# Trajectory Tracking for Nonlinear Systems Using Composed Recursive Controllers

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**Abstract:** The paper deals with a new recursive controller for trajectory tracking of MIMO nonlinear affine in control systems. The proposed controller comprises stabilization and compensation sub-controllers and dose not require knowledge of the physical model parameters. The stabilization sub-controller is developed using the theory of a particular class of hybrid systems called piecewise continuous systems and characterized by autonomous switchings and controlled impulses. The compensation sub-controller is designed based on the time delay estimation theory. The proposed recursive controller is tested and compared with a classical PID controller for a three tank system.

*Keywords:* Nonlinear affine in control system, Piecewise continuous systems, Recursive controllers.

## 1. INTRODUCTION

With few exceptions the existing nonlinear control methods are based on mathematical models of the plants (Isidori, 1995; Khalil, 2002; Tahir et al., 2009). The model based control design provides mathematically rigorous results concerning the controller structure and properties. The potential limitation of this approach is that the controller performances strongly depend on the availability of an accurate plant model. Plant modelling and identification are always difficult problems (Gédouin et al., 2008). Moreover, the use of highly accurate models taking into account many coupled nonlinear effects, gives rise to substantial difficulties in parameter and state estimation.

Despite the development of more and more sophisticated model based control methods, their practical application remains very limited (Sun, 2007). Over 90% of industrial processes are operated by Proportional-Integral-Derivative (PID) controllers (Aström and Haglund, 1995) for their conceptual simplicity and usage of uncomplicated mathematical model. However, the quite tedious tuning and poor performance of PID controllers in case of severe coupling plant nonlinearities have prompted the introduction of model-free control techniques like fuzzy logic, artificial intelligence and neural nets. Recently, a numerical differential algebraic method has been proposed using a phenomenological model to approximate the plan dynamics in a short amount of time. This model is updated step by step and the desired system behavior is obtained using a standard PID controller (Gédouin et al., 2008; Fliess et al., 2006b; Fliess et al., 2006a). Moreover, an experimental method called "active disturbance rejection control" and based on a nonlinear combination of PID errors and an extended state observer has been developed in (Han, 1998; Han, 2009; Gao, 2006).

In this paper a Composed Recursive Controller (CRC) is proposed for MIMO nonlinear affine in control systems. This controller consists of stabilization and compensation sub-controllers and does not require identification of process parameters. The stabilization sub-controller is developed by using the theory of a particular class of hybrid systems called piecewise continuous systems (Koncar and Vasseur, 2003). In turn, the compensation sub-controller is developed based on the time delay estimation theory (Cho et al., 2005).

The paper is organized as follows. The problem statement is given in Section 2 and the existing piecewise continuous controllers are briefly described in Section 3. The development of CRC is presented in Section 4. The CRC application to a nonlinear three tank system is considered in Section 5. Some concluding remarks are given in Section 6.

## 2. PROBLEM STATEMENT

The following MIMO nonlinear, affine in control system is considered

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x,t) \\ f_2(x_1,x_2,t) \end{bmatrix} + \begin{bmatrix} g(x,t) \\ 0 \end{bmatrix} u(t) \quad (1)$$

$$y(t) = x_1(t) \quad (2)$$

where $x(t) = \left[x_1^T(t),\ x_2^T(t)\right]^T$ is the system state vector with $x_1(t) \in \Re^m$, $x_2(t) \in \Re^{n-m}$; $u(t) \in \Re^m$ is the control vector, $y(t) \in \Re^m$ is the output vector, $f_1(x,t)$, $g(x,t) \in C^1$ in $\Re^n \times \Re$ and $f_2(x_1,x_2,t) \in C^1$ in $\Re^m \times \Re^{n-m} \times \Re$. It is assumed that the subsystem

$$\dot{x}_2(t) = f_2(x_1,x_2,t) \quad (3)$$

is input-to-state stable (Pan et al., 2005).

The system output $y(t)$ is required to track a known bounded reference signal $y_d(t)$:

$$y(t) \to y_d(t). \tag{4}$$

In particular, $y_d(t)$ can be generated by a dynamical reference model

$$\dot{x}_r(t) = f_r(x_r(t), r(t), t) \tag{5}$$

$$y_d(t) = h_r(x_r, t) \tag{6}$$

where $r(t)$ is a smooth reference input.

## 3. PIECEWISE CONTINUOUS SYSTEMS AND CONTROLLERS

The Piecewise Continuous Systems (PCS) introduced first in (Koncar and Vasseur, 2003) and then developed in (Wang et al., 2008; Wang et al., 2010a; Wang et al., 2010c), are hybrid systems with autonomous switchings and controlled impulses. PCS are characterized by two input spaces and two time spaces. The first time space is the discrete time space $S = \{t_k, k = 0, 1, 2, \ldots\}$ called switching space, where $t_k$ are the switching instants. The second time space is the continuous time space $\Phi_t = \{\Im - S\}$ where $\Im = \{t \in [0, \infty)\}$. In reference to these time spaces, PCS are controlled by two types of inputs: sampled inputs at switching and continuous inputs between two switching instants. For a constant switching period $t_e$ one has $S = \{kt_e, k = 0, 1, 2, \ldots\}$.

Based on the PCS theory, a Piecewise Continuous Controller (PCC) was developed in (Koncar and Vasseur, 2003), (Wang et al., 2006), enabling sampled tracking of linear time-invariant systems. For constant switching period PCC can be described as

$$\lambda(kt_e^+) = B_{c2}\psi(kt_e), \forall k \in S \tag{7}$$

$$\dot{\lambda}(t) = A_c\lambda(t) + B_{c1}\varphi(t), \forall t \in \Phi_t \tag{8}$$

$$u_s(t) = C_c\lambda(t), \forall t \in \Im. \tag{9}$$

Equation (7) defines the controller state $\lambda(t) \in \Re^p$ at switching instants by means of the sampled input $\psi(kt_e) \in \Re^q$. Equation (8) describes the continuous-time evolution of the controller state under the action of the continuous input $\varphi(t) \in \Re^r$. Equation (9) is the PCC output equation, the controller output $u_s(t) \in \Re^m$ being the plant control signal. $B_{c2}$, $A_c$, $B_{c1}$ and $C_c$ are constant matrices with appropriate dimensions. Fig. 1(a) and Fig. 1(b) show respectively the PCC realization diagram and state evolution. Note that normally $\lambda(kt_e^-) \neq \lambda(kt_e^+)$ according to equations (7), (8). Further on we shall denote $f(kt_e)$ by $f_k$ and $f(kt_e^+)$ by $f_k^+$.

A simplified PCC can be obtained choosing $B_{c1} = 0$ and $C_c$ diagonal matrix with positive diagonal elements. Thus the only parameter defining the controller behavior between two switching instants is the matrix $A_c$ which is chosen to ensure the PCC stability. In this case the tuning of PCC consists of determining $B_{c2}$ and $\psi(t)$ in order to achieve a sampled tracking of a desired state trajectory $x_d(t)$ by the plant state $x_s(t)$ with one sampling period of delay:

$$x_{s,k+1} = x_{d,k}, \quad k = 0, 1, 2, \ldots \tag{10}$$

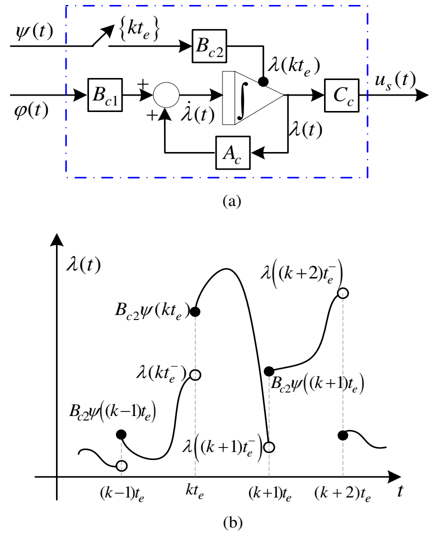


Fig. 1. PCC realization diagram and state evolution

Let the plant be modelled as

$$\dot{x}_s(t) = Ax_s(t) + Bu_s(t) \tag{11}$$

$$y_s(t) = Cx_s(t) \tag{12}$$

where $x_s(t) \in \Re^n$, $u_s(t) \in \Re^m$ and $y_s(t) \in \Re^m$ are the plant state, input and output, respectively, and $A$, $B$ and $C$ are constant matrices. As shown in (Wang et al., 2008), the sampled state tracking for such plants is ensured by a simplified PCC with $B_{c2} = M^{-1}$ and $\psi(t) = x_d(t) - A_d x_s(t)$, where $A_d = e^{At_e}$ and $M = A_d \int_0^{t_e} e^{-A\tau} BC_c e^{A_c\tau}\, d\tau$.

PCC can be further simplified for sampled tracking of a desired output trajectory $y_d(t)$ :

$$y_{s,k+1} = y_{d,k}, \quad k = 0, 1, 2, \ldots \tag{13}$$

by enabling switching at high frequencies ($t_e \to 0^+$). In this case one obtains

$$\lambda_k^+ = I_m^- \lambda_k^- + e_k \tag{14}$$

where $I_m^- = I_m - CBC_c t_e - \varsigma(t_e^2)$ and $e_k = y_{d,k} - y_{s,k}$ (see Wang et al., 2008). Equation (14) can be interpreted algorithmically as an iterative evaluation of $\lambda_k^+$ at each calculation step:

$$\lambda_k^+ \leftarrow I_m^- \lambda_k^- + e_k. \tag{15}$$

The evolution of the controller state for $t_e \to 0^+$ being negligible, PCC can be regarded as a zero order hold. Furthermore, if switching occurs at each calculation step of the computer, PCC can be realized by a simple circuit as shown in Fig. 2.

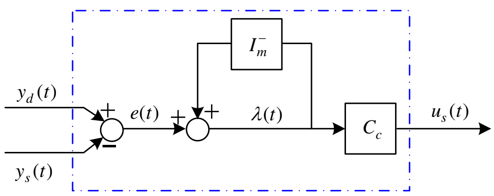


Fig. 2. Derived Piecewise Continuous Controller

In practice $I_m^-$ can be chosen as a diagonal matrix $I_m^- = diag(i_1^-, i_2^-, \ldots, i_m^-)$ with $\|i_k^-\| < 1$, $k = 1, 2, \ldots, m$ and thus the knowledge of the plant parameters is not necessary.

## 4. COMPOSED RECURSIVE CONTROLLER

### *4.1 Controller Design*

In this section a new controller, called Composed Recursive Controller (CRC) is proposed to solve the nonlinear control problem (1), (2), (4). The proposed controller is defined as

$$u(t) = G^{-1}\big(u_s(t) + u_c(t)\big) \tag{16}$$

where $u_s(t)$ is a stabilization sub-control, $u_c(t)$ is a sub-control for compensation of unknown system nonlinearities and $G \in \Re^{m\times m}$ is a nonsingular diagonal weighting matrix. Both $u_s(t)$ and $u_c(t)$ are determined using recursive calculation loops like in PCC, see Fig.2. It is important to note that the stabilization sub-control $u_s(t)$ is derived from (15) as shown in (Wang et al., 2010b). The CRC realization diagram is given in Fig. 3.

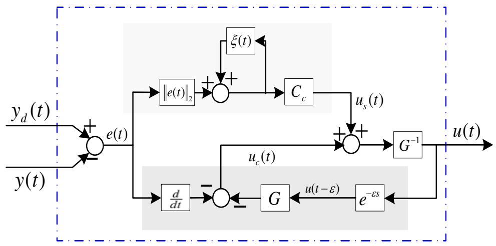


Fig. 3. Composed Recursive Controller

As shown in Fig. 3, the stabilization control component $u_s(t)$ is computed by

$$\lambda(t) = \|e(t)\|_2 e(t) + \xi(t)\lambda(t) \tag{17}$$

$$u_s(t) = C_c\lambda(t) \tag{18}$$

where $\lambda(t) \in \Re^m$ is the stabilization sub-controller state, $e(t) = y_d(t) - y(t)$ is the output trajectory tracking error, $C_c \in \Re^{m\times m}$ is the sub-controller output matrix, and $\xi(t)$ is the sub-controller tracking coefficient. To realize $e(t) \to 0$, the value of $\xi(t)$ is tuned as

$$\xi(t) = \exp\big(\frac{-e^T(t)e(t)}{2\sigma^2}\big) \tag{19}$$

with $0 < \sigma \le 1$.

The compensation sub-control $u_c(t)$ is determined using time delay estimation techniques (Cho et al., 2005) as

$$u_c(t) = \dot{y}_d(t) - P^*(t) \tag{20}$$

where

$$P^*(t) = Gu(t-\epsilon) - \dot{y}(t-\epsilon) \tag{21}$$

and $\epsilon$ is the numerical integration step.

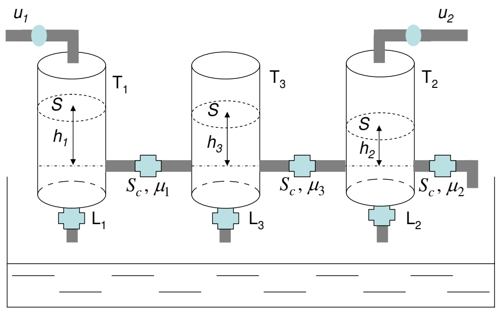


Fig. 4. Three tank system

## 5. SIMULATION RESULTS

In order to illustrate the CRC design and trajectory tracking performances, the controller is applied to the three tank system shown in Fig.4.

The system is composed of three cylindrical tanks of same section $S$, connected serially by pipes of cross section $S_c$ . The inputs of the system are $u_1$ and $u_2$ , the flow rates of pumps 1 and 2 (not shown in Fig.4), and the outputs are $h_1$ and $h_2$ , the liquid level of tanks $T_1$ and $T_2$. The outflows through valves are $L_1$, $L_2$ and $L_3$ which are zeros in the modelling procedure but are considered as disturbance in the numerical simulation. The output-flowing liquid is collected in a reservoir, which is the source of pumps 1 and 2. The pumps 1 and 2 supply liquid for tank $T_1$ and tank $T_2$, respectively, and tank $T_3$ only gain liquid by coupling effect from tank $T_1$ and tank $T_2$. The open-loop three tank system response time is comparatively long (Bouzouita et al., 2008), (Yang et al., 2008) and the main control problem is to realize tracking of desired trajectories at a shorter time.

The liquid level evolutions in tanks $T_1$, $T_2$ and $T_3$ depend on the input and output flows and can be expressed as (Fliess et al., 2006a; Yang et al., 2008; Benayache et al., 2008):

$$\dot{h}_1 = -C_1\mathrm{sign}(h_1 - h_3)\sqrt{|h_1 - h_3|} + \frac{u_1}{S} \tag{22}$$

$$\dot{h}_2 = C_3\mathrm{sign}(h_3 - h_2)\sqrt{|h_3 - h_2|} - C_2\mathrm{sign}(h_2)\sqrt{|h_2|} + \frac{u_2}{S} \tag{23}$$

$$\dot{h}_3 = C_1\mathrm{sign}(h_1 - h_3)\sqrt{|h_1 - h_3|} - C_3\mathrm{sign}(h_3 - h_2)\sqrt{|h_3 - h_2|} \tag{24}$$

with $C_i = \mu_i S_c\sqrt{(2g)}/S$, $i = 1, 2, 3$. The corresponding parameter values are given in Table 1.

Table 1. Physical Parameters of the Three Tank System

| Symbol | Value | Meaning |
|---|---|---|
| $S$ | $0.0154$ m$^2$ | section of the tanks |
| $S_c$ | $5.10^{-5}$ m$^2$ | section of pipes |
| $g$ | $9.81$ m/s$^2$ | gravity coefficient |
| $\mu_i, i = 1, 2, 3$ | $\mu_1 = \mu_2 = 0.5$, $\mu_3 = 0.675$ | flowing coefficient |

The system output is

$$y(t) = [h_1(t) \;\; h_2(t)]^T . \tag{25}$$

The CRC parameters are tuned as $G = diag(2, 4)$, $\sigma = 0.98$, $C_c = diag(100, 100)$ and the time delay $\epsilon$ is chosen equal to the sampling period used $t_e = 0.0001s$.

CRC performances are compared with those of a classical PID controller

$$u(t) = k_p e(t) + k_i \int_0^t e(t)dt + k_d \dot{e}(t)$$

tuned according to Ziegler-Nichols method with $k_p = 1.87$, $k_i = 0.02$, $k_d = 0.005$.

CRC and PID controllers are tested and compared for tracking three types of reference trajectories (a piecewise constant reference, a multiple frequency sinusoidal reference and a transcendental reference) under the outflow disturbances of $L_1$ and $L_2$ shown in Fig. 5. Denoting $e_1(t) = h_{1d}(t) - h_1(t)$, $e_2(t) = h_{2d}(t) - h_2(t)$, where $y_d(t) = [h_{1d}(t) \;\; h_{2d}(t)]^T$ is the the desired output trajectory, the control objective is to realize $e(t) = [e_1(t) \; e_2(t)]^T \to 0$.

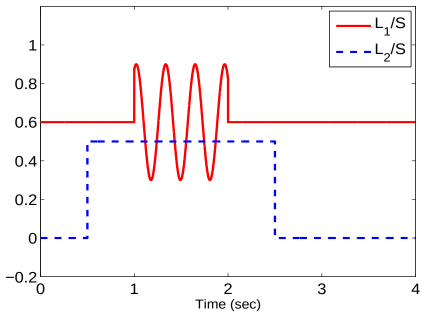


Fig. 5. Outflow disturbances

### 5.1 *Tracking Piecewise Constant Reference Trajectories*

The desired and obtained output trajectories using CRC and PID controllers are given in Fig. 6 and Fig. 7, respectively. The corresponding tracking errors $e_1(t)$ and $e_2(t)$ are shown in Fig. 8 and Fig. 9.

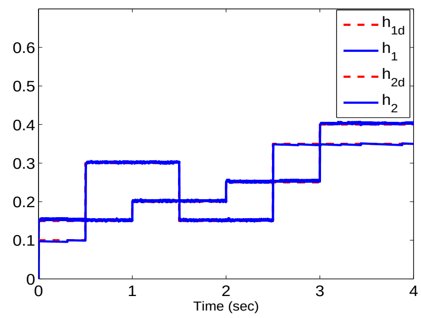


Fig. 6. CRC control for piecewise constant reference trajectories

### 5.2 *Tracking Multiple Frequency Sinusoidal Reference Trajectories*

Under the same outflow disturbances, the desired reference trajectories are defined by a 3rd-order Bézier polynomial and a sinus function as

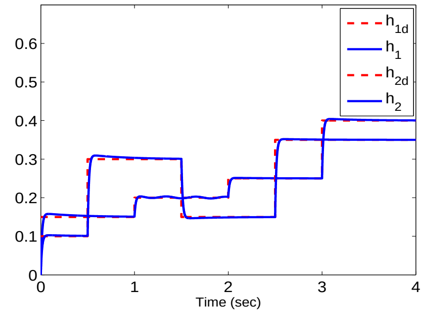


Fig. 7. PID control for piecewise constant reference trajectories

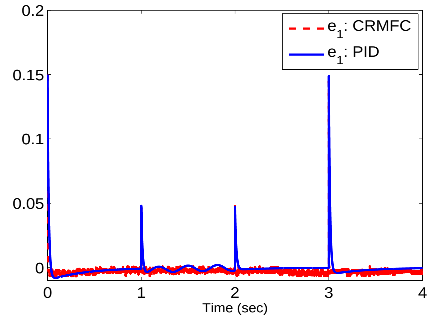


Fig. 8. Tracking error $e_1(t)$ for piecewise constant reference trajectories

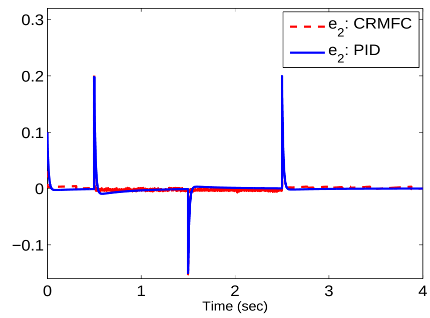


Fig. 9. Tracking error $e_2(t)$ for piecewise constant reference trajectories

$$y_d = \begin{bmatrix} 0.05\Big(\big(\sin(8t)\big)^3 - 3\big(\sin(8t)\big)^2 + 3\sin(8t)\Big) + 0.4 \\ 0.1\sin(20t) + 0.1 \end{bmatrix}$$

The corresponding results for CRC and PID control are given in Fig. 10 – Fig.13. As in the previous case, it can be seen that comparing to PID the proposed controller ensures faster responses with small tracking errors.

### 5.3 *Tracking Transcendental Reference Trajectories*

Finally, CRC and PID tracking performances were tested for transcendental reference trajectories defined as

$$\dot{x}_r(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_r(t) + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r(t) \tag{26}$$

$$x_{1d}(t) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} E x_r(t) \tag{27}$$

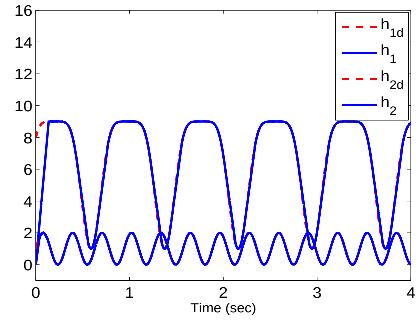


Fig. 10. CRC control for multiple frequency reference trajectories

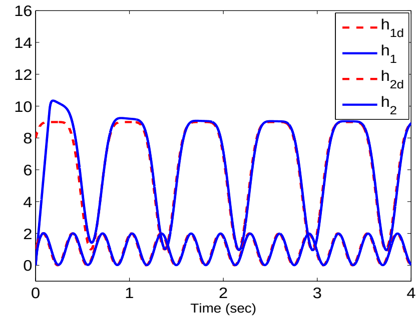


Fig. 11. PID control for multiple frequency reference trajectories

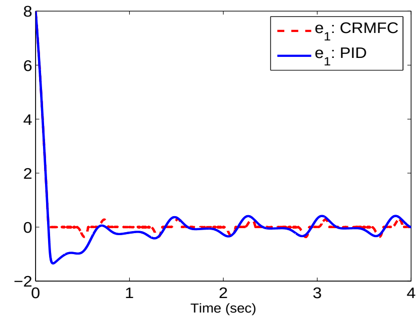


Fig. 12. Tracking error $e_1(t)$ for multiple frequency reference trajectories

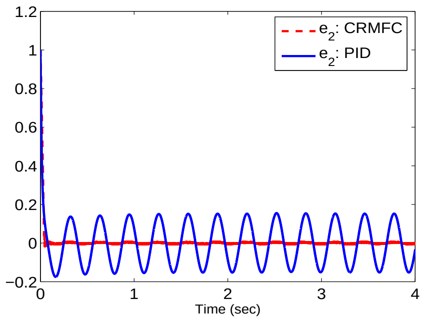


Fig. 13. Tracking error $e_2(t)$ for multiple frequency reference trajectories

with $x_r(0) = [1 \;\; 2\pi]^T$, $E = [1 \;\; 0]$ and $r(t)$ being an external signal defined by

$$r(t) = \frac{1}{25}\pi^2 \exp(\sin(2\pi t)) \left[\frac{1+\cos(4\pi t)}{2} - \sin(2\pi t)\right] \quad (28)$$

The results obtained are presented in Fig. 14 – Fig. 17.

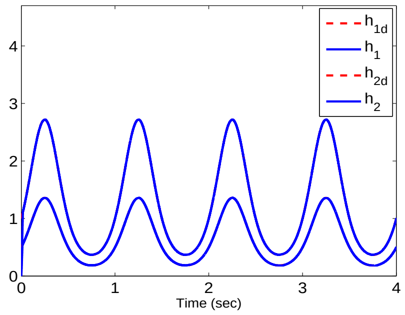


Fig. 14. CRC control for transcendental reference trajectories

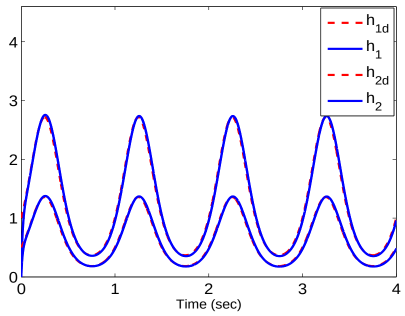


Fig. 15. PID control for transcendental reference trajectories

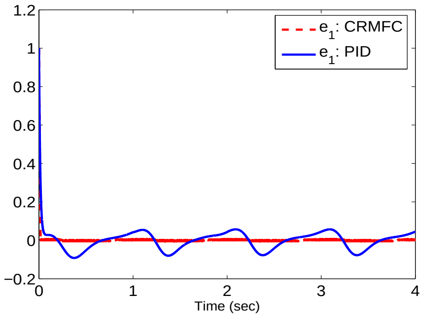


Fig. 16. Tracking error $e_1(t)$ for transcendental reference trajectories

The test results in all considered cases show that the proposed controller ensures faster response time and smaller trajectory tracking errors comparing to the classical PID controller.

## 6. CONCLUSION

In this paper a composed controller with two recursive calculation loops is proposed for MIMO nonlinear affine in control systems. The new controller does not require identification of the process model parameters and is easy

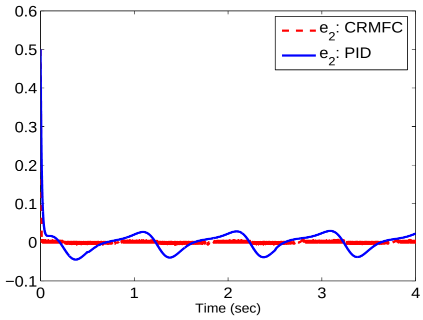


Fig. 17. Tracking error $e_2(t)$ for transcendental reference trajectories

to implement. The performances of the proposed controller are tested and compared with standard PID controller performances for a three tank system.

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