Stability of a Class of Switched Linear Systems with Infinite Number of Subsystems

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Abstract: This paper proposes a new approach for globally uniformly asymptotically stability (GUAS) analysis. This method applies for discrete-time switched linear systems with infinite number of switching. A sufficient condition for GUAS of switched linear systems with infinite subsystem is proposed as a theorem.

Keywords: Global uniform asymptotic stability, Infinite number of subsystem, switched linear System.

1. INTRODUCTION

A switched system is a dynamical system that consists of finite or infinite number of subsystems and a logical rule that orchestrates switching between these subsystems. Mathematically, these subsystems are usually described by a collection of indexed differential or difference equations. One convenient way to classify switched systems is based on the dynamics of their subsystems, for example continuous-time or discrete-time, linear or nonlinear and so on (Lin and Antsaklis, 2009). In the recent years many researchers have been investigated on stability of switching systems. Lin and Antsaklis reviewed some results on stability and stabilizability of switched linear systems. They outlined briefly some necessary and sufficient conditions for the asymptotic stability of switched linear systems under arbitrary switching then they explored necessary and sufficient condition for the switching stabilizability of continuous-time switched linear systems.

Also they proved a necessary and sufficient condition for asymptotic stabilizability of switched linear systems (Lin and Antsaklis, 2009). Cheban et al. proposed absolute asymptotic stability of discrete linear inclusions in Banach (both finite and infinite dimensional) space also they established the relation between absolute asymptotic stability, asymptotic stability, uniform asymptotic stability and uniform exponential stability. They proved that for asymptotical compact discrete linear inclusions the concepts of asymptotic stability and uniform exponential stability are equivalent (Cheban and Mammana, 2005).

(Muller et al., 2010) considered the concept of state-norm estimators for switched nonlinear systems under average dwell-time switching signals. State-norm estimators are closely related to the concept of input/output-to-state stability (IOSS). They showed that if the average dwell-time is large enough, there exists a nonswitched state-norm estimator for a switched system which each of its constituent subsystems is IOSS (Muller et al., 2010). (Liberzon et al., 2009) studied linear switched differential algebraic equations (DAEs), i.e., systems defined by a finite family of linear DAE subsystems and a switching signal that governs the switching between them. They showed by examples that switching between stable subsystems may lead to instability and that the presence of algebraic constraints lead to a larger variety of possible instability mechanisms compared to those observed in switched systems described by ordinary differential equations (ODEs). They prove two sufficient conditions for stability of switched DAEs based on the existence of suitable Lyapunov functions (Liberzon et al., 2009).

(Agrachev et al., 2010) presented new sufficient conditions for exponential stability of switched linear systems under arbitrary switching, which involve the commutator (Lie brackets) among the given matrices generating the switched system. Their proposed stability criteria was robust with respect to small perturbations of the system parameters. They investigated both discrete and continuous switched linear systems (Agrachev et al., 2010). (Hien et al., 2009) investigated the problem of exponential stability and stabilization of switched linear time-delay systems which the system parameter uncertainties was time-varying and unknown but they was norm-bounded. The delay in the system states was also time-varying. They designed a switching rule for the exponential stability and stabilization By using an improved Lyapunov–Krasovskii functional by using of the solution of Riccati-type equations (Hien et al., 2009). (Raouf et al., 2009) proposed a new sufficient condition that guarantees global exponential stability of switched linear systems based on Lyapunov-Metzler inequalities. The condition relies on the solution of a set of bilinear matrix inequalities (BMI) (Raouf et al., 2009).

(Hante et al., 2011) considered switched systems on Banach and Hilbert spaces governed by strongly continuous one-parameter semi groups of linear evolution operators. They
provided necessary and sufficient conditions for their global exponential stability, uniform with respect to the switching signal with arbitrary switching, in terms of the existence of a Lyapunov function common to all modes (Hante et al., 2011).

(Du et al., 2009) proposed finite-time stability and stabilization problems for switched linear systems. They extended the concept of finite-time stability to switched linear systems, and then they represented a necessary and sufficient condition for finite-time stability of switched linear systems based on the state transition matrix of the system. They designed state feedback controllers and a class of switching signals with average dwell-time to stabilize the switched linear control systems (Du et al., 2009).

(Zhai et al., 2007) studied stability and $L_2$ gain properties for a class of switched systems that are composed of normal discrete-time subsystems. They showed that When all subsystems are Schur stable, a common quadratic Lyapunov function exists for all subsystems and then the switched normal system is exponentially stable under arbitrary switching.

As regards $L_2$ gain analysis, they introduced an expanded matrix including each subsystem’s coefficient matrices, and they showed that if the expanded matrix is normal and Schur stable so that each subsystem is Schur stable and has unity $L_2$ gain, then the switched normal system also has unity $L_2$ gain under arbitrary switching (Zhai et al., 2007).

(Santarelli et al., 2010) designed a switched feedback controller for second order switched systems and then developed the switched state feedback control law for the stabilization of LTI systems of arbitrary dimension. They proposed switched state feedback control law by examining relevant geometric properties of the phase portraits in the case of two-dimensional systems for the stabilization of LTI systems of arbitrary dimension. The control law operates by switching between two static gain vectors in such a way that the state trajectory is driven onto a stable (n-1) dimensional hyper plane (where $n$ represents the system dimension) then they derived a necessary and sufficient conditions to ensure stabilizability of the resulting switched system and they applied their new control condition to the problem of minimizing the maximal Lyapunov exponent of the corresponding closed-loop state trajectories (Santarelli et al., 2010).

(Zhai et al., 2010) proposed a unified approach to stability analysis for switched linear descriptor systems under arbitrary switching in both continuous-time and discrete-time domains. The approach is based on common quadratic Lyapunov functions incorporated with Linear Matrix Inequalities (LMIs). They show that if there exists a common quadratic Lyapunov function for the stability of all subsystems, then the switched system is stable under arbitrary switching (Zhai et al., 2010).

In (Araghi et al., 2009) the authors investigated the stability of switched linear systems then proposed three methods for existence of a common quadratic Lyapunov function for robust stability analysis of fuzzy Elman neural network. These methods have been considered using stabilizing state feedback control in closed loop switching system.

In (Suratgar et al., 2002; Suratgar et al., 2003) the authors proposed some theorems for stability analysis of TSK and linguistic fuzzy models, (Guo et al., 2012) investigated the stability of a class of switched linear systems and they proposed a novel analysis method by using the 2-norm technique. Their proposed method guarantees the stability of the systems under arbitrary switching, and also provides an algorithm to find the minimum dwell time (MDT) with which switches make the switched systems stable (Guo et al., 2012).

In (Ratchagit et al., 2012) proposed a switching design for the asymptotic stability of switched linear discrete-time systems with interval time-varying delays and he designed a switching rule for the asymptotic stability for the switched system via linear matrix inequalities. The system which he concerned was with the delay with a fast time-varying function and the lower bound was not restricted to zero. (Wang et al., 2012) investigated the finite-time stability problem for a class of discrete-time switched linear systems with impulse effects. They established a sufficient condition which ensures that the state trajectory of the system remains in a bounded region of the state space over a pre-specified finite time interval. They showed that the total activation time of unstable subsystems can be greater than that of stable subsystems. In addition, the finite-time stability degree may be also greater than one (Wang et al., 2012). (Su et al., 2012) established a stability result for a class of linear switched systems involving Kronecker product. This problem is interesting in that the system matrix does not have to be Hurwitz at any time instant. As applications of this stability result, they proposed the solvability conditions for both the leaderless and the leader-following consensus problem for general marginally stable linear multi-agent systems under switching network topology. Their results only assume that the dynamic graph is uniformly connected (Su et al., 2012). (Liu et al., 2012) concerned with the stability problem of discrete-time positive switched linear systems with delays. The states of the systems under consideration are confined in the positive orthant, and the delays can be time-varying and not necessarily bounded. They established a delay independent stability criterion for switched linear systems with delays systems (Liu et al., 2012). (Kermani et al., 2012) investigated new stability conditions for discrete-time switched linear systems based on overvaluing systems built on vector norms and the application of Borne-Gentina criterion. This stability conditions issued from vector norms correspond to a vector Lyapunov function. In fact, the switched system to be controlled will be represented in the Companion form. A comparison system relative to a regular vector norm was used in order to get the simple arrow form of the state matrix that yields to a suitable use of Borne-Gentina criterion for the establishment of sufficient conditions for global asymptotic stability (Kermani et al., 2012). Also they studied the stability and stabilization problems for continuous-time switched linear systems and they established a new stability conditions based on the comparison, the overvaluing principle, the application of Borne-Gentina criterion and the Kotelyanski conditions (Kermani et al., 2012). In (Cimochowski et al., 2012) studied...
the positive switched discrete-time linear systems. He established a new necessary and sufficient conditions for asymptotic stability of the positive linear discrete-time switched systems with delays in states. The result of his proposal was that asymptotic stability of the switched system is equivalent to asymptotic stability of the corresponding positive discrete-time switched system without delays.

In the recent years there are many articles about stability of systems with finite number of switching but in this paper stability analysis of discrete-time switched linear systems with infinite number of switching is investigated.

2. PROBLEM STATEMENT

Consider a set \( \Sigma \) of matrices \( A_{ik} \) and pick an initial point \( x_0 \), at \( t = 0 \). A switched linear system is a dynamical system of the type:

\[
x_{k+1} = A_{i_k}x_k , A_{i_k} \in \Sigma, i_k \in \mathbb{Z}^+ \quad (1)
\]

Where \( I \) is an infinite index set, \( i_k \) is switching law and the state \( x \in \mathbb{R}^n \) and \( A_{i_k} \in \mathbb{R}^{n \times n} \).

This notation means that at every instant, the matrix \( A_{i_k} \) defining the evolution of the system can be replaced by another one from of the set \( \Sigma \).

The stability of switched system when there is no restriction on the switching signals is usually called stability analysis under arbitrary switching. For this analysis, it is necessary that all the subsystems are asymptotically stable. However, even when all the subsystems of a switched system are exponentially stable, it is still possible to construct a divergent trajectory from any initial condition. Therefore, in general, the assumption of subsystems’ stability is not sufficient to assure stability of switched systems under arbitrary switching, except for some special cases, such as pairwise commutative systems, symmetric or normal systems (all subsystems). Consequently, if there exists a common Lyapunov function for all the subsystems, then the stability of the switched system is guaranteed under arbitrary switching (Lin and Antsaklis, 2009).

Let us recall the following definition and lemma that will be helpful through the rest of the paper.

**Definition 1.** The linear switched system (1) is globally uniformly asymptotically stable (GUAS) if for any initial condition \( x_0 \in \mathbb{R}^n \) and any switching law \( i_k \), (Monovich et al., 2011):

\[
\lim_{k \to \infty} x_k = 0, \quad \forall (i_k)
\]

As this is supposed to hold for any initial vector \( x_0 \), it is equivalent to saying that all matrix products taken from the set converge to the zero matrix, i.e.,

\[
\lim_{k \to \infty} A_{i_k}A_{i_{k-1}}...A_{i_1} = 0, \quad \forall (i_k)
\]

The GUAS problem is closely related to determining the joint spectral radius (JSR) of the set of matrices \( \Sigma = \{A_1, ..., A_n\} \), denoted by \( \rho(\Sigma) \) (Monovich et al., 2011).

The formal definition of the joint spectral radius was first introduced by Rota and Strang in the 60’s. In the 90’s, Daubechies & Lagarias defined the generalised spectral radius, and Berger & Wang proved later these two values to be equal for bounded sets of matrices (Jungers et al., 2008).

The joint spectral radius characterizes the maximal asymptotic growth rate of a point submitted to a switching linear system in discrete time. The maximal growth rate one can ensure the stability of the system, provided that this growth rate is less than one (Jungers et al., 2008).

Let \( \| . \| : \mathbb{R}^n \to \mathbb{R}_+ \) denote the Euclidean vector norm and denote (Hartfiel, 2002)

\[
\rho_k(\Sigma) = \max \left\{ \| A_{i_1}A_{i_2}...A_{i_k} \| ^{\frac{1}{k}} , \ i_j \in \{0, 1, ..., n\} \right\} \quad (2)
\]

Then the joint spectral radius is defined as,

\[
\rho(\Sigma) = \lim_{k \to \infty} \rho_k(\Sigma).
\]

And let,

\[
\beta_k(\Sigma) = \sup \left\{ \rho(A_{i_1}A_{i_2}...A_{i_k})^{\frac{1}{k}}, \ i_j \in \{0, 1,..., n\} \right\}
\]

The generalized spectral radius is defined as,

\[
\bar{\rho}(\Sigma) = \lim_{k \to \infty} \beta_k(\Sigma).
\]

For bounded set of matrices the joint spectral radius and the generalized spectral radius are equal.

In general (Hartfiel, 2002),

\[
\beta_k(\Sigma)^{\frac{1}{k}} \leq \rho(\Sigma) \leq \beta(\Sigma)^{\frac{1}{k}}
\]

The switched system (1) is GUAS if and only if \( \rho(\Sigma) < 1 \) (Monovich et al., 2011).

Some results show that computing or even approximating the JSR is extremely hard (Jungers et al., 2008). In this paper we propose matrix structure for the subsystems in switched linear systems that the stability of switched system is guaranteed.

Through the rest of the paper, the discrete-time switched linear system is considered as follow:

\[
x(k+1) = \Sigma_k x(k), \quad k = 1, 2, ...
\]

where \( \Sigma_k \in \Sigma, \Sigma = \{ \Sigma_1, \Sigma_2, ... \} \) such that,

\[
\Sigma_k = \begin{bmatrix} A_{i_k} & B_{i_k} \\ 0 & C_{i_k} \end{bmatrix}
\]

and \( k \) is an infinite index set, \( k = 1, 2, ..., [A_{i_k}] \in \mathbb{R}^{n_1 \times n_1}, \ [B_{i_k}] \in \mathbb{R}^{n_1 \times n_2} \) and \( [C_{i_k}] \in \mathbb{R}^{n_2 \times n_2} \), the state \( x \in \mathbb{R}^n, \Sigma_k \in \mathbb{R}^{n \times n} \).

If the joint spectral radius of system (3), \( \rho(\Sigma) < 1 \), then the dynamical system is stable, because \( x_k = \Sigma^k x_0 \), where \( \Sigma^k \triangleq \{ \Sigma_1 \Sigma_2 \Sigma_3 \} \) and so \( \| x_k \| \leq \| \Sigma^k \| \| x_0 \| \to 0 \) (Jungers et al., 2008).
First we prove the following lemma.

**Lemma 1.** Let \((A_k)_{k \in \mathbb{N}}\) be a set of square matrices. If there exist \(\alpha < 1\) such that for all \(k \in \mathbb{N}\), \(\|A_k\| \leq \alpha\), Then,
\[
\lim_{k \to \infty} A_k A_{k-1} \ldots A_1 = 0.
\]

**Proof:**
From the submultiplicity property of norms,
\[
\|A\| \leq \|A\| \|B\|
\]
the following equation is correct:
\[
\|A_k A_{k-1} \ldots A_1\| \leq \|A_k\| \|A_{k-1}\| \ldots \|A_1\|
\]
Since \(\forall k \in \mathbb{N}\), \(\|A_k\| \leq \alpha\), therefore,
\[
\lim_{k \to \infty} \|A_k A_{k-1} \ldots A_1\| \leq \lim_{k \to \infty} \alpha^k \to 0
\]
And consequently,
\[
\lim_{k \to \infty} \|A_k A_{k-1} \ldots A_1\| \to 0 \text{ then } \lim_{k \to \infty} A_k A_{k-1} \ldots A_1 \to 0
\]

### 3. INFINITE PRODUCT OF MATRICES

**Theorem 1.** Let \((\Sigma_k)_{k \in \mathbb{N}}\) be a sequence of Matrices of the form (4) and let there exist two numbers \(\gamma \) such that \(\gamma < 1\), \(\alpha < 1\) and \(\|A_k\| \leq \alpha\), \(\|C_k\| \leq \gamma\) for some matrix norm \(\|\cdot\|\). The sequence \(P_k = \Sigma_1 \Sigma_2 \ldots \Sigma_k\) (infinite product of matrices) converges to zero if and only if
\[
A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1} \text{ converges to zero.}
\]

**Proof:**
To prove the sufficient condition, by construction \(P_k\) from \(\Sigma_k\) as follow:
\[
P_k = \Sigma_1 \Sigma_2 \ldots \Sigma_k, \Sigma_k \in \Sigma, k \in \mathbb{N}
\]
\[
P_k = \begin{bmatrix}
A_1 & A_2 & \ldots & A_k & X_k \\
C_1 & C_2 & \ldots & C_k
\end{bmatrix}
\]
Where
\[
X_k = A_1 A_2 \ldots A_{k-1} B_k + A_1 A_2 \ldots A_{k-2} B_{k-1} C_k + \ldots
\]
By hypothesis of the theorem and the Lemma1 since \(\|A_k\| \leq \alpha < 1\), \(\|C_k\| \leq \gamma < 1\),
\[
\lim_{k \to \infty} A_1 A_2 \ldots A_k = \lim_{k \to \infty} C_1 C_2 \ldots C_k = 0.
\]
Therefore \(\lim_{k \to \infty} X_k = 0\) implies that \(\lim_{k \to \infty} X_k \to 0\) and therefore,
\[
\lim_{k \to \infty} P_k = 0
\]
By some calculations, between \(X_k\) and \(X_{k-1}\), the following relation is achieved:
\[
X_k = X_{k-1} C_k + A_1 A_2 \ldots A_{k-1} B_k
\]
By substitution \(X_{k-1}\), the above equation is:
\[
X_k - X_{k-1} = X_{k-1} C_k + A_1 A_2 \ldots A_{k-2} B_{k-1} B_k - X_{k-1}
\]
Because of \(\|C_k\| \leq \gamma < 1\), therefore \(\|C_k - 1\| \neq 0\) and \(C_k - 1\) is invertible. So,
\[
(X_k - X_{k-1})(C_k - 1)^{-1} = X_{k-1} + A_1 A_2 \ldots A_{k-2} B_k (C_k - 1)^{-1}
\]
\[
(X_k - X_{k-1})(I - C_k)^{-1} = A_1 A_2 \ldots A_{k-2} B_k (I - C_k)^{-1} - X_{k-1}
\]
From (7) and (8);
\[
\lim_{k \to \infty} X_k = \lim_{k \to \infty} A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}
\]
and consequently,
\[
\lim_{k \to \infty} P_k = \begin{bmatrix} 0 & \lim_{k \to \infty} A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1} \\ 0 & 0 \end{bmatrix}
\]
Namely, \(P_k\) converges to zero if \(A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}\) converges to zero.

To prove the necessary condition of the theorem, it must be proved that if \(\lim_{k \to \infty} P_k = 0\) then \(\lim_{k \to \infty} X_k = 0\).

To prove this, first it must be proved that:
\[
\lim_{k \to \infty} \|X_k\| = \lim_{k \to \infty} \|A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}\|
\]
By defining \(D_k\) as the difference between \(X_k\) in (6) and \(A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}\),
\[
D_k = X_k - A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}; k \in \mathbb{N}
\]
and by defining \(Y_k\) as:
\[
Y_k = A_1 A_2 \ldots A_k B_{k+1} (I - C_{k+1})^{-1} - A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}
\]
From (9) and (10) it is obtained that,
\[
D_{k+1} = X_{k+1} - A_1 A_2 \ldots A_k B_{k+1} (I - C_{k+1})^{-1} = (D_k - Y_k) C_{k+1}
\]
and thus,
\[
\|D_{k+1}\| \leq (\|D_k\| + \|Y_k\|) \|C_{k+1}\|, \text{ such that } \|C_k\| \leq \gamma,
\]
So,
\[
\|D_{k+1}\| \leq (\|D_k\| + \|Y_k\|) \gamma,
\]
\[
\|D_k\| \leq (\|D_{k-1}\| + \|Y_{k-1}\|) \gamma,
\]
\[
\|D_{k-1}\| \leq (\|D_{k-2}\| + \|Y_{k-2}\|) \gamma,
\]
By repeating the above inequalities it is obtained:
\[
\|D_k\| \leq \|D_{k-1}\| \gamma + \|Y_{k-1}\| \gamma^2 + \|Y_{k-2}\| \gamma^3 + \ldots + \|Y_{k-1}\| \gamma^k
\]
and as a result,\n\[
\|D_k\| \leq \|D_1\| \gamma^k + \sum_{i=1}^{k-1} \|Y_{k-1}\| \gamma^i, i = 1, 2, \ldots, k - 1
\]
Therefore,
\[
\lim_{k \to \infty} \|D_k\| \leq \lim_{k \to \infty} \sum_{i=1}^{k-1} \|Y_{k-1}\| \gamma^i, \quad 0 \leq \gamma < 1
\]
Because of \(\lim_{k \to \infty} \|C_k\| \gamma^k = \lim_{k \to \infty} \|C_k\| \gamma^k = 0\).
By considering \(S = \lim \sup_{k \to \infty} \|D_k\| < \infty\) and since \(\lim Y_k = 0\), and therefore,
\[
\lim \sup_{k \to \infty} \|D_k\| \leq \lim \sup_{k \to \infty} \sum_{i=1}^{k-1} \|Y_{k-1}\| \gamma^i,
\]
and consequently,
\[
S \leq 0 \implies S = 0.
\]
The above equation means that \(\lim_{k \to \infty} D_k = 0\) and from (9),
\[ \lim_{k \to \infty} X_k = \lim_{k \to \infty} A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}, k \in \mathbb{N} \]

As a result if \( \lim_{k \to \infty} \|A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1}\| = 0 \) then \( \lim_{k \to \infty} \|X_k\| = 0 \) also By hypothesis of the theorem and the lemma,

\[ \lim_{k \to \infty} A_1 A_2 \ldots A_k = \lim_{k \to \infty} C_1 C_2 \ldots C_k = 0, \]

Therefore,

\[ \lim_{k \to \infty} P_k = 0. \]

So it completes the proof.

4. STABILITY OF SWITCHED LINEAR WITH INFINITE NUMBER OF SUBSYSTEM

**Theorem 2.** Let \( (\Sigma_k)_{k \in \mathbb{N}} \) be a sequence of switching system of the form (4) with \( \|A_i\| \leq \alpha < 1, \|C_i\| \leq \gamma < 1 \). The discrete time switched linear system (3) with infinite number of switching system is GUAS under arbitrary switching if \( \lim_{k \to \infty} A_1 A_2 \ldots A_{k-1} B_k (I - C_k)^{-1} = 0 \).

Arbitrary switching refers to switched systems that there are no restrictions on the discrete event dynamics.

Proof: By definition1 and theorem1, any infinite product of this kind of switched linear system under these conditions converges to 0 so \( \rho(\Sigma) < 1 \) and the switches linear system is GUAS.

**Lemma 2.** Suppose the discrete-time switched linear system (3) where \( B_k = 0 \). The discrete-time switched linear system (3) is GUAS if and only if \( \{A_1, A_2, \ldots, A_k, \ldots\} \) \( \{C_1, C_2, \ldots, C_k, \ldots\} \) are GUAS.

**Proof.** Let \( \{\Sigma_k\} \) be of the form \( \begin{bmatrix} A_k & 0 \\ 0 & C_k \end{bmatrix} \) where \( [A_k] \in \mathbb{R}^{n_1 \times n_1} \) and \( [C_k] \in \mathbb{R}^{n_2 \times n_2} \).

Then,

\[ P_k = \Sigma_1 \Sigma_2 \ldots \Sigma_k = \begin{bmatrix} A_1 A_2 \ldots A_k & 0 \\ 0 & C_1 C_2 \ldots C_k \end{bmatrix} \]

that the state \( x \in \mathbb{R}^n, \Sigma_k \in \mathbb{R}^{n \times n} \). So \( \lim_{k \to \infty} P_k \to 0 \) if and only if,

\[ \lim_{k \to \infty} \|A_1 A_2 \ldots A_k \| \to 0 \text{ and } \lim_{k \to \infty} \|C_1 C_2 \ldots C_k \| \to 0. \]

So in fact the discrete-time switched linear system (3) with \( B_k = 0 \) is GUAS if and only if \( \{A_1, A_2, \ldots, A_k, \ldots\}, \{C_1, C_2, \ldots, C_k, \ldots\} \) are GUAS.

**Result 1.** The discrete time switched linear system (3) with infinite number of switching system with the form (4) is GUAS under arbitrary switching if there exist two numbers \( \alpha \) and \( \gamma \) such that, \( \|A_i\| \leq \alpha < 1, \|C_i\| \leq \gamma < 1 \) and \( \|B_i\| \leq \beta < \infty \) is bounded.

**Result 2.** Because of \( \rho(\cdot) \leq \|\cdot\| \), the discrete time switched linear system (3) with infinite number of switching system with the form (4) is GUAS under arbitrary switching if there exist two numbers \( \alpha \) and \( \gamma \) such that, \( \|A_i\| \leq \alpha < 1, \|C_i\| \leq \gamma < 1 \) and \( \|B_i\| \leq \beta < \infty \) is bounded.

Example 1. Consider switched linear system (3) with two subsystems of the form (4) and let:

\[ \Sigma_k = \begin{bmatrix} -1 & 3 & 5 \\ -5 & 1 & 2 \\ -4 & -2 & 4 \end{bmatrix}, K = 1, 2. \]

According to the result 2 this infinite switched system is stable because

\[ \rho(A_k) = \rho \left( \begin{bmatrix} 1 & \cos(k) \\ \frac{1}{3k} & 1 \end{bmatrix} \right) < 1, \]

\[ \rho \left( \frac{1}{2k} \right) < 1, \|3 \sin(k)\| < \infty \text{ for } k = 1, 2, \ldots. \]

Example 2. Consider a discrete time switched linear system of the form 4 that,

\[ \Sigma_k = \begin{bmatrix} 1 & \cos(k) \\ \frac{1}{3k} & 1 \end{bmatrix}, K = 1, 2, \ldots. \]

So in fact the discrete time switched linear system (3) with infinite number of switching system with the form (4) is GUAS under arbitrary switching if there exist two numbers \( \alpha \) and \( \gamma \) such that, \( \|A_i\| \leq \alpha < 1, \|C_i\| \leq \gamma < 1 \) and \( \|B_i\| \leq \beta < \infty \) is bounded.

Figure 1 shows some simulation result of example 1.
5. CONCLUSIONS

This paper proposed a new analytic method for globally uniformly asymptotically stability analysis of the linear switched systems with infinite number of switching. A sufficient condition based on the GUAS definition was established.

REFERENCES


Consider the time-invariant dynamic system
\[ \dot{x} = f(x), \quad x \in \mathbb{R}^n \]  
and let the initial time be \( t_0 = 0 \) without loss of generality. The origin \( x^* = 0 \) is said to be a stable equilibrium point of (A.1) in the sense of Lyapunov, if for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[ \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \varepsilon, \forall t \geq 0 \]  
In this case it also will be said simply that the system (A.1) is stable. Lyapunov stability does not require that trajectories starting close to the origin tend to the origin asymptotically. The system (A.1) is called asymptotically stable if it is stable and \( \delta \) can be chosen so that
\[ \|x(0)\| \leq \delta \Rightarrow \lim_{t \to \infty} x(t) = 0 \]  
The set of all initial states from which the trajectories converge to the origin is called the region of attraction. If the condition (A.3) holds for all \( \delta \), i.e., if the origin is a stable equilibrium and its region of attraction is the entire state space, then the system (A.1) is called globally asymptotically stable.

If the system is not necessarily stable but has the property that all solutions with initial conditions in some neighborhood of the origin converge to the origin, then it is called locally attractive.

The system (A.1) is globally attractive if its solutions converge to the origin from all initial conditions.

Uniform stability is a concept which guarantees that the equilibrium point is not losing stability.

Uniform asymptotic stability of the system (A.1) requires that \( x^* = 0 \) is uniformly stable and the convergence in equation (A.3) holds and is uniform.

The system (A.1) is called exponentially stable if there exist positive constants \( \delta \), \( c \), and \( \lambda \) such that all solutions of \( \dot{x} = f(x) \) with \( |x(0)| \leq \delta \) satisfy the inequality
\[ |x(t)| \leq c|x(0)|e^{-\lambda t}, \quad \forall t \geq 0 \]  
If this exponential decay estimate holds for all \( \delta \), the system is said to be globally exponentially stable. The constant \( \lambda \) in (A.4) is occasionally referred to as a stability margin (Liberzon, 2003).