A Coning Compensation Algorithm for SINS in High Dynamic Motion[1]

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Abstract: The accuracy of coning compensation in attitude updating algorithm directly affects the navigation accuracy of SINS. A new generalized coning compensation algorithm is presented based on dividing the minor coning compensation updating interval into a number of sub-minor intervals. Unlike other existing algorithms, the updating rate of the proposed algorithm is independent of the number of gyro incremental angles for calculating the coning compensation. The relationship between the attitude errors and the coning compensation updating rate as well as the number of gyro incremental angle samples for calculating the coning compensation is deduced for quantitative analysis of the new coning compensation algorithm under typical coning movement conditions. The result reveals that the attitude errors caused by the vehicle coning movement may be decreased by increasing the updating speed of coning compensation or increasing the number of gyro incremental angle samples in each sampling interval for calculating the coning compensation. But the selection of the number of gyro incremental angle samples has more complex effects than the selection of the updating speed of coning compensation on the attitude error.

Keywords: SINS, attitude algorithm, coning compensation, quantitative analysis of error.

1. INTRODUCTION

Recent advances in technology have allowed ever-increasing speed and acceleration of vehicles, e.g., aircrafts, space shuttles, rockets and missiles. With this tendency comes the need for enhanced systems to navigate and control these vehicles to meet precise attitude requirements. In a strapdown inertial navigation system (SINS) used in those vehicles, inertial sensors (gyros and accelerometers) are rigidly attached to the vehicle, which leads to the system suffering from the high dynamics environment of vehicle movement. In addition, inertial sensors may be subjected to high frequency tremors as a result of body bending and engine-induced vibration. The ability of the attitude updating algorithm in a SINS to keep track of its body attitude accurately in severe maneuvering and vibratory environment may be the critical factor in determining the performance of the SINS, if accurate navigation is required.

Most attitude updating algorithms commonly used in modern SINS are the quaternion algorithms based on rotation vector (Savage 2006; Savage 2007; Savage 2009; Savage 2009; Savage 2010) with a two-speed structure by which each attitude updating calculation is divided into two parts (Savage 1998; Titterton and Weston 2004): an exact moderate-speed updating routine typically designed to update the attitude based on the maximum angular rate considerations of vehicles, whose input is a rotation vector equal to integrated gyro sensed angular rate over the updating cycle; and a coning compensation calculated on the basis of gyro data in a separate algorithm executed in a high-speed routine. The high-speed updating routine is designed to accurately account for vibration induced coning effects based on the anticipated vibration environment of SINS. The coning compensation is one of the key calculations performed in the quaternion algorithms because the accuracy of the compensation directly determines the performance of a SINS, particularly when it is in high dynamic motion.

Most of the algorithms for the coning compensation adopt truncated Taylor series expansion approximations for the calculation of angular rate of system body over the coning compensation computation interval (Lee and Yoon 1990; Jiang and Lin 1992; Musoff and Murphy 1995; Ignagni 1996; Park, Kim et al. 1996; Park, Kim et al. 1999; Titterton and Weston 2004; Savage 2007). The accuracy of a coning compensation algorithm is determined by the updating rate of coning compensation calculation and the order of truncated Taylor series expansion. Generally, in order to improve the accuracy of an attitude updating algorithm, the updating rate of the involved coning compensation algorithm must be increased to keep track of the body angular motion more accurately (Yang and Fang 2007; Pan, Wu et al. 2008; Tan,

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Huang et al. 2008; Weng, Zhao et al. 2008; Yan, Yan et al. 2008; Yu and Zhang 2008; Yang and Fang 2010). Among these existing algorithms, however, when the sampling rate of inertial sensors remains constant and the number of gyro incremental angle for calculating the coning compensation (i.e., the order of the truncated Taylor series expansion) is selected, the updating rate of the algorithm is also determined. Increasing the updating rate needs to decrease the number of gyro incremental angle, while decreasing the number of gyro incremental angle will in turn reduce the accuracy of the algorithm, thus limiting the further improvement of the attitude accuracy. Therefore increasing the updating rate of coning compensation calculation and increasing the order of truncated Taylor series expansion are the contradictory tasks.

In this paper, a new generalized coning compensation algorithm is presented based on dividing the minor coning compensation updating interval into a number of sub-minor intervals. Unlike other existing algorithms, the updating rate of the proposed algorithm is independent of the number of gyro incremental angles for calculating the coning compensation. Thus, when the sampling rate of inertial sensors remains constant, this algorithm allows the updating speed to be increased, at the same time increasing the number of gyro incremental angle samples, in order to improve the accuracy of the related attitude updating algorithms. The relationship between the attitude errors and the coning compensation updating rate as well as the number of gyro incremental angle samples for calculating the coning compensation is deduced for quantitative analysis of the new coning compensation algorithm under typical coning movement conditions. The result reveals that the attitude errors caused by the vehicle coning movement may be decreased by increasing the updating speed of coning compensation (indirectly, increasing the updating rate of rotation vector and attitude) or selecting an appropriate number of gyro incremental angle samples in each sampling interval for calculating the coning compensation.

2. QUATERNION ATTITUDE UPDATING ALGORITHM BASED ON A NEW GENERALIZED CONING COMPENSATION

In the quaternion-based attitude updating algorithms used in most modern SINS, each major attitude updating interval is divided into a number of minor rotation vector (and coning compensation) intervals. Each of the intervals is in turn split into a number of sub-minor sampling intervals over which the gyro incremental angles are measured, as shown in Fig.1.

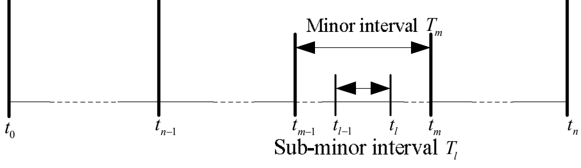

Fig.1. Intervals associated with attitude updating algorithm

According to the quaternion chain rule, the updating algorithm for the attitude quaternion $\mathbf{Q}_B^N$ is generally constructed as follows (Savage 1998; Titterton and Weston 2004):

$$\begin{aligned} \mathbf{Q}_{B(m)}^{N(n-1)} &= \mathbf{Q}_{B(m-1)}^{N(n-1)} \otimes \mathbf{q}_{B(m)}^{B(m-1)}, \\ \mathbf{Q}_{B(m)}^{N(n)} &= \mathbf{q}_{N(n-1)}^{N(n)} \otimes \mathbf{Q}_{B(m)}^{N(n-1)}, \end{aligned} \tag{1}$$

where $\mathbf{Q}_{B(m-1)}^{N(n-1)}$ is the attitude quaternion $\mathbf{Q}_B^N$ relating the B frame at time t_{m-1} to the N frame at time t_{n-1}; $\mathbf{Q}_{B(m)}^{N(n)}$ is the attitude quaternion $\mathbf{Q}_B^N$ relating the B frame at time t_m to the N frame at time t_n; $\mathbf{q}_{B(m)}^{B(m-1)}$ is the attitude updating quaternion that accounts for the angular motion of the B frame from time t_{m-1} to time t_m; $\mathbf{q}_{N(n-1)}^{N(n)}$ is the attitude updating quaternion that accounts for the N frame rotation from time t_{n-1} to time t_n.

According to the relationship between rotation vector and attitude updating quaternion, the attitude updating quaternion $\mathbf{q}_{B(m)}^{B(m-1)}$ can be formulated in the following form of rotation vector:

$$\mathbf{q}_{B(m)}^{B(m-1)} = \cos\frac{\Phi_m}{2} + \frac{\mathbf{\Phi}_m}{\Phi_m}\sin\frac{\Phi_m}{2} \tag{2}$$

where $\mathbf{\Phi}_m$ is the rotation vector that accounts for the angular motion of the B frame from time t_{m-1} to time t_m; Φ_m is the magnitude of $\mathbf{\Phi}_m$.

The rotation vector $\mathbf{\Phi}_m$ can be computed by treating $\mathbf{\Phi}$ as a general rotation vector defining the angular motion of the B frame. In practice, it can be obtained from the following rotation vector rate equation (Bortz 1971):

$$\dot{\mathbf{\Phi}} = \boldsymbol{\omega}_{IB}^B + \frac{1}{2}\mathbf{\Phi}\times\boldsymbol{\omega}_{IB}^B + \frac{1}{12}\mathbf{\Phi}\times\left(\mathbf{\Phi}\times\boldsymbol{\omega}_{IB}^B\right) \tag{3}$$

where $\boldsymbol{\omega}_{IB}^B$ represents the angular rate of the B frame. The last two terms in equation (3) are referred as the non-commutativity terms. In order to improve the accuracy of attitude updating algorithms, the second term must be compensated with the gyro incremental angles, while the third term (i.e., the triple-cross-product term in equation (3)) is usually quite small and can be neglected (Bortz 1971).

Under the second order accuracy, the rotation vector $\mathbf{\Phi}_m$ in equation (2) can be approximately obtained by the integral of equation (3) from time t_{m-1} to time t_m:

$$\mathbf{\Phi}_m = \int_{t_{m-1}}^{t_m} [\boldsymbol{\omega}_{IB}^B + \frac{1}{2}(\boldsymbol{\alpha}(t)\times\boldsymbol{\omega}_{IB}^B)]dt = \boldsymbol{\alpha}_m + \boldsymbol{\beta}_m \tag{4}$$

where $\boldsymbol{\beta}_m$ represents the coning compensation for coning effect from time t_{m-1} to time t_m, and

$$\boldsymbol{\alpha}_m = \boldsymbol{\alpha}(t_m),\quad \boldsymbol{\alpha}(t) = \int_{t_{m-1}}^{t} \boldsymbol{\omega}_{IB}^{B} d\tau,$$
$$\boldsymbol{\beta}_m = \boldsymbol{\beta}(t_m),\quad \boldsymbol{\beta}(t) = \frac{1}{2}\int_{t_{m-1}}^{t} (\boldsymbol{\alpha}(\tau)\times\boldsymbol{\omega}_{IB}^{B}) d\tau. \tag{5}$$

In order to develop the new generalized coning compensation algorithm where the updating rate is independent of the number of gyro incremental angle samples, first let equation (5) be divided into two portions up to and after a general time t_{l-1} within the interval $[t_{m-1}, t_m]$. Then equation (5) can be rewritten as:

$$\boldsymbol{\alpha}_l = \boldsymbol{\alpha}_{l-1} + \Delta\boldsymbol{\alpha}_l,\quad \Delta\boldsymbol{\alpha}_l = \int_{t_{l-1}}^{t_l} \boldsymbol{\omega}_{IB}^{B} d\tau, \tag{6a}$$

$$\boldsymbol{\alpha}_m = \boldsymbol{\alpha}_l\ \ (t_l = t_m),\quad \boldsymbol{\alpha}_l = 0\ \ (t_l = t_{m-1}), \tag{6b}$$

$$\boldsymbol{\beta}_l = \boldsymbol{\beta}_{l-1} + \Delta\boldsymbol{\beta}_l,\quad \Delta\boldsymbol{\beta}_l = \frac{1}{2}\int_{t_{l-1}}^{t_l} (\boldsymbol{\alpha}(t)\times\boldsymbol{\omega}_{IB}^{B}) dt, \tag{6c}$$

$$\boldsymbol{\beta}_m = \boldsymbol{\beta}_l\ \ (t_l = t_m),\quad \boldsymbol{\beta}_l = 0\ \ (t_l = t_{m-1}). \tag{6d}$$

Note that equations (6a) - (6d) constitute a digital recursive algorithm in the sub-minor sampling interval $T_l = t_l - t_{l-1}$ for the calculation of the coning compensation term $\boldsymbol{\beta}_m$ as a summation of changes in $\boldsymbol{\beta}$ over the interval $[t_{m-1}, t_m]$. Substituting equation (6a) into equation (6c), $\Delta\boldsymbol{\beta}_l$ can be expressed as follows:

$$\begin{aligned}\Delta\boldsymbol{\beta}_l &= \frac{1}{2}\left(\boldsymbol{\alpha}_{l-1}\times\Delta\boldsymbol{\alpha}_l\right) + \frac{1}{2}\int_{t_{l-1}}^{t_l} (\Delta\boldsymbol{\alpha}(t)\times\boldsymbol{\omega}_{IB}^{B}) dt \\ &= \frac{1}{2}\left(\boldsymbol{\alpha}_{l-1}\times\Delta\boldsymbol{\alpha}_l\right) + \delta\boldsymbol{\beta}_l.\end{aligned} \tag{7}$$

For a SINS, the coning movement is the worst working condition which will induce a serious attitude errors (Miller 1983; Lee and Yoon 1990; Jiang and Lin 1992; Musoff and Murphy 1995). In other words, if in the case of coning movement the attitude errors are made minimal by a certain algorithm, the errors in the other cases will also be minimal by the same algorithm. So in order to develop the generalized algorithm, it is assumed that the vehicle is undergoing a pure coning movement, defined by the following angular rate:

$$\boldsymbol{\omega}_{IB}^{B}(t) = \left[a\Omega\cos(\Omega t)\quad b\Omega\sin(\Omega t)\quad 0\right]^T \tag{8}$$

where Ω is the frequency associated with the coning movement; a and b are the amplitudes of coning movement.

According to equation (6a) and equation (8), the integral term $\delta\boldsymbol{\beta}_l$ in equation (7) has the following form:

$$\delta\boldsymbol{\beta}_l = \left[0\quad 0\quad \frac{ab}{2}\left(\Omega T_l - \sin\Omega T_l\right)\right]^T. \tag{9}$$

Equation (9) shows an interesting property that the integral term $\delta\boldsymbol{\beta}_l$ is constant over all sub-minor intervals, regardless of the absolute time at which the interval begins. It only depends on the duration of the sub-minor interval.

According to the concept of distance between the cross products (Lee and Yoon 1990; Park, Kim et al. 1999), the cross products with equal distance behave exactly the same in a pure coning environment defined equation (8). The coning compensation algorithms that use the enhancement concept will have the simplest form, optimal accuracy characteristics and the minimal computational throughput. Taking advantage of this property, a generalized algorithm for equation (7) consists of the sum of all possible cross products formed from the gyro incremental angle samples for the l sub-minor sampling intervals, making up the computation over the minor computational interval of rotation vector, such as (Ignagni 1996),

$$\delta\hat{\boldsymbol{\beta}}_l = \sum_{i=1}^{N-1} k_i \Delta\boldsymbol{\alpha}_{l-i}\times\Delta\boldsymbol{\alpha}_l \tag{10}$$

where N is the number of gyro incremental angle samples for the calculation of $\delta\hat{\boldsymbol{\beta}}_l$; $\Delta\boldsymbol{\alpha}_{l-i}\left(i = 1, 2, \cdots, N-1\right)$ is the gyro incremental angle in the $(l-i)$-th sub-minor interval; $k_i\left(i = 1, 2, \cdots, N-1\right)$ is the constant coefficients for the cross product of $\Delta\boldsymbol{\alpha}_{l-i}$ and $\Delta\boldsymbol{\alpha}_l$.

Substituting equations (6a) and (8) into equation (10), and expanding each terms using Taylor series, the integral term $\delta\hat{\boldsymbol{\beta}}_l$ over the sub-minor interval is obtained as:

$$\delta\hat{\boldsymbol{\beta}}_l = \left[0\quad 0\quad ab\sum_{i=1}^{\infty}(-1)^{i+1}\sum_{j=1}^{N-1} A_{ij}k_j\lambda_l^{2i+1}\right]^T \tag{11}$$

where $\lambda_l = \Omega T_l$; A_{ij} is a constant defined by

$$A_{ij} = \frac{(j+1)^{2i+1} - 2\bullet j^{2i+1} + (j-1)^{2i+1}}{(2i+1)!}, i = 1, 2, \cdots, \infty, j = 1, 2, \cdots, N-1. \tag{12}$$

In order to derive $k_i\left(i = 1, 2, \cdots, N-1\right)$ in equation (10), expanding equation (9) by using Taylor series yields:

$$\delta\boldsymbol{\beta}_l = \left[0\quad 0\quad ab\sum_{i=1}^{\infty}(-1)^{i+1} C_i\lambda_l^{2i+1}\right]^T \tag{13}$$

where C_i is a constant defined by

$$C_i = \frac{1}{(2i+1)!\times 2}. \tag{14}$$

Combining equation (11) with equation (13), the following simultaneous equations for constant coefficients k_i, $i = 1, 2, \cdots, N-1$, can be obtained:

$$\sum_{j=1}^{N-1} A_{ij} k_j = C_i \,, i = 1, 2, \cdots, N-1. \tag{15}$$

In a matrix form, equation (14) is equivalent to

$$\left[A_{ij} \right]_{(N-1)\times(N-1)} \cdot \left[k_j \right]_{(N-1)\times 1} = \left[C_i \right]_{(N-1)\times 1} . \tag{16}$$

So, according to equation (16), coefficients k_i can be solved as follows:

$$\left[k_j \right]_{(N-1)\times 1} = \left[A_{ij} \right]^{-1}_{(N-1)\times(N-1)} \cdot \left[C_i \right]_{(N-1)\times 1} . \tag{17}$$

where $\left[x_i \right]_{m\times 1}$ and $\left[x_{ij} \right]_{m\times n}$ are m-dimensional column vector and the m-by-n matrix, respectively.

Once N is selected, the corresponding optimal coning compensation can be designed by the following procedures: firstly, the constants A_{ij} and C_i are calculated according to equations (12) and (14); secondly, the constant coefficients k_i are solved by equation (17); finally, the optimal coning compensation is obtained by equations (7) and (10). The new generalized optimum algorithm for the coning compensation $\boldsymbol{\beta}_m$ over the minor interval is a composite of the following two recursive computations:

a) Angular rate integral recursive computation:

$$\boldsymbol{\alpha}_l = \boldsymbol{\alpha}_{l-1} + \Delta\boldsymbol{\alpha}_l , \tag{18a}$$

$$\boldsymbol{\alpha}_m = \boldsymbol{\alpha}_l \ \ (t_l = t_m), \ \ \boldsymbol{\alpha}_l = 0 \ \ (t_l = t_{m-1}). \tag{18b}$$

b) Coning compensation recursive computation:

$$\boldsymbol{\beta}_l = \boldsymbol{\beta}_{l-1} + \Delta\boldsymbol{\beta}_l , \quad \Delta\boldsymbol{\beta}_l = \frac{1}{2}\left(\boldsymbol{\alpha}_{l-1} + \sum_{i=1}^{N-1} k_i \Delta\boldsymbol{\alpha}_{l-i} \right) \times \Delta\boldsymbol{\alpha}_l , \tag{19a}$$

$$\boldsymbol{\beta}_m = \boldsymbol{\beta}_l \ \ (t_l = t_m), \ \ \boldsymbol{\beta}_l = 0 \ \ (t_l = t_{m-1}). \tag{19b}$$

3. ATTITUDE ERROR ANALYSIS

In a quaternion-based attitude updating algorithm to which the new generalized coning compensation proposed in Section 2 is applied, the attitude errors mainly depend on the updating rate of the coning compensation and the number of gyro incremental angle samples for the coning compensation. In order to analyze the attitude error caused by the new generalized coning compensation, the following assumptions are made:

(1) The system body is undergoing a pure coning movement, defined by the angular rate in equation (8);

(2) The navigation coordinate coincides with the body coordinate at the initial moment of navigation, and the initial attitude quaternion by the initial alignment is error-free, namely, $\hat{\mathbf{Q}}^{N(0)}_{B(0)} = \mathbf{Q}^{N(0)}_{B(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$;

(3) The navigation coordinate is stationary, ignoring the navigation coordinate angular motion;

(4) The attitude updating interval, the rotation vector updating interval and the coning compensation updating interval are equal, namely, $T_l = T_m = T_n$.

Proposition 1. Under the assumptions (1)-(4), the relationship between the attitude error and the updating rate of the new generalized coning compensation algorithm satisfies the following equations:

$$e_\gamma = 0, \tag{20a}$$

$$e_\theta = 0, \tag{20b}$$

$$e_\psi = \frac{N!}{2^{N+1} \times \prod_{k=1}^{N+1} (2k-1)} ab \left(\Omega T_l \right)^{2N+1} , \tag{20c}$$

where e_γ, e_θ and e_ψ are the errors in the angles of roll, pitch and yaw, respectively, normalized by the total time of navigation to eliminate its impact; N is the number of gyro incremental angle samples for the calculation of the coning compensation.

Proof: According to the new generalized coning compensation algorithm (equations (18) - (19)) proposed in Section 2 and the assumptions (1)-(4), the true and computed values of the coning compensation term over the updating interval T_l can be expressed as follows:

$$\boldsymbol{\beta}_n = \begin{bmatrix} 0 & 0 & \frac{ab}{2}\left(\Omega T_l - \sin\Omega T_l \right) \end{bmatrix}^T , \tag{21a}$$

$$\hat{\boldsymbol{\beta}}_n = \begin{bmatrix} 0 & 0 & ab \sum_{i=1}^{\infty} (-1)^{i+1} \sum_{j=1}^{N-1} A_{ij} k_j \lambda_l^{2i+1} \end{bmatrix}^T . \tag{21b}$$

Then according to equation (4) and equation (18), the true and computed values of rotation vector over the updating interval T_l can be obtained as follows:

$$\boldsymbol{\Phi}_n = \Delta\boldsymbol{\alpha}_n + \boldsymbol{\beta}_n . \tag{22a}$$

$$\hat{\boldsymbol{\Phi}}_n = \Delta\boldsymbol{\alpha}_n + \hat{\boldsymbol{\beta}}_n . \tag{22b}$$

So if the true and computed values of rotation vector over the updating interval T are expressed by

$$\boldsymbol{\Phi}_n = \begin{bmatrix} \Phi_{n,x} & \Phi_{n,y} & \Phi_{n,z} \end{bmatrix}^T , \tag{23a}$$

$$\hat{\mathbf{\Phi}}_n = \begin{bmatrix} \hat{\Phi}_{n,x} & \hat{\Phi}_{n,y} & \hat{\Phi}_{n,z} \end{bmatrix}^T, \tag{23b}$$

respectively, then according to equations (21) and (22), the computed error of rotation vector over the m-th minor interval can be obtained as follows:

$$\begin{aligned} \Delta\mathbf{\Phi}_n &= \mathbf{\Phi}_n - \hat{\mathbf{\Phi}}_n \\ &= \begin{bmatrix} \Phi_{n,x} - \hat{\Phi}_{n,x} & \Phi_{n,y} - \hat{\Phi}_{n,y} & \Phi_{n,z} - \hat{\Phi}_{n,z} \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 0 & \Delta\Phi_{n,z} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & \boldsymbol{\beta}_n - \hat{\boldsymbol{\beta}}_n \end{bmatrix}^T. \end{aligned} \tag{24}$$

Equation (24) shows that the computed error of rotation vector only exists in the z-axis direction. So for analysis conveniently, it is assumed that $\Phi_{n,x} = \Phi_{n,y} = 0$ and $\hat{\Phi}_{n,x} = \hat{\Phi}_{n,y} = 0$. Then according to equation (2), the true and computed values of the attitude updating quaternion over the updating interval T are

$$\mathbf{q}_{B_{(n)}}^{B_{(n-1)}} = \begin{bmatrix} \cos\frac{\Phi_{n,z}}{2} & 0 & 0 & \sin\frac{\Phi_{n,z}}{2} \end{bmatrix}^T, \tag{25a}$$

$$\hat{\mathbf{q}}_{B_{(n)}}^{B_{(n-1)}} = \begin{bmatrix} \cos\frac{\hat{\Phi}_{n,z}}{2} & 0 & 0 & \sin\frac{\hat{\Phi}_{n,z}}{2} \end{bmatrix}^T, \tag{25b}$$

respectively. Equations (21) - (25) show that the attitude updating quaternion over the updating interval T_l depends only on the duration of the interval, regardless of the absolute time at which the interval begins.

In addition, because relative to the body coordinate the angular movement of the navigation coordinate is small, the angular movement of the navigation coordinate can be ignored. Namely, the navigation coordinate can be assumed stationary. Thus according to equation (1), the attitude quaternion at time t_n can be written as:

$$\begin{aligned} \mathbf{Q}_{B_{(n)}}^{N_{(n)}} &= \mathbf{q}_{N_{(n-1)}}^{N_{(n)}} \otimes \mathbf{Q}_{B_{(n-1)}}^{N_{(n-1)}} \otimes \mathbf{q}_{B_{(n)}}^{B_{(n-1)}} \\ &\approx \mathbf{Q}_{B_{(n-1)}}^{N_{(n-1)}} \otimes \mathbf{q}_{B_{(n)}}^{B_{(n-1)}} = \mathbf{Q}_{B_{(n-1)}}^{N_{(n-1)}} \otimes \mathbf{q}(T_l), \end{aligned} \tag{26}$$

where $\mathbf{Q}_{B_{(n-1)}}^{N_{(n-1)}}$ and $\mathbf{Q}_{B_{(n)}}^{N_{(n)}}$ are the attitude quaternion of vehicle at time t_n and t_{n-1}, respectively; $\mathbf{q}_{B_{(n)}}^{B_{(n-1)}} = \mathbf{q}(T_l)$.

Therefore, according to equation (26), the true and computed values of attitude quaternion at different time are

$$\begin{aligned} &\mathbf{Q}_{B_{(1)}}^{N_{(1)}} \approx \mathbf{Q}_{B_{(0)}}^{N_{(0)}} \otimes \mathbf{q}(T_l), \\ &\mathbf{Q}_{B_{(2)}}^{N_{(2)}} \approx \mathbf{Q}_{B_{(1)}}^{N_{(1)}} \otimes \mathbf{q}(T) \approx \mathbf{Q}_{B_{(0)}}^{N_{(0)}} \otimes \underbrace{\left[\mathbf{q}(T_l) \otimes \mathbf{q}(T_l)\right]}_{\text{Cross-Product of 2-}\mathbf{q}(T_l)}, \\ &\cdots\cdots\cdots \\ &\mathbf{Q}_{B_{(n)}}^{N_{(n)}} \approx \mathbf{Q}_{B_{(n-1)}}^{N_{(n-1)}} \otimes \mathbf{q}(T) \approx \mathbf{Q}_{B_{(0)}}^{N_{(0)}} \otimes \underbrace{\left[\mathbf{q}(T_l) \otimes \cdots \otimes \mathbf{q}(T_l)\right]}_{\text{Cross-Product of } n\text{-}\mathbf{q}(T_l)}, \end{aligned} \tag{27a}$$

$$\begin{aligned} &\hat{\mathbf{Q}}_{B_{(1)}}^{N_{(1)}} \approx \hat{\mathbf{Q}}_{B_{(0)}}^{N_{(0)}} \otimes \hat{\mathbf{q}}(T_l), \\ &\hat{\mathbf{Q}}_{B_{(2)}}^{N_{(2)}} \approx \hat{\mathbf{Q}}_{B_{(1)}}^{N_{(1)}} \otimes \hat{\mathbf{q}}(T) \approx \hat{\mathbf{Q}}_{B_{(0)}}^{N_{(0)}} \otimes \underbrace{\left[\hat{\mathbf{q}}(T_l) \otimes \hat{\mathbf{q}}(T_l)\right]}_{\text{Cross-Product of 2-}\hat{\mathbf{q}}(T_l)}, \\ &\cdots\cdots\cdots \\ &\hat{\mathbf{Q}}_{B_{(n)}}^{N_{(n)}} \approx \hat{\mathbf{Q}}_{B_{(n-1)}}^{N_{(n-1)}} \otimes \hat{\mathbf{q}}(T) \approx \hat{\mathbf{Q}}_{B_{(0)}}^{N_{(0)}} \otimes \underbrace{\left[\hat{\mathbf{q}}(T_l) \otimes \cdots \otimes \hat{\mathbf{q}}(T_l)\right]}_{\text{Cross-Product of } n\text{-}\hat{\mathbf{q}}(T_l)}, \end{aligned} \tag{27b}$$

respectively.

Then under the assumption (2), the true and computed values of the attitude quaternion at time t_n are

$$\hat{\mathbf{Q}}_{B_{(n)}}^{N_{(n)}} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T \otimes \underbrace{\left[\hat{\mathbf{q}}(T_l) \otimes \cdots \otimes \hat{\mathbf{q}}(T_l)\right]}_{\text{Cross-Product of } n\text{-}\hat{\mathbf{q}}(T_l)}, \tag{28a}$$

$$\mathbf{Q}_{B_{(n)}}^{N_{(n)}} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T \otimes \underbrace{\left[\mathbf{q}(T_l) \otimes \cdots \otimes \mathbf{q}(T_l)\right]}_{\text{Cross-Product of } n\text{-}\mathbf{q}(T_l)}, \tag{28b}$$

respectively. Furthermore, according to equation (25a), we obtain

$$\underbrace{\left[\mathbf{q}(T_l) \otimes \cdots \otimes \mathbf{q}(T_l)\right]}_{\text{Cross-Product of } n\text{-}\mathbf{q}(T_l)} = \begin{bmatrix} \cos\frac{n\Phi_{n,z}}{2} & 0 & 0 & \sin\frac{n\Phi_{n,z}}{2} \end{bmatrix}^T. \tag{29}$$

Similar to the derivation of equation (29), according to equation (25b), we have

$$\underbrace{\left[\hat{\mathbf{q}}(T_l) \otimes \cdots \otimes \hat{\mathbf{q}}(T_l)\right]}_{\text{Cross-Product of } n\text{-}\hat{\mathbf{q}}(T_l)} = \begin{bmatrix} \cos\frac{n\hat{\Phi}_{n,z}}{2} & 0 & 0 & \sin\frac{n\hat{\Phi}_{n,z}}{2} \end{bmatrix}^T. \tag{30}$$

Substituting equations (29) and (30) into equation (28), the true and computed values of the attitude quaternion at time t_n can be further simplified in the following:

$$\mathbf{Q}_{B_{(n)}}^{N_{(n)}} = \begin{bmatrix} \cos\frac{n\Phi_{n,z}}{2} & 0 & 0 & \sin\frac{n\Phi_{n,z}}{2} \end{bmatrix}^T, \tag{31a}$$

$$\hat{\mathbf{Q}}_{B_{(n)}}^{N_{(n)}} = \begin{bmatrix} \cos\frac{n\hat{\Phi}_{n,z}}{2} & 0 & 0 & \sin\frac{n\hat{\Phi}_{n,z}}{2} \end{bmatrix}^T. \tag{31b}$$

According to the relationship between attitude matrix $\mathbf{C}_n^b$ and attitude quaternion $\mathbf{Q}$, the attitude matrix is

$$\mathbf{C}_n^b = \begin{bmatrix} Q_1^2 + Q_0^2 - Q_3^2 - Q_2^2 & 2(Q_1Q_2 + Q_0Q_3) & 2(Q_1Q_3 - Q_0Q_2) \\ 2(Q_1Q_2 - Q_0Q_3) & Q_2^2 - Q_3^2 + Q_0^2 - Q_1^2 & 2(Q_2Q_3 + Q_0Q_1) \\ 2(Q_1Q_3 + Q_0Q_2) & 2(Q_2Q_3 - Q_0Q_1) & Q_3^2 - Q_2^2 - Q_1^2 + Q_0^2 \end{bmatrix} \quad (32)$$

where $Q_i (i = 0,1,2,3)$ are the components of quaternion $\mathbf{Q}$. Without loss of generality, if we select the east-north-up geographic coordinate as the navigation coordinate, then the attitude matrix $\mathbf{C}_n^b$ can be expressed in the form of attitude angles as follows:

$$\mathbf{C}_n^b = \begin{bmatrix} \cos\psi\cos\gamma + \sin\psi\sin\theta\sin\gamma & -\sin\psi\cos\gamma + \cos\psi\sin\theta\sin\gamma & -\cos\theta\sin\gamma \\ \sin\psi\cos\theta & \cos\psi\cos\theta & \sin\theta \\ \cos\psi\sin\gamma - \sin\psi\sin\theta\cos\gamma & -\sin\psi\sin\gamma - \cos\psi\sin\theta\cos\gamma & \cos\theta\cos\gamma \end{bmatrix} \quad (33)$$

where γ, θ and ψ are the angles of roll, pitch and yaw at time t_n, respectively.

Then, according equations (32) and (33), the principal value of the attitude angles can be calculated as follows:

$$\gamma_p = \tan^{-1}\left[-\frac{2(Q_1Q_3 - Q_0Q_2)}{Q_0^2 - Q_1^2 - Q_2^2 + Q_3^2}\right], \quad (34a)$$

$$\theta_p = \sin^{-1}\left[2\left(Q_2Q_3 + Q_0Q_1\right)\right], \quad (34b)$$

$$\psi_p = \tan^{-1}\left[\frac{2(Q_1Q_2 - Q_0Q_3)}{Q_0^2 - Q_1^2 + Q_2^2 - Q_3^2}\right]. \quad (34c)$$

If we define the true and computed values of the attitude quaternion at time t_n in the form of components as follows:

$$\mathbf{Q}_{B(n)}^{N(n)} = \left[Q_0(t_n) \quad Q_1(t_n) \quad Q_2(t_n) \quad Q_3(t_n)\right]^T, \quad (35a)$$

$$\hat{\mathbf{Q}}_{B(n)}^{N(n)} = \left[\hat{Q}_0(t_n) \quad \hat{Q}_1(t_n) \quad \hat{Q}_2(t_n) \quad \hat{Q}_3(t_n)\right]^T, \quad (35b)$$

then according to equations (31a) and (35a), the true value of the related items in equation (34) can be expressed as follows:

$$-\frac{2\left[Q_1(t_n)Q_3(t_n) - Q_0(t_n)Q_2(t_n)\right]}{Q_0^2(t_n) - Q_1^2(t_n) - Q_2^2(t_n) + Q_3^2(t_n)} = 0, \quad (36a)$$

$$2\left[Q_2(t_n)Q_3(t_n) + Q_0(t_n)Q_1(t_n)\right] = 0, \quad (36b)$$

$$\frac{2\left[Q_1(t_n)Q_2(t_n) - Q_0(t_n)Q_3(t_n)\right]}{Q_0^2(t_n) - Q_1^2(t_n) + Q_2^2(t_n) - Q_3^2(t_n)} = -\tan\left(n\Phi_{n,z}\right). \quad (36c)$$

Similarly, according to equations (31b) and (35b), the computed values of the related items in equation (34) can be expressed as follows:

$$-\frac{2\left[\hat{Q}_1(t_n)\hat{Q}_3(t_n) - \hat{Q}_0(t_n)\hat{Q}_2(t_n)\right]}{\hat{Q}_0^2(t_n) - \hat{Q}_1^2(t_n) - \hat{Q}_2^2(t_n) + \hat{Q}_3^2(t_n)} = 0, \quad (37a)$$

$$2\left[\hat{Q}_2(t_n)\hat{Q}_3(t_n) + \hat{Q}_0(t_n)\hat{Q}_1(t_n)\right] = 0, \quad (37b)$$

$$\frac{2\left[\hat{Q}_1(t_n)\hat{Q}_2(t_n) - \hat{Q}_0(t_n)\hat{Q}_3(t_n)\right]}{\hat{Q}_0^2(t_n) - \hat{Q}_1^2(t_n) + \hat{Q}_2^2(t_n) - \hat{Q}_3^2(t_n)} = -\tan\left(n\hat{\Phi}_{n,z}\right). \quad (37c)$$

Thus according to equations (34), (36) and (37), the true and computed principal values of the attitude angles can be obtained as follows:

$$\gamma_p = 0, \quad (38a)$$

$$\theta_p = 0, \quad (38b)$$

$$\psi_p = \tan^{-1}\left[\tan\left(n\Phi_{n,z}\right)\right] = n\Phi_{n,z}; \quad (38c)$$

$$\hat{\gamma}_p = 0, \quad (39a)$$

$$\hat{\theta}_p = 0, \quad (39b)$$

$$\hat{\psi}_p = \tan^{-1}\left[\tan\left(n\hat{\Phi}_{n,z}\right)\right] = n\hat{\Phi}_{n,z}. \quad (39c)$$

In order to eliminate the impact of navigation time on the attitude angle errors, we define the normalized attitude angle errors as follows:

$$e_\gamma \triangleq \left(\gamma_p - \hat{\gamma}_p\right)/n, \quad (40a)$$

$$e_\theta \triangleq \left(\theta_p - \hat{\theta}_p\right)/n, \quad (40b)$$

$$e_\psi \triangleq \left(\psi_p - \hat{\psi}_p\right)/n. \quad (40c)$$

Then according to equations (21), (22), (23), (39) and (40), we obtain equation (20). □

Similarly, if we assume that the vehicle is undergoing a pure coning movement defined by the angular rate

$$\boldsymbol{\omega}_{ib}^b(t) = \left[a\Omega\cos(\Omega t) \quad 0 \quad b\Omega\sin(\Omega t)\right]^T,$$

then the normalized attitude angle errors are

$$e_\gamma = 0, \tag{41a}$$

$$e_\theta = \frac{N!}{2^{N+1} \times \prod_{k=1}^{N+1}(2k-1)} ab\left(\Omega T_l\right)^{2N+1}, \tag{41b}$$

$$e_\psi = 0. \tag{41c}$$

If the vehicle is undergoing a pure coning movement defined by the angular rate

$$\boldsymbol{\omega}_{ib}^b(t) = \begin{bmatrix} 0 & a\Omega\cos(\Omega t) & b\omega\sin(\Omega t) \end{bmatrix}^T,$$

then the normalized attitude angle errors are

$$e_\gamma = \frac{N!}{2^{N+1} \times \prod_{k=1}^{N+1}(2k-1)} ab\left(\Omega T_l\right)^{2N+1}, \tag{42a}$$

$$e_\theta = 0, \tag{42b}$$

$$e_\psi = 0. \tag{42c}$$

Equations (20), (41) and (42) show that the attitude angle errors $(e_\gamma, e_\theta, e_\psi)$ are related to the coning compensation updating interval T_l and the number N of gyro incremental angle samples for the coning compensation, and, for a given N, the attitude error can be reduced by increasing the updating rate of coning compensation, or, equivalently, decreasing the coning compensation updating interval T_l. However, the effects of the number of gyro incremental angle samples on the attitude error are not so clear from those equations. In the following we give an estimation of the effect based on equation (20c) without loss of generality.

According to Stirling's formula (Romik 2000), the estimation of factorial can be approximately expressed as,

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}. \tag{43}$$

Or according to the improvement of Stirling's formula, it can be more accurately expressed as (Li 2006; Peng 2006),

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}}, \quad \theta \in (\theta_1, \theta_2), \quad \theta_1 = 1 - \frac{1}{30n^2}, \quad \theta_2 = 1 - \frac{1}{30n^2} + \frac{1}{60n^4}. \tag{44}$$

In addition, we know that the following relationship exists,

$$\prod_{k=1}^{n+1}(2k-1) = \frac{(2n+1)!}{2^n \times n!}. \tag{45}$$

Based on equation (44), we have the following approximation:

$$\frac{(N!)^2}{(2N)!} \sim \frac{2N\pi\left(\frac{N}{e}\right)^{2N} e^{\frac{\theta_1}{6N}}}{\sqrt{4N\pi}\left(\frac{2N}{e}\right)^{2N} e^{\frac{\theta_2}{24N}}} = \frac{\sqrt{N\pi}}{4^N} e^{\frac{\theta_1}{6N} - \frac{\theta_2}{24N}} \approx \frac{\sqrt{N\pi}}{4^N} e^{\frac{1}{8N}} \tag{46}$$

Combining with equations (45) and (46), equation (20c) can be simplified as,

$$\begin{aligned} e_\psi &\sim \frac{1}{2\times(2N+1)} \frac{(N!)^2}{(2N)!} ab\Omega^{2N+1} T_l^{2N+1} \\ &\sim \frac{\sqrt{N\pi}}{(2N+1)\times 2^{2N+1}} e^{\frac{1}{8N}} ab\Omega^{2N+1} T_l^{2N+1} \\ &= ab\lambda(N) \end{aligned} \tag{47}$$

where

$$\lambda(N) \triangleq \frac{\sqrt{N\pi}}{2N+1}\left(\frac{\Omega T_l}{2}\right)^{2N+1} e^{\frac{1}{8N}}. \tag{48}$$

Taking a numerical analysis for $\lambda(N)$ yields the results that, when $\Omega T_l < 1.93$, the attitude angle error e_ψ will decrease with the increase of N, the number of gyro incremental angle samples for the coning compensation; and, when $1.94 \le \Omega T_l \le 2.03$, there is a minimal error e_ψ for a specific value of N; and when $\Omega T_l > 2.03$, the attitude angle error e_ψ will increase with the increase of N. Fig. 2 and Tab. 1 demonstrate the phenomena mentioned above.

These results reveal the fact that the selection of N has more complex effects on the attitude error than the selection of T_l. When a vehicle moves with "normal" dynamics (characterized by a small or moderate coning movement frequency Ω), both the increase of the updating rate of the coning compensation and the increase of the number of gyro incremental angle samples benefit the reduction of the attitude error. When a vehicle is in a high dynamic motion described by a larger frequency Ω, however, in consideration of the technical limitations on the updating rate of the coning compensation (e.g., due to the limited computation speed of the navigation computer), the choice of the sampling number of gyro incremental angles must be made carefully.

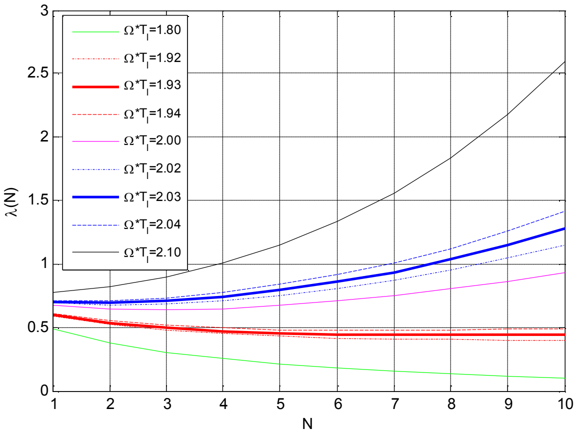

Fig. 2. The relationship between $\lambda(N)$ and N for different values of ΩT_I

for quantitative analysis of the new coning compensation algorithm under typical coning movement conditions.

Quantitative analysis reveals that the attitude errors caused by the vehicle coning movement may be decreased by increasing the updating speed of coning compensation (indirectly, increasing the updating rate of rotation vector and attitude) or increasing the number of gyro incremental angle samples in each sampling interval for calculating the coning compensation. But the selection of the number of gyro incremental angle samples has more complex effects on the attitude error than the selection of the updating speed of coning compensation. When a vehicle moves with high dynamics (characterized by a larger coning movement frequency), in consideration of the technical limitations on the updating rate of the coning compensation (e.g., due to the limited computation speed of the navigation computer), the choice of the sampling number of gyro incremental angles and the updating rate of the coning compensation must be made carefully, in order to improve the accuracy of the related attitude updating algorithms.

Tab. 1 The relationship between $\lambda(N)$ and N for different values of ΩT_I

ΩT_I \ N	1	2	3	4	5	6	7	8	9	10
1.80	0.4881	0.3801	0.3052	0.2516	0.2112	0.1797	0.1544	0.1337	0.1164	0.1019
1.92	0.5923	0.5249	0.4795	0.4497	0.4296	0.4159	0.4065	0.4005	0.3969	0.3953
1.93	0.6016	0.5387	0.4973	0.4713	0.4549	0.4449	0.4395	0.4375	0.4381	0.4409
1.94	0.6110	0.5528	0.5156	0.4937	0.4815	0.4758	**0.4749**	0.4776	0.4833	0.4914
2.00	0.6695	0.6437	0.6381	0.6494	0.6731	0.7070	0.7500	0.8016	0.8620	0.9316
2.02	0.6898	0.6766	0.6841	0.7102	0.7510	0.8046	0.8707	0.9494	1.0414	1.1481
2.03	0.7001	**0.6935**	0.7082	0.7425	0.7929	0.8580	0.9376	1.0325	1.1439	1.2735
2.04	0.7105	0.7107	0.7330	0.7761	0.8370	0.9146	1.0094	1.1225	1.2558	1.4120
2.10	0.7750	0.8216	0.8979	1.0074	1.1513	1.3332	1.5591	1.8373	2.1783	2.5954

4. CONCLUSIONS

In this paper, a new generalized coning compensation algorithm is presented based on dividing the minor coning compensation updating interval into a number of sub-minor intervals. Unlike other existing algorithms, the updating rate of the proposed algorithm is independent of the number of gyro incremental angles for calculating the coning compensation. Thus, when the output sampling rate of inertial sensors remains constant, this algorithm allows increasing the updating speed, yet with more numbers of gyro incremental angle in order to improve the accuracy of the related attitude updating algorithms. The relationship between the attitude errors and the coning compensation updating rate as well as the number of gyro incremental angle samples for calculating the coning compensation is deduced

ACKNOWLEDGEMENT

This work is supported partially sponsored by the National Natural Science Foundation of China under Grant No.61070003 and the Natural Science Foundation of Zhejiang Province of China under Grant No.R1090052.

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