Transforming non-autonomous systems to p-normal forms: an approach to reduce the computation loads of cyber-physical Systems

Duan Zhang, Xiongxiong He

Zhejiang University of Technology, No. 288, Liuhe Road, Westlake district 310023, Hangzhou, Zhejiang, China (dzhang@zjut.edu.cn)

Abstract: A key purpose of studying the problem of equivalence between two control systems is to transform a complex nonlinear system to a simple system whose controller is easy to realize real-time response in cyber-physical systems. Using the theories of singular distributions, necessary and sufficient conditions under which single-input non-autonomous systems are feedback equivalent to non-autonomous low-triangular systems. Moreover, we presented a necessary and sufficient condition under which single-input non-autonomous systems are feedback equivalent to non-autonomous p-normal forms, which are special cases of low-triangular systems. Two examples are given to illustrate how to realize those equivalent transformations via state feedback and coordinate transformation.

Keywords: non-autonomous system, low-triangular system, p-normal form, feedback equivalence, coordinates change.

1. INTRODUCTION

Cyber-physical systems (CPS) are integrations of computation and physical processes [1, 2]. Distributed sensors, embedded computers and actuators in the CPS networks monitor and control the physical processes [3]. So there is a need for methods to reduce the computation loads of controllers working real-time in the embedded computers [4]. By using the equivalence between two control systems it is possible to transform a complex nonlinear system into a simple system whose controller is easy to realize real-time response. Differential geometry has universally regarded as a successful approach to study the equivalence problem [5]. Low-triangular forms are nonlinear systems attracting great attention. Backstepping, a technique for designing a controller for a nonlinear system, was advanced in the 1990s and now it is applied to low-triangular forms in general. If a nonlinear system is equivalent to a low-triangular form by state feedbacks and coordinate transformations, it is possible to design a controller by the backstepping technique.

For the single-input case, Celikovsky and Nijmeijer provided necessary and sufficient conditions for the equivalence of nonlinear systems to low-triangular forms in [6]. The p-normal forms are special nonlinear systems. Cheng and Lin provided necessary and sufficient conditions for nonlinear systems to be equivalent to non-autonomous p-normal forms by using coordinate transformations and state feedback of the type $u=\alpha(\xi)+v$ in [7]. Subsequently Respondek dealt with this problem using coordinate transformations and state feedback of the type $u=\alpha(\xi)+\beta(\xi)v$ in [8]. On the other hand, a series of exciting results on the issue of designing controller for the p-normal forms, not only the autonomous case but also the non-autonomous case, have been obtained [9, 10].

In this paper, first we provide the necessary and sufficient conditions for nonlinear systems to be equivalent to the non-autonomous low-triangular forms via state feedback and coordinate transformation. Second, we provide the necessary and sufficient conditions for nonlinear systems to be equivalent to the non-autonomous p-normal forms via state feedback and coordinate transformation. Those necessary and sufficient conditions are more convenient to be check than the conditions given in [6, 8], see Remark 1 and 2.

The rest of the paper is organized as follows. Section 2 illuminates the system equivalence problems discussed in this paper. The main results are formulated in Section 3 and Section 4. We conclude the paper in Section 5.

2. PROBLEM FORMULATION

Consider the non-autonomous nonlinear system

$$\begin{aligned}\dot{\xi}_1 &= F_1(t,\xi)+G_1(t,\xi)u\\ \dot{\xi}_2 &= F_2(t,\xi)+G_2(t,\xi)u\\ &\cdots\\ \dot{\xi}_n &= F_n(t,\xi)+G_n(t,\xi)u\end{aligned} \tag{1}$$

where $\xi=(\xi_1,\xi_2,\cdots,\xi_n)^{\mathrm{T}}$, $u\in\mathrm{R}$ is input, F_i and F_i are all smooth functions. The first question we pose in this note is whether the nonlinear systems is equivalent to the non-autonomous low-triangular form via state feedback and coordinate transformations. The state feedback considered here is

$$u=\alpha(t,\xi)+\beta(t,\xi)v, \tag{2}$$

where $\alpha:\mathrm{R}\times\mathrm{R}^n\to\mathrm{R}$ and $\beta:\mathrm{R}\times\mathrm{R}^n\to\mathrm{R}(i,j=1,2,\cdots,m)$ are both smooth functions, and v are the new input under feedbacks. The smooth coordinate transformations are expressed as

$$x = T(\xi) = \left(T_1(t,\xi),\cdots,T_n(t,\xi)\right)^{\mathrm{T}}. \tag{3}$$

The non-autonomous low-triangular form is expressed as

$$\begin{aligned}
\dot{x}_1 &= f_1(t,x_2,x_1)\\
&\vdots\\
\dot{x}_{n-1} &= f_{n-1}(t,x_n,\cdots,x_1)\\
\dot{x}_n &= f_n(t,x_1,x_2,\cdots,x_n) + g_n(t,x_1,x_2,\cdots,x_n)v
\end{aligned} \tag{4}$$

where $x_1,\cdots,x_n$ are state variables, v is scalar input, $f_i : \mathrm{R}\times\mathrm{R}^{i+1} \to \mathrm{R}(i=1,\cdots,n-1)$ are all smooth functions satisfying $\partial f_i / \partial x_{i+1} \overset{\text{a.e.}}{\neq} 0\,(i=1,\cdots,n-1)$ in a neighbourhood of the original point, the symbol *a.e.* means almost everywhere, that is, $\partial f_i/\partial x_{i+1}$ does not equal 0 almost everywhere or it equals 0 only in a null set, which is a set of measure zero.

The second question we pose in this note is whether the nonlinear system is equivalent to, via state feedback (3) and coordinate transformations (4), the following non-autonomous p-normal form

$$\begin{aligned}
\dot{y}_1 &= y_2^{\,p_1} + \sum_{i=0}^{p_1-1}\varphi_1^i(t,y_1)y_2^{\,i}\\
&\vdots\\
\dot{y}_{n-1} &= y_n^{\,p_{n-1}} + \sum_{i=0}^{p_{n-1}-1}\varphi_{n-1}^i(t,y_1,\cdots,y_{n-1})y_n^{\,i}\\
\dot{y}_n &= v
\end{aligned} \tag{5}$$

where φ_j^i are all smooth functions, and $p_1,p_2,\cdots,p_{n-1}$ are positive integer. Supposing that for a given non-autonomous p-normal form, $p_1 = p_2 = \cdots = p_{n-1} = 1$ is hold, it is easy to verify that the system can be exactly linearized by state feedback and coordinate changes.

According to the symbols used in differential geometry, the time-variant vector fields corresponding to System (1) are

$$\begin{aligned}
F &= \frac{\partial}{\partial t} + F_1\frac{\partial}{\partial \xi_1} + \cdots + F_n\frac{\partial}{\partial \xi_n},\\
G &= G_1\frac{\partial}{\partial \xi_1} + \cdots + G_n\frac{\partial}{\partial \xi_n};
\end{aligned} \tag{6}$$

The vector fields corresponding to the non-autonomous low-triangular form are

$$\begin{aligned}
f &= \frac{\partial}{\partial t} + f_1\frac{\partial}{\partial x_1} + f_2\frac{\partial}{\partial x_2} + \cdots\cdots + f_n\frac{\partial}{\partial x_n},\\
g &= g_n\frac{\partial}{\partial x_n}.
\end{aligned} \tag{7}$$

And the vector fields corresponding to the non-autonomous p-normal form are

$$\begin{aligned}
f &= \frac{\partial}{\partial t} + f_{p,1}\frac{\partial}{\partial y_1} + f_{p,2}\frac{\partial}{\partial y_2} + \cdots\cdots + f_{p,n-1}\frac{\partial}{\partial y_{n-1}},\\
g &= \frac{\partial}{\partial y_n}.
\end{aligned} \tag{8}$$

where $f_{p,j} = y_{j+1}^{\,p_j} + \sum_{i=0}^{p_j-1}\varphi_j^i(t,y_1,\cdots,y_j)y_{j+1}^{\,i}\,(j=1,2,\cdots,n-1)$.

Let $X = \sum_{i=1}^{n} X^i\,\partial/\partial x_i$ and $Y = \sum_{i=1}^{n} Y^i\,\partial/\partial x_i$ be two vector fields, the Lie bracket of X and Y is a third vector field defined by

$$\mathrm{ad}_X Y = \mathrm{ad}_X^1 Y = [X,Y] = \sum_{j=1}^{m}\sum_{i=1}^{m}\left(X^i\frac{\partial Y^j}{\partial x_i} - Y^i\frac{\partial X^j}{\partial x_i}\right)\frac{\partial}{\partial x_j}. \tag{9}$$

Let $\mathrm{ad}_X^0 Y = Y$ and $\mathrm{ad}_X Y = \mathrm{ad}_X^1 Y$. Then $\mathrm{ad}_X^i Y = \mathrm{ad}_X(\mathrm{ad}_X^{i-1}Y)$ holds for any $i=1,2,\cdots$. A distribution generated by the vector fields $X_1,\cdots,X_h$ is written as $\mathrm{span}\{X_1,\cdots,X_h\}$. Let D^1 and D^2 be two smooth distributions. Then $D^1\oplus D^2$ expresses the distribution generated by all the vector fields belonging to D^1 or D^2. A distribution is called nonsingular in U, which is an open subset of R^n, if the dimension of distribution $\dim(D)$ is fixed in U; otherwise, it is called singular. The nonsingular distribution $\mathrm{span}\{X_1,\cdots,X_h\}$ is involutive if and only if $\mathrm{ad}_{X_i}X_j$ belongs to this distribution for any $i,j=1,\cdots,h$. Let D be a singular distribution. A nonsingular distribution is written as $\overline{D}$ if $\overline{D} \overset{\text{a.e.}}{=} D$ is satisfied. For vector field X as defined above, the coordinate transformations induces a map

$$T_*(X)\big|_y = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial T_i}{\partial x_j}X_j\bigg|_x\right)\frac{\partial}{\partial y_i} \tag{10}$$

where T_* is called push forward map [11].

3. TRANSFORMING A NONLINEAR SYSTEM TO A NON-AUTONOMOUS LOW-TRIANGULAR FORM

In this section, we will provide the necessary and sufficient conditions for nonlinear systems to be equivalent to the non-autonomous low-triangular forms. In order to formulate the conditions we will need the following lemmas [11].

Lemma 1. Let $(z,x) = (z_1,z_2,\cdots,z_m,x_1,x_2,\cdots,x_n)$ be the coordinates in space $\mathrm{R}^m\times\mathrm{R}^n$. If vector field

$$X = a_1(z,x)\frac{\partial}{\partial x_1} + a_2(z,x)\frac{\partial}{\partial x_2} + \cdots + a_n(z,x)\frac{\partial}{\partial x_n}, \tag{11}$$

is identically one dimensional, then there exist partial coordinate transformations $\left(z, y = y(z,x)\right)$ such that

$$X = \frac{\partial}{\partial y_1}. \tag{12}$$

Lemma 2. Let

$$X^i = a_1^i(z,x)\frac{\partial}{\partial x_1} + a_2^i(z,x)\frac{\partial}{\partial x_2} + \cdots + a_n^i(z,x)\frac{\partial}{\partial x_n}, \quad i = 1,2,\cdots,n \tag{13}$$

be vector fields. Suppose that the distributions

$$D^i = \text{span}\{X^1, X^2, \cdots, X^i\}, \quad i = 1,2,\cdots,n. \tag{14}$$

are identically i dimensional, then there exist partial coordinate transformations $(z, y = y(z,x))$ such that

$$D^i = \text{span}\left\{\frac{\partial}{\partial y_n}, \frac{\partial}{\partial y_{n-1}}, \cdots, \frac{\partial}{\partial y_{n-i+1}}\right\}, \quad i = 1,2,\cdots,n. \tag{15}$$

Theorem 1. System (1) can be converted into the non-autonomous low-triangular form via feedback (2) and coordinate changes (3) if and only if it satisfies the following two conditions.

(i) the distributions

$$\begin{aligned} &D^1 = \text{span}\{G\}, \\ &D^2 = D^1 \oplus \text{span}\{\text{ad}_F G^1 | G^1 \in \bar{D}^1\}, \\ &\cdots, \\ &D^i = D^{i-1} \oplus \text{span}\{\text{ad}_F G^{i-1} | G^{i-1} \in \bar{D}^{i-1}\}, \\ &i = 1,2,\cdots,n. \end{aligned} \tag{16}$$

satisfy $\dim(D^i) \overset{a.e.}{=} i$;

(ii) the nonsingular distributions $\bar{D}^i (i = 1,\cdots,\text{n})$ satisfy

$$\bar{D}^n \supset \cdots \supset \bar{D}^1. \tag{17}$$

Proof: (Necessity) Suppose System (1) can be converted into the non-autonomous low-triangular form via feedbacks (2) and coordinate changes (3). Then System (1) can be converted into the latter only via coordinate changes $x = T(t,\xi)$. Hence we must verify that conditions (i, ii) hold for System (4).

According to Eq. (16), the following equality holds.

$$\bar{D}^1 = \overline{\text{span}\{g\}} = \text{span}\left\{\frac{\partial}{\partial x_n}\right\}. \tag{18}$$

The low-triangular form of System (4) implies that for every $g^1 \in \bar{D}^1$ we have

$$\text{ad}_f g^1 \in \text{span}\left\{\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}\right\} = \bar{D}^2, \ \text{ad}_f g^1 \notin \bar{D}^1. \tag{19}$$

The general step follows an inductive argument j. Assuming that $g^{i-1} \in \bar{D}^{i-1}$,

$$\begin{aligned} &\text{ad}_f g^{i-1} \in \text{span}\left\{\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \cdots, \frac{\partial}{\partial x_{n-i+1}}\right\} = \bar{D}^i, \\ &\text{ad}_f g^{i-1} \notin \bar{D}^{i-1}, i = 2,3,\cdots,n. \end{aligned} \tag{20}$$

Thus, we have shown that the conditions (i, ii) hold for System (4).

(Sufficiency) It follows from condition (ii) that there exist n nonsingular vector fields $X^1, X^2, \cdots, X^n$ such that

$$\bar{D}^i = \text{span}\{X_n, X_{n-1}, \cdots, X_{n-i+1}\}, \quad i = 1,2,\cdots,n. \tag{21}$$

Under proper coordinate changes, the equality

$$\bar{D}^i = \text{span}\left\{\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \cdots, \frac{\partial}{\partial x_{n-i+1}}\right\} \tag{22}$$

holds in a neighbourhood of the original point. According to condition (i), we obtain

$$G^1 = d_n^1(t,x)\frac{\partial}{\partial x_n}, \tag{23}$$

and

$$\text{ad}_F G^1 = d_{n-1}^2(t,x)\frac{\partial}{\partial x_{n-1}} + d_n^2(t,x)\frac{\partial}{\partial x_n}, \tag{24}$$

under the coordinates (t,x). From Eq. (24), it is clear that

$$\frac{\partial F_i}{\partial x_n} = 0, \ \forall (t,x) \in N_0 \ \forall i \in \{1,2,\cdots,n-2\}. \tag{25}$$

Furthermore, in general, condition (i) implies that

$$\begin{aligned} G^{j-1} = {} & d_{n-j}^{j+1}(t,x)\frac{\partial}{\partial x_{n-j}} + d_{n-j+1}^{j+1}(t,x)\frac{\partial}{\partial x_{n-j+1}} + \cdots \\ & + d_n^{j+1}(t,x)\frac{\partial}{\partial x_n}, \end{aligned} \tag{26}$$

and

$$\begin{aligned} ad_F^{j+1} G^{j-1} = {} & d_{n-j}^{j+2}(t,x)\frac{\partial}{\partial x_{n-j}} + d_{n-j+1}^{j+2}(t,x)\frac{\partial}{\partial x_{n-j+1}} + \cdots \\ & + d_n^{j+2}(t,x)\frac{\partial}{\partial x_n}, \end{aligned} \tag{27}$$

for $j = 2,\cdots,n$. It means that

$$\frac{\partial F_i}{\partial x_j} = 0 \ \ \forall (t,x) \in N_0, \forall j - i > 1. \tag{28}$$

Together with Eqs. (27) and (28), System (1) is in the form of System (4) under proper coordinates.

Remark 1. For the autonomous case, Ref. [6] provided necessary and sufficient conditions for nonlinear systems to be equivalent to the low-triangular forms. For checking those conditions, we have to select the following vector fields as $G^1 = G, G^2 = \mathrm{ad}_F G^1, \cdots, G^{n-1} = \mathrm{ad}_F G^{n-2}$. So condition (i) in Theorem 1 provides a possibility to simplify the computational processes of Lie brackets by choosing especial vector fields.

The proof of Theorem 1 is constructive. The following example is given to illustrate how to realize the equivalent transformations mentioned in Theorem 1 by the method implied in the proof.

Example 1. Consider the following 3 dimensional nonlinear system.

$$\begin{aligned} \dot{\xi}_1 &= \xi_3^4 + \xi_3^2 + \xi_2^3 + \xi_2 \cos t + 4\xi_1 \cos t + \xi_1 + (\xi_2 - \xi_3^2)u \\ \dot{\xi}_2 &= \xi_3^4 + \xi_3^2 + \xi_2 \cos t + 2\xi_1\xi_3 + 2\xi_3(\xi_2 - \xi_3^2)u \\ \dot{\xi}_3 &= \xi_1 + (\xi_2 - \xi_3^2)u \end{aligned} \tag{29}$$

Let

$$\begin{aligned} F = {} & \frac{\partial}{\partial t} + (\xi_3^4 + \xi_3^2 + \xi_2^3 + \xi_2 \cos t + 4\xi_1 \cos t + \xi_1)\frac{\partial}{\partial \xi_1} \\ & + (\xi_3^4 + \xi_3^2 + \xi_2 \cos t + 2\xi_1\xi_3)\frac{\partial}{\partial \xi_2} + \xi_3 \frac{\partial}{\partial \xi_3} \end{aligned} \tag{30}$$

and

$$G = (\xi_2 - \xi_3^2)\frac{\partial}{\partial \xi_1} + 2\xi_3(\xi_2 - \xi_3^2)\frac{\partial}{\partial \xi_2} + (\xi_2 - \xi_3^2)\frac{\partial}{\partial \xi_3}. \tag{31}$$

The distribution generated by singular vector field G equals another distribution generated by the following nonsingular vector field almost everywhere.

$$G^1 = \frac{\partial}{\partial \xi_1} + 2\xi_3 \frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial \xi_3} \tag{32}$$

Since computing $\mathrm{ad}_F G^1$ is easier than $\mathrm{ad}_F G$ we compute it.

$$\mathrm{ad}_F G^1 = (4\xi_3^3 + 2\xi_3^2)\frac{\partial}{\partial \xi_1} + (4\xi_3^3 + 2\xi_3^2)\frac{\partial}{\partial \xi_2}. \tag{33}$$

The distribution generated by singular vector field $\mathrm{ad}_F G^1$ equals another distribution generated by the following nonsingular vector field almost everywhere.

$$G^2 = \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \tag{34}$$

Due to $\overline{\mathrm{span}\{\mathrm{ad}_F G^1, G^1\}} = \mathrm{span}\{G^2, G^1\}$, we compute $\mathrm{ad}_F G^2$ instead of $\mathrm{ad}_F\left(\mathrm{ad}_F G^1\right)$.

$$\mathrm{ad}_F G^2 = (\cos t + 3\xi_3^4 - 6\xi_3^2\xi_2 + 3\xi_2^2)\frac{\partial}{\partial \xi_1} + \cos t \frac{\partial}{\partial \xi_2} \tag{35}$$

Note that $\mathrm{ad}_F G^2 - \cos(t)\cdot G^2 = (3\xi_3^4 - 6\xi_3^2\xi_2 + 3\xi_2^2 t)\frac{\partial}{\partial \xi_1}$, it is feasible to choose

$$G^3 = \frac{\partial}{\partial \xi_3}. \tag{36}$$

It is clear that $\overline{\mathrm{span}\{\mathrm{ad}_F G^2, G^2, G^1\}} = \mathrm{span}\{G^3, G^2, G^1\}$ holds. By direct calculation $\mathrm{span}\{G_1\}$, $\mathrm{span}\{G^2, G^1\}$, $\mathrm{span}\{G^3, G^2, G^1\}$ is a nested sequence of nonsingular involutive distributions. It follows from Theorem 1 that System (29) can be converted into the non-autonomous low-triangular form. Suppose the coordinate changes to realize the conversion are $x = (x_1, x_2, x_3) = \left(T_1(t,\xi), T_2(t,\xi), T_3(t,\xi)\right)$, we can choose $T_1(t,\xi)$, $T_2(t,\xi)$ and $T_3(t,\xi)$ by the following equations

$$\begin{aligned} &< \mathrm{d}T_i, G^j >= 0,\ i < j,\ j = 2,3 \\ &< \mathrm{d}T_i, G^j >\neq 0,\ i = j,\ j = 1,2,3. \end{aligned} \tag{37}$$

A solution of Eqs. (37) is

$$\begin{aligned} x_1(t) &= \xi_1(t) - \xi_2(t) - \xi_3(t) + \xi_3^2(t) \\ x_2(t) &= \xi_2(t) - \xi_3^2(t) \\ x_3(t) &= \xi_3(t). \end{aligned} \tag{38}$$

Rewrite the system equations in terms of the new coordinates.

$$\begin{aligned} \dot{x}_1 &= x_2^3 + 4x_1 \cos t \\ \dot{x}_2 &= x_3^4 + x_3^2 + x_2 \cos t \\ \dot{x}_3 &= x_1 + x_2 u. \end{aligned} \tag{39}$$

Now the system is in the form of System (4).

4. TRANSFORMING A NONLINEAR SYSTEM TO A NON-AUTONOMOUS P-NORMAL FORM

In this section, we will provide the necessary and sufficient conditions for nonlinear systems to be equivalent to the non-autonomous p-normal forms. The following lemma is important in singularity theory [12].

Lemma 3. Let $\pi = \pi(z,w)$ be a C^∞ function from $\mathrm{R}\times\mathrm{R}^{i+1}$ to R. Suppose, for a positive integer k, $\frac{\partial^k \pi}{\partial z^k}(0,0) \neq 0$ holds, but $\frac{\partial^i \pi}{\partial z^i}(0,0) = 0$ for all integers $1 \le i \le k-1$. Then there exists a coordinate change $\tilde{z} = \rho(z,w)$ such that

$$\pi(z,w) = \pi\left(\rho^{-1}(z,w), w\right) = \delta \tilde{z}^k + \sum_{i=0}^{k-1} a_i(w)\tilde{z}^i \tag{40}$$

where $a_i(w)$ are smooth functions satisfying $a_i(0)=0$ for $0 \le i \le k-1$, and

$$\delta = \begin{cases} 1 & k = 2i+1,\ i=0,1,\cdots \\ \pm 1 & k = 2i,\ i=1,2,\cdots\ . \end{cases} \tag{41}$$

Theorem 3. System (1) can be converted into the non-autonomous p-normal form via feedback (2) and coordinate changes (3) if and only if it satisfies the following two conditions.

(i) System (1) can be converted into non-autonomous low-triangular form;

(ii) The low-triangular system converted from System (1) satisfies $f_i(0,0)=0 (1 \le i \le n-1)$ and

$$\min_j \left\{ \left. \frac{\partial^j f_i(t,x)}{\partial x_{i+1}^j} \right|_{x=0} \ne 0 \right\} = p_i,\ 1 \le i \le n-1. \tag{42}$$

Proof: (Necessity) Since the non-autonomous p-normal forms are special low-triangular form and System (5) satisfies condition (ii) obviously, it suffices to prove that if a non-autonomous low-triangular form can be converted into a non-autonomous p-normal form then it satisfies condition (ii). We first show that it is only the low-triangular coordinate transformations to help us to convert a low-triangular form to another low-triangular system, then, by calculating selected partial derivatives we check the condition (ii).

For converting System (4) to into System (5), we need a "low-triangular" coordinate transformation

$$\begin{aligned} y_1 &= T_1(t,x_1) \\ &\vdots \\ y_{n-1} &= T_{n-1}(t,x_1,\cdots,x_{n-1}) \\ y_n &= T_n(t,x_1,\cdots,x_n). \end{aligned} \tag{43}$$

The inverse transformation is also a "low-triangular" coordinate transformation

$$\begin{aligned} x_1 &= S_1(t,y_1) \\ &\vdots \\ x_{n-1} &= S_{n-1}(t,y_1,\cdots,y_{n-1}) \\ x_n &= S_n(t,y_1,\cdots,y_n). \end{aligned} \tag{44}$$

From System (43) and System (44), it is clear that the following equalities

$$\frac{\partial y_j}{\partial x_i} = 0\ \ (j<i),\ \frac{\partial x_j}{\partial y_i} = 0\ \ (j<i). \tag{45}$$

must be hold. For System (4), we compute

$$\begin{aligned} \frac{\partial f_i}{\partial x_{i+1}} &= \frac{\partial f_i}{\partial t}\cdot\frac{\partial t}{\partial x_{i+1}} + \sum_{j=1}^{n}\frac{\partial f_i}{\partial y_j}\cdot\frac{\partial y_j}{\partial x_{i+1}} \\ &= \sum_{j=1}^{i+1}\frac{\partial f_i}{\partial y_j}\cdot\frac{\partial y_j}{\partial x_{i+1}} = \frac{\partial f_i}{\partial y_{i+1}}\cdot\frac{\partial y_{i+1}}{\partial x_{i+1}} \\ &= \left(\partial\left(\frac{\partial x_i}{\partial t} + \sum_{j=1}^{i}\frac{\partial x_i}{\partial y_j}\cdot f_{p,j}\right)\Big/\partial y_{i+1}\right)\cdot\frac{\partial y_{i+1}}{\partial x_{i+1}} \\ &= \left(\frac{\partial^2 x_i}{\partial t\partial y_{i+1}} + \partial\left(\sum_{j=1}^{i}\frac{\partial x_i}{\partial y_j}\cdot f_{p,j}\right)\Big/\partial y_{i+1}\right)\cdot\frac{\partial y_{i+1}}{\partial x_{i+1}} \\ &= \sum_{j=1}^{i}\left(\frac{\partial^2 x_i}{\partial y_j\partial y_{i+1}}\cdot f_{p,j} + \frac{\partial x_i}{\partial y_j}\cdot\frac{\partial f_{p,j}}{\partial y_{i+1}}\right)\cdot\frac{\partial y_{i+1}}{\partial x_{i+1}} \\ &= \frac{\partial x_i}{\partial y_i}\cdot\frac{\partial f_{p,i}}{\partial y_{i+1}}\cdot\frac{\partial y_{i+1}}{\partial x_{i+1}} \end{aligned} \tag{46}$$

and

$$\begin{aligned} \frac{\partial^k x_i}{\partial y_i^k\partial x_{i+1}} &= \frac{\partial^{k+1} x_i}{\partial y_i^k\partial t}\frac{\partial t}{\partial x_{i+1}} + \sum_{j=1}^{i}\frac{\partial^{k+1} x_i}{\partial y_i^k\partial y_j}\frac{\partial y_j}{\partial x_{i+1}} = 0 \\ \frac{\partial^{k+1} f_{p,i}}{\partial y_{i+1}^k\partial x_{i+1}} &= \frac{\partial^{k+1} f_{p,i}}{\partial y_{i+1}^k\partial t}\frac{\partial t}{\partial x_{i+1}} + \sum_{j=1}^{i+1}\frac{\partial^{k+1} f_{p,i}}{\partial y_{i+1}^k\partial y_j}\frac{\partial y_j}{\partial x_{i+1}} \\ &= \frac{\partial^{k+1} f_{p,i}}{\partial y_{i+1}^{k+1}}\frac{\partial y_{i+1}}{\partial x_{i+1}}. \end{aligned} \tag{47}$$

From Eqs. (46) and (47), we see that $\partial^k f_i / \partial x_{i+1}^k$ is the sum of the following items

$$\begin{aligned} &\frac{\partial x_i}{\partial y_i}\cdot\left(\left(\frac{\partial f_{p,i}}{\partial y_{i+1}}\right)^{n_f^1}\cdot\left(\frac{\partial f_{p,i}}{\partial y_{i+1}}\right)^{n_f^2}\cdots\cdots\left(\frac{\partial^k f_{p,i}}{\partial y_{i+1}^k}\right)^{n_f^k}\right) \\ &\cdot\left(\left(\frac{\partial y_{i+1}}{\partial x_{i+1}}\right)^{n_x^1}\cdot\left(\frac{\partial y_{i+1}}{\partial x_{i+1}}\right)^{n_x^2}\cdots\cdots\left(\frac{\partial^j y_{i+1}}{\partial x_{i+1}^j}\right)^{n_x^j}\right), \\ &k \ge 1,\ j \ge 1,\ n_f^k \ge 1,\ n_x^k \ge 1. \end{aligned} \tag{48}$$

It implies that for every integer satisfying $1 \le k < p_i$ $\left.\partial^k f_i / \partial x_{i+1}^k\right|_{x=0} = 0$ and $\left.\partial^{p_i} f_i / \partial x_{i+1}^{p_i}\right|_{x=0} \ne 0$.

(Sufficiency) We try to prove the only if part of the theorem by a constructive approach. From condition (i), System (1) are in the low-triangular form

$$\begin{aligned} \dot{x}_1 &= f_1(t,x_1,x_2) \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(t,x_1,x_2,\cdots,x_n) \\ \dot{x}_n &= f_n(t,x_1,x_2,\cdots,x_n) + g_n(t,x_1,x_2,\cdots,x_n)u. \end{aligned} \tag{49}$$

And It is guaranteed by condition (ii) that

$$\frac{\partial^i f_1}{\partial x_2^i}(0,0)\begin{cases} =0 & 0\le i\le p_1-1 \\ \neq 0 & i=p_1. \end{cases} \quad (50)$$

By using Lemma (3) there exist partial coordinate changes

$$\begin{aligned} & x_{1,j}=x_j \quad 1\le j\le n \text{ and } j\neq 2 \\ & x_{1,2}=\rho_1(t,x_1,x_2), \end{aligned} \quad (51)$$

where ρ_1 is invertible respect to x_2, such that

$$f_1(t,x_1,x_2)=\delta_1\hat{x}_{1,2}^{p_1}+\sum_{i=0}^{p_1-1}\varphi_1^i(t,x_{1,1})x_{1,2}^i, \quad (52)$$

where $\delta_1=\pm 1$, $\varphi_1^i(0,0)=0$. The System can be express as

$$\begin{aligned} \dot{x}_{1,1} &= \delta_1 x_{1,2}^{p_1}+\sum_{i=0}^{p_1-1}\varphi_1^i(t,x_{1,1})x_{1,2}^i \\ \dot{x}_{1,2} &= f_{1,2}(t,x_{1,1},x_{1,2},x_{1,3}) \\ &\vdots \\ \dot{x}_{1,n-1} &= f_{1,n-1}(t,x_{1,1},x_{1,2},\cdots,x_{1,n}) \\ \dot{x}_{1,n} &= f_{1,n}(t,x_1,x_2,\cdots,x_n)+g_{1,n}(t,x_1,x_2,\cdots,x_n)u. \end{aligned} \quad (53)$$

According to condition (i),

$$\frac{\partial^i f_{1,2}}{\partial x_{1,3}^i}(0,0)\begin{cases} =0 & 0\le i\le p_2-1 \\ \neq 0 & i=p_2. \end{cases} \quad (54)$$

holds. There exist partial coordinate changes

$$\begin{aligned} & \hat{x}_{2,j}=x_{1,j} \quad 1\le j\le n \text{ and } j\neq 3 \\ & \hat{x}_{2,3}=\rho_2(t,x_{1,1},x_{1,2},x_{1,3}), \end{aligned} \quad (55)$$

where ρ_2 is invertible respect to $x_{1,3}$, such that

$$f_{2,2}(t,x_{2,1},x_{2,2})=\delta_2 x_{2,3}^{p_2}+\sum_{i=0}^{p_2-1}\varphi_2^i(s,x_{2,1},x_{2,2})x_{2,3}^i, \quad (56)$$

where $\delta_2=\pm 1$, $\varphi_2^i(0,0)=0$. The System can be rewritten as

$$\begin{aligned} \dot{x}_{2,1} &= \delta_1 x_{2,2}^{p_1}+\sum_{i=0}^{p_1-1}\varphi_1^i(t,x_{2,1})x_{2,2}^i \\ \dot{x}_{2,2} &= \delta_{2,2} x_{2,3}^{p_2}+\sum_{i=0}^{p_2-1}\varphi_2^i(t,x_{2,1},x_{2,2})x_{2,3}^i \\ \dot{x}_{2,3} &= f_{2,3}(t,x_{2,1},x_{2,2},x_{2,3},x_{2,4}) \\ &\vdots \\ \dot{x}_{2,n-1} &= f_{2,n-1}(t,x_{2,1},x_{2,2},\cdots,x_{2,n}) \\ \dot{x}_{2,n} &= f_{2,n}(t,x_{2,1},x_{2,2},\cdots,x_{2,n})+g_n(t,x_{2,1},x_{2,2},\cdots,x_{2,n})u. \end{aligned} \quad (57)$$

Repeat the previous steps, and then System (1) is converted into the form of

$$\begin{aligned} \dot{x}_{n-1,1} &= \delta_1 x_{n-1,2}^{p_1}+\sum_{i=0}^{p_1-1}\varphi_1^i(t,x_{n-1,1})x_{n-1,2}^i \\ &\vdots \\ \dot{x}_{n-1,n-1} &= \delta_{n-1}x_{n-1,n}^{p_{n-1}}+\sum_{i=0}^{p_{n-1}-1}\varphi_{n-1}^i(t,x_{n-1,1},\cdots,x_{n-1,n-1})x_{n-1,n}^i \\ \dot{x}_{n-1,n} &= f_{n-1,n}(t,x_{n-1,1},\cdots,x_{n-1,n})+g_{n-1,n}(t,x_{n-1,1},\cdots,x_{n-1,n})u. \end{aligned} \quad (58)$$

Let $y_n=x_{n-1,n}$ and $y_{n-1}=\delta_{n-1}x_{n-1,n-1}$, we obtain

$$\begin{aligned} \dot{x}_{n-1,n-2} &= \frac{\delta_{n-2}}{\delta_{n-1}^{p_{n-2}}}y_{n-1}^{p_{n-2}}+\sum_{i=0}^{p_{n-2}-1}\frac{\varphi_{n-2}^i(t,x_1,\cdots,x_{n-2})}{\delta_{n-1}^i}y_{n-1}^i \\ \dot{y}_{n-1} &= y_n^{p_{n-1}}+\delta_{n-1}\sum_{i=0}^{p_{n-1}-1}\varphi_{n-1}^i(t,x_{n-1,1},\cdots,x_{n-1,n-1},y_{n-1})y_n^i. \end{aligned} \quad (59)$$

Let $y_{n-2}=\left(\delta_{n-2}/\delta_{n-1}^{p_{n-2}}\right)x_{n-1,n-1}$. Hence the two equations in Eq. (59) are both in the form of System (5). Repeat the previous steps and choose proper feedback, then System (1) is converted into System (5).

Remark 2. For the autonomous case, [8] provided necessary and sufficient conditions for nonlinear systems to be equivalent to the p-normal forms. For checking those conditions, we have to compute Lie bracket $p_1+p_2+\cdots+p_{n-1}$ times. So the conditions given in Theorem 2 are easy to check.

Example 2. Consider the following non-autonomous system slightly different from System (29).

$$\begin{aligned} \dot{\xi}_1 &= \xi_3^4+\xi_3^2+\xi_2^3+\xi_2\cos t+4\xi_1\cos t+\xi_1+u \\ \dot{\xi}_2 &= \xi_3^4+\xi_3^2+\xi_2\cos t+2\xi_1\xi_3+2\xi_3 u \\ \dot{\xi}_3 &= \xi_1+u. \end{aligned} \quad (60)$$

Followed Example 1 we can convert System (60) into a non-autonomous low-triangular form

$$\begin{aligned} \dot{x}_1 &= f_1(t,x_1,x_2)=x_2^3+4x_1\cos t \\ \dot{x}_2 &= f_2(t,x_1,x_2,,x_3)=x_3^4+x_3^2+x_2\cos t \\ \dot{x}_3 &= f_3(t,x_1,x_2,,x_3)+g_3(t,x_1,x_2,,x_3)u=x_1+u. \end{aligned} \quad (61)$$

System (61) satisfies $\partial f_1/\partial x_2|_{\mathbf{x=0}}=0$, $\partial^2 f_1/\partial x_2^2|_{\mathbf{x=0}}=0$, $\partial^3 f_1/\partial x_2^3|_{\mathbf{x=0}}=1$, $\partial f_2/\partial x_3|_{\mathbf{x=0}}=0$, $\partial^2 f_2/\partial x_3^2|_{\mathbf{x=0}}=1$. By Theorem 2 System (60) can be further convert into a non-autonomous p-normal form. The first equation of System (61) is in the form of p-normal form. So we choose the partial coordinate changes as $y_1=x_1$, $y_2=x_2$. Using the approach provided in [12] we know that the second equation of System (5) is in the form of p-normal form by setting the new state variable $y_3=(x_3^4+x_3^2)^{1/2}$. Choosing a proper feedback System (60) can be express as the following equations in terms of the new coordinates.

$$\begin{aligned}\dot{y}_1 &= y_2^3 + 4y_1 \cos t \\ \dot{y}_2 &= y_3^2 + y_2 \cos t \\ \dot{y}_3 &= v.\end{aligned} \tag{62}$$

6. CONCLUSIONS

One purpose of studying the problem of equivalence between two control systems is to transform a complex nonlinear system to a simple system whose controller is easy to realize real-time response in cyber-physical systems. It is of immediate significance to the design and analysis of a class of nonlinear systems. By using differential geometric control theory we provide the necessary and sufficient conditions under which non-autonomous systems are feedback equivalent to non-autonomous low-triangular systems and non-autonomous p-normal systems respectively. We also discuss how to realize the conversions and how to simplify the computation procedure of checking those conditions.

ACKNOWLEDGEMENT

This work was supported by the National High-Tech Research \& Development Program of China (Grant No. 2009AA04z138) and the National Natural Science Foundation of China (Grant Nos. 60604015, 61074054, 60774021).

REFERENCES

Celikovsky, S., Nijmeijer, H. (1996). Equivalence of nonlinear systems to triangular form: the singular case. *Systems & Control Letters*, 27, 135–144.

Cheng, D., Lin, W. (2003). On p-normal forms of Nonlinear System. *IEEE Transactions on Automatic Control*, 48(7), 1242–1248.

Gupta, S.K.S., Mukherjee, T., Varsamopoulos, G., Banerjee, A. (2011). Research directions in energy-sustainable cyber–physical systems. *Sustainable Computing: Informatics and Systems*, 1, 54-71.

Hong, Y., Wang, J., Cheng, D. (2006). Adaptive finite-time control of nonlinear systems with parametric uncertainty. *IEEE Transactions on Automatic Control,* 51(5), 858-862.

Ikeda, K., Murota, K. (2010). Imperfect Bifurcation in Structures and Materials. *Engineering Use of Group-Theoretic Bifurcation Theory.* Springer-Verlag, New York.

Ilic, M.D., Xie, L., Khan, U.A., Moura, J.M.F. (2010). Modeling of Future Cyber–Physical Energy Systems for Distributed Sensing and Control. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, 40(4), 825-883.

Lee, J.M. (2009). *Manifolds and Differential Geometry*. American Mathematical Society, New York.

Poovendran, R. (2010). Cyber–Physical Systems: Close Encounters Between Two Parallel World. *Proceedings of the IEEE*. 98(8), 1363-1366.

Respondek, W. (2003). Transforming a single-input system to a p-normal form via feedback. *Proceedings of 42nd IEEE CDC, Maui*, 1574-1579.

Sastry, S. (2010). Nonlinear Systems. *Analysis, Stability, and Control.* Springer-Verlag, New York.

Sha, L., Gopalakrishnan, S., Liu, X., Wang, Q. (2008). Cyber-physical systems: A New Frontier. *IEEE International Conference on Sensor Networks, Ubiquitous, and Trustworthy Computing*, 1-9.

Yang, B., Lin, W. (2005). Robust output feedback stabilization of uncertain nonlinear systems with uncontrollable and unobservable linearization. *IEEE Transactions on Automatic Control,* 50(5), 619-630.