2\right]s^{n-1} \\ &+ \left[\binom{n-1}{1}w_n^2(\lambda\zeta w_n) + \binom{n-1}{2}(2\zeta w_n)(\lambda\zeta w_n)^2 + \binom{n-1}{3}(\lambda\zeta w_n)^3\right]s^{n-2} \\ &+ \left[\binom{n-1}{2}w_n^2(\lambda\zeta w_n)^2 + \binom{n-1}{3}(2\zeta w_n)(\lambda\zeta w_n)^3 + \binom{n-1}{4}(\lambda\zeta w_n)^4\right]s^{n-3} \\ &\vdots \\ &+ \left[\binom{n-1}{n-3}w_n^2(\lambda\zeta w_n)^{n-3} + \binom{n-1}{n-2}(2\zeta w_n)(\lambda\zeta w_n)^{n-2} + \binom{n-1}{n-1}(\lambda\zeta w_n)^{n-1}\right]s^2 \\ &+ \left[\binom{n-1}{n-2}w_n^2(\lambda\zeta w_n)^{n-2} + \binom{n-1}{n-1}(2\zeta w_n)(\lambda\zeta w_n)^{n-1}\right]s \\ &+ \binom{n-1}{n-1}w_n^2(\lambda\zeta w_n)^{n-1} \end{aligned} \tag{67}$$

where $n \geq 2$ and the binomial coefficient for $k \leq n$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Using (64) and equating coefficients on both sides of polynomials (65) and (67) give the optimal gains of the $\mathrm{PID}^{(n-1)}$-ADRC, that is

$$\begin{aligned} k_i &= \binom{n-1}{n-1}w_n^2(\lambda\zeta w_n)^{n-1} \\ k_p &= \binom{n-1}{n-2}w_n^2(\lambda\zeta w_n)^{n-2} + \binom{n-1}{n-1}(2\zeta w_n)(\lambda\zeta w_n)^{n-1} \\ k_{d_1} &= \binom{n-1}{n-3}w_n^2(\lambda\zeta w_n)^{n-3} + \binom{n-1}{n-2}(2\zeta w_n)(\lambda\zeta w_n)^{n-2} + \binom{n-1}{n-1}(\lambda\zeta w_n)^{n-1} \\ &\vdots \\ k_{d_{n-2}} &= \binom{n-1}{0}w_n^2 + \binom{n-1}{1}(2\zeta w_n)(\lambda\zeta w_n) + \binom{n-1}{2}(\lambda\zeta w_n)^2 \\ k_{d_{n-1}} &= \binom{n-1}{0}2\zeta w_n + \binom{n-1}{1}\lambda\zeta w_n \end{aligned} \tag{68}$$

Remark 7: As expected, if (68) is evaluated for n = 2 one obtains the PID-ADRC gains computed previously.

The optimal RESO gain $L = [\beta_1 \quad \dots \quad \beta_n]^T$ in (58) is found by minimizing the quadratic cost (46), that is

$$\boldsymbol{L} = \rho_o{}^{-1}\boldsymbol{M}\boldsymbol{C}^T = [\beta_1 \quad \dots \quad \beta_n]^T = \begin{bmatrix} \frac{m_{11}}{\rho_o} & \dots & \frac{m_{1,n+1}}{\rho_o} \end{bmatrix}^T.$$

Following a similar procedure, it can be obtained

$$\beta_i = \frac{n!}{i!\,(n-i)!} w_o^i; \quad i = 1, \dots, n. \tag{69}$$

If it is considered the presence of measurement noise $\eta(t)$ in the controlled output $y(t)$, the estimation error dynamics of the ESO (12) takes the form

$$\dot{\boldsymbol{e}}(t) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}(\boldsymbol{t}) + \boldsymbol{E}\dot{f}(t) - \boldsymbol{L}\eta(t) \,.$$

It is evident that a large value for $\boldsymbol{L}$ will enhance the effect of the measurement noise, since this is usually a high-frequency signal. It is needed a compromise between speed of response and noise immunity. In PI/PID/PID$^{(n-1)}$-ADRC, for controller design, the requirements of the closed-loop performance in time domain (PO and Ts) are converted into a pair of conjugate poles $p_{1,2} = -a \pm jb$ where $a = \zeta w_n$. When one has a derivative order $n \geq 2$ in (6) the condition (66) requires that the ratio of the real part of any other poles to $-a$ exceeds λ. For GDPP, the constant λ is usually chosen following the rule

$$\lambda = 3 \; to \; 5 \; times \; (\zeta w_n) \,. \tag{70}$$

The larger ζw_n, the faster the response, the larger the control signal and a system more susceptible to noise. For RESO design, it can be choosen the bandwidth w_0 by fixing α_1 in (56) for $n = 1$ in (6) or α_2 in (57) for $n \geq 2$ in (6). In the case of α_1 one fixes

$$\alpha_1 = 2 \; to \; 6 \; times \; (\zeta w_n). \tag{71}$$

And for α_2 it is used

$$\alpha_2 = 2 \; to \; 6 \; times \; (\lambda). \tag{72}$$

This ensures the observer errors decay faster than the desired closed-loop dynamics allowing the controller poles to dominate the total response. If sensor noise if a problem then the observer poles may be chosen slower than two times (ζw_n or λ). This would yield a system with lower bandwidth, more noise smoothing and less control energy expenditure.

6. SOME EXAMPLES

In this section, some numerical examples are considered to show the effectiveness of the method. In all the computer simulations, the values of the controller's gains predicted by the design equations are employed without doing any post tuning refinement.

6.1 Mass-spring system with dynamic friction

The motion equation of a mass-spring system with dynamic friction is adapted from (Fliess et al., 2011) as

$$\ddot{y} = -6y - 20y^3 - 5\dot{y} + 2\mathcal{F}(\dot{y}) + 2u. \tag{73}$$

Where the dynamic friction is given by

$$\mathcal{F}(\dot{y}) = \begin{cases} 0.3 + 0.4(\dot{y} + 0.25)^2 - 5\dot{y} & if \; \dot{y} < 0 \\ -0.3 - 0.4(\dot{y} + 0.25)^2 - 5\dot{y} & if \; \dot{y} > 0 \end{cases} \tag{74}$$

A PI-ADRC is designed for (6) considering $PO = 0.1, Ts = 4$ and $\alpha_1 = 5$ and a PID-ADRC for (32) when the condition $\alpha_2 = 3$ is added. The Fig. 4 shows the results for a step set point and Fig. 5 the resulting control inputs. In Fig. 6, it is added a constant disturbance of amplitude 0.1 applied since $t = 10$. Finally, in Fig. 7 a zero mean Gaussian white noise of variance 0.01 is added at the output for testing the robustness property of the design. It is evident from the results, the good performance of both controllers.

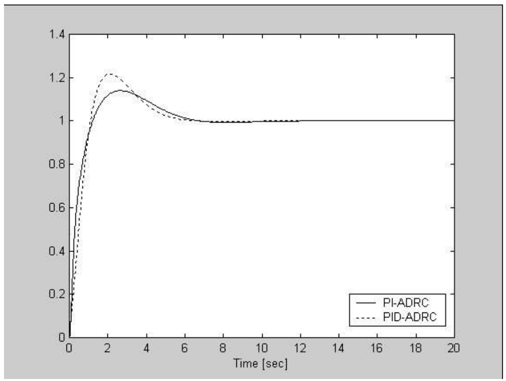

Fig. 4. PI/PID-ADRC tracking for a mass-spring system with dynamic friction.

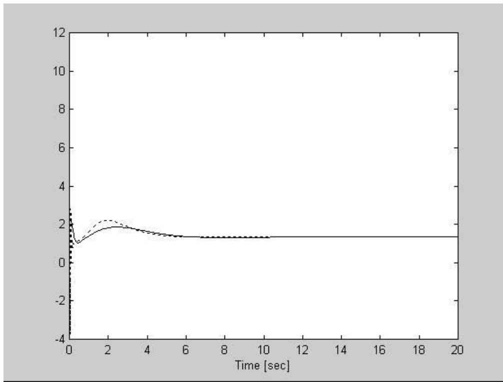

Fig. 5. PI/PID-ADRC command inputs for the tracking problem of a mass-spring system with dynamic friction.

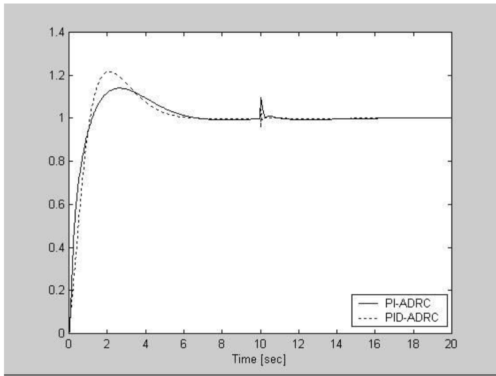

Fig. 6. PI/PID-ADRC tracking and disturbance rejection for a mass-spring system with dynamic friction.

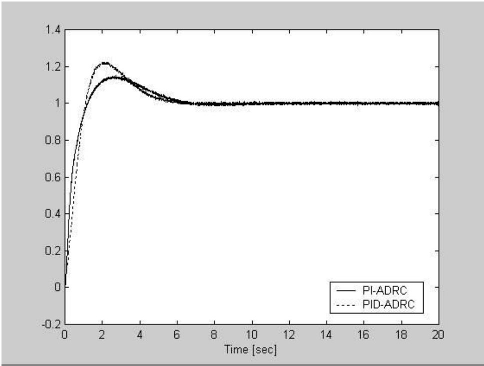

Fig. 7. PI/PID-ADRC tracking for a mass-spring system with dynamic friction considering a noisy output.

6.2 Unstable linear system with a large spectrum

The system is characterized by the transfer function

$$\frac{s^5}{(s+10)(s+1)(s+0.1)(s-0.2)(s-2)(s-20)}$$

A PI-ADRC is designed for $PO = 0.1, Ts = 0.4$ and $\alpha_1 = 5$ when the system is locally described by (6). The Fig. 8 shows the effectiveness of the tracking of a square signal of amplitude 1 and frequency 0.1 Hz.

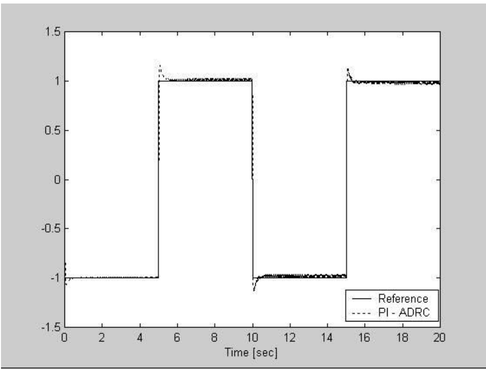

Fig. 8. PI-ADRC tracking for an unstable linear system with a large spectrum.

6.3 Inverted pendulum

The motion equation of the inverted pendulum of Fig. 9 is given by

$$J\ddot{y}(t) - mglSin[y(t)] + b\dot{y}(t) = u(t), \tag{75}$$

where $y(t)$ describes the angular deviation from the upright position. It is assumed a damping term proportional to the angular velocity and that it is possible to affect the pendulum by a torque $u(t)$ at its base. The parameters of the system are in Table 4. The controller's gains computed in the design of the mass-spring control system are used for the inverted pendulum. The Fig. 10 exhibits the tracking performance of a square signal of amplitude 1 and frequency 0.01 Hz. The Fig. 11 (zoomed) considers the disturbance rejection of a 0.1 amplitude step signal applied since $t = 10$ in the output channel. The results show that PI-ADRC has a little better performance in the case of tracking and PID-ADRC has a better one in the case of step disturbance rejection. If it is done a post tuning adjustment of the controllers gains or it is modified the estimate of the scaling factor $\hat{\alpha}$, the results may be distinctly different.

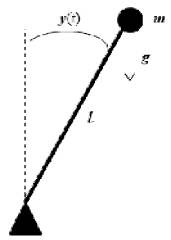

Fig. 9. Inverted pendulum.

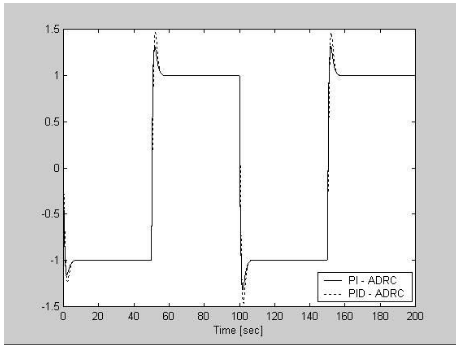

Fig. 10. PI/PID - ADRC tracking for the inverted pendulum.

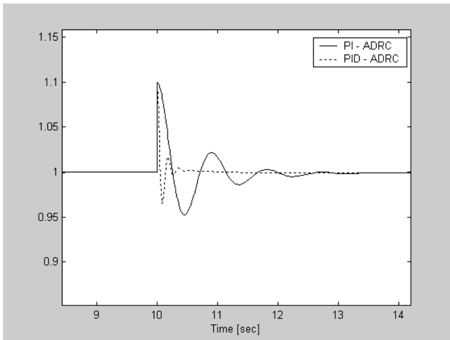

Fig. 11. PI/PID - ADRC disturbance rejection for the inverted pendulum.

Table 4. Parameters of the inverted pendulum.

Symbol	Description	Value
m	Mass of the sphere	$0.1\ Kg$
g	Gravity on the surface of the earth	$9.8\ m/s^2$
l	Length of the rigid rod	$1\ m$
J	Moment of the inertia	$0.1\ Kg.m^2$

7. CONCLUSIONS

It has been developed a method that allows tuning a PI/PID/PID$^{(n-1)}$ controller into the ADRC framework. The method makes use of the LQR approach and by imposing the condition of guaranteed dominant pole placement; it is defined a criterion for selecting the Q and R matrices in order to have a desired percentage overshoot and settling time of the closed loop response. Some numerical examples are considered to show the effectiveness of the approach.

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